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Research Article

COMPUTATION IN GENERALIZED METRIC SPACESAND QUASI-PSEUDO-METRIC PARTIAL

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ABSTRACT
In this paper we consider the GPQ-metric spaces which are a generalized Partial quasi-metrics and GQ-metric spaces which are a generalized quasi-metrics.

Keywords: *Generalized Partial Quasi-metric; Quasi-metric; GPQ-metric; G-metric*

INTRODUCTION
Partial metrics were introduced by Matthews (1992) as a generalization of metrics where self-distances are not necessarily zero, Mustafa and Sims introduced a new structure of generalized metric spaces called G-metric spaces. We introduce a generalization of a G-metric space which is also a generalization of a partial metric space.

Let X be a nonempty set and g_{pq}: X ×X ×X → [0,∞), be a function

1a) $g_{pq}(x, x, x) \leq g_{pq}(x, x, y)$ whenever $x \neq y$;
1b) $g_{pq}(x, x, x) \leq g_{pq}(x, y, y)$;
1c) $g_{pq}(x, x, x) \leq g_{pq}(y, x, x)$;
1d) $g_{pq}(x, x, x) \leq g_{pq}(x, y, x)$;
2a) $g_{pq}(x, x, y) \leq g_{pq}(x, y, z)$ whenever $y \neq z$;
2b) $g_{pq}(x, y, x) \leq g_{pq}(x, z, y)$ whenever $x \neq z$;
2c) $g_{pq}(y, x, x) \leq g_{pq}(x, y, z)$ whenever $x \neq y$, $x \neq z$;
2d) $g_{pq}(y, x, x) \leq g_{pq}(x, z, y)$ whenever $x \neq y$, $x \neq z$;
2e) $g_q(x, y, y) \leq g_q(z, x, y)$ whenever $x \neq z$, $x \neq y$;
2f) $g_q(x, y, y) \leq g_q(z, y, x)$ whenever $x \neq z$, $x \neq y$;
3) $g_{pq}(x, y, z) \leq g_{pq}(x, a, a)+g_{pq}(a, y, z)- g_{pq}(a, a, a)$ whenever $x, y, z \in X$ and
4) $x = y = z$ iff ($g_{pq}(x, x, x) = g_{pq}(x, y, z)$ and $g_{pq}(y, y, y) = g_{pq}(y, x, z)$ and $g_{pq}(z, z, z) = g_{pq}(z, y, x)$) whenever $x, y, z \in X$.

Then the function g_{pq} is called a generalized partial quasi-metric and then (X, g_{pq}) is called a GPQ-metric space.

1.1. Defnition
A function q: X×X → [0,∞) is called a quasi-metric iff (Bukatin *et al.*, 2006; Künzi *et al.*, 2006),
1) $x = y$ iff $q(x, y) = 0 = q(y, x)$ whenever $x, y \in X$,
2) $q(x, z) \leq q(x, y) + q(y, z)$ whenever $x, y, z \in X$.

2. GQ-metric Spaces
2.1. Definition
Let X be a nonempty set and g_q: X×X×X → [0,∞)
is called a quasi-metric iff
1) $x = y = z$ iff $g_q(x, y, z) = g_q(z, y, x) = g_q(x, x, x) = g_q(y, y, y) = g_q(z, z, z) = 0$ whenever $x, y, z \in X$,
2a) $g_q(x, x, y) \leq g_q(x, y, z)$ whenever $y \neq z$;
2b) $g_q(x, y, x) \leq g_q(x, z, y)$ whenever $x \neq z$;
2c) $g_q(y, x, x) \leq g_q(x, y, z)$ whenever $y \neq z$;
2d) $g_q(y, x, x) \leq g_q(x, z, y)$ whenever $y \neq z$;
2e) $g_q(x, y, y) \leq g_q(z, x, y)$ whenever $x \neq z$, $x \neq y$;

2f) $g_q(x, y, y) \leq g_q(z, y, x)$ whenever $x \neq z$, $x \neq y$;
3) $g_q(x, y, z) \leq g_q(x, a, a) + g_q(a, y, z)$ whenever $x, y, z, a \in X$.
Then (X, g_q) is called a GQ-metric space.
cleary every G-metric is a GQ-metric space, but every GQ-metric is not a G-metric space.

2.2. Examples

Let $X = \{a, b\}$ and $g_q: X \times X \times X \to [0,\infty)$ be defined as following:
$g_q(a, a, a) = g_q(b, b, b) = 0$;
$g_q(a, a, b) = g_q(a, b, a) = 1$;
$g_q(b, a, a) = 2$;
$g_q(a, b, b) = g_q(b, a, b) = g_q(b, b, a) = 3$;
Then (X, g_q) is a GQ-metric space,but Clearly (X, g_q) is not a G-metric space.

2.3. Examples

Let $X = [0,\infty)$ and $g_q : X \times X \times X \to [0,\infty)$ for all
$x, y, z \in X$ be defined as following:

$$g_q(x, y, z) = \begin{cases} 0 & x = y = z, \\ \max\{x, y, z\} & \text{otherwise} \end{cases}$$

Then (X, g_q) is a GQ-metric space,and a G-metric space.

3. GPQ-METRIC SPACES

3.1. Definition

A partial quasi-metric on a set X is a function $pq : X \times X \to [0,\infty)$ such that (Heckmann, 1999),
1a) $pq(x, x) \leq pq(x, y)$ whenever $x, y \in X$.
1b) $pq(x, x) \leq pq(y, x)$ whenever $x, y \in X$.
2) $pq(x, z) \leq pq(x, y) + pq(y, z) - pq(y, y)$ whenever $x, y, z \in X$.
3) $x = y$ iff $(pq(x, x) = pq(x, y)$ and $pq(y, y) = pq(y, x))$ whenever $x, y \in X$.
If pq satis_es all these conditions except possibly (1b), we shall speak
of a lopsided partial quasi-metric.

3.2. Definition

Let X be a nonempty set and $g_{pq} : X \times X \times X \to [0,\infty)$, be a function
1a) $g_{pq}(x, x, x) \leq g_{pq}(x, x, y)$ whenever $x \neq y$;
1b) $g_{pq}(x, x, x) \leq g_{pq}(x, y, y)$;
1c) $g_{pq}(x, x, x) \leq g_{pq}(y, x, x)$;
1d) $g_{pq} (x, x, x) \leq g_{pq} (x, y, x)$;
2a) $g_{pq} (x, x, y) \leq g_{pq} (x, y, z)$ whenever $y \neq z$;
2b) $g_{pq} (x, y, x) \leq g_{pq} (x, z, y)$ whenever $x \neq z$;
2c) $g_{pq} (y, x, x) \leq g_{pq} (x, y, z)$ whenever $x \neq y$, $x \neq z$;
2d) $g_{pq} (y, x, x) \leq g_{pq} (x, z, y)$ whenever $x \neq y$, $x \neq z$;
2e) $g_q(x, y, y) \leq g_q(z, x, y)$ whenever $x \neq z$, $x \neq y$;
2f) $g_q(x, y, y) \leq g_q(z, y, x)$ whenever $x \neq z$, $x \neq y$;
2d) $0 \leq g_{pq}(y, x, x) \leq g_{pq}(x, z, y)$ whenever $y \neq z$.
3) $g_{pq}(x, y, z) \leq g_{pq}(x, a, a)+g_{pq}(a, y, z)- g_{pq}(a, a, a)$ whenever $x, y, z \in X$ and
4) $x = y = z$ iff $(g_{pq}(x, x, x) = g_{pq}(x, y, z)$ and $g_{pq}(y, y, y) = g_{pq}(y, x, z)$
and $g_{pq}(z, z, z) = g_{pq}(z, y, x))$ whenever $x, y, z \in X$.
Then the function g_{pq} is called a generalized partial quasi-metric and
then (X, g_{pq}) is called a GPQ-metric space.
if g_{pq} is a generalized partial quasi-metric on X satisfying
$g_{pq}(x, y, z) = g_{pq}(z, x, y) = g_{pq}(x, z, y) = \ldots$ (symmetry in all three variables) whenever $x, y, z \in X$
then g_{pq} is called a generalized partial metric on X.
Künzi, Pajoohesh and Schellekens studied another variant of partial metrics, namely partial quasi-metrics, by dropping the symmetry condition in the definition of a partial metric .we in this study a generalization

of partial quasi-metrics. If g_q is a generalized quasi-metric on X then g_q is a generalized partial quasi-metric on X but it is not necessary to hold vice versa (Künzi and Vanjer, 1994).

3.3. Examples

let $X = [0,\infty)$ and (X, d) is an ordinary metric space, let $g_{pq} : X \times X \times X \to [0,\infty)$, be a function, (X, g_{pq}) can define GPQ- metrics on X by

1)$g_{pq}(x, y, z) = \max\{x, y, z\}$,

2)$g_{pq}(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$ whenever $x, y, z \in X$.

then g_{pq} is a GPQ-metric space and is a GP-metric space, Clearly(X, g_{pq}) is not a G-metric space and is not a GQ-metric space.

3.4. Examples

Let X be a set and let $f : X \to [0,\infty)$ not be an one- to-one function. Set $g_{qf}(x, y, z) = \max\{f(z) - f(y) - f(x)\}$ whenever $x, y, z \in X$. Then g_{qf} is not a GQ-metric on X. Clearly g_{qf} is a GPQ- metric and a GP-metric, but (X, g_{qf}) is not a G-metric

3.5. Examples

if $a = b = c \Rightarrow g_{pq}(a, b, c) = 1$;
$g_{pq}(a, a, b) = g_{pq}(a, b, b) = 2$;
$g_{pq}(a, a, c) = g_{pq}(a, c, c) = 4$;
$g_{pq}(b, c, c) = g_{pq}(c, b, b) = 6$;
$g_{pq}(a, b, c) = 8$;
$g_{pq}(b, a, c) = g_{pq}(b, c, a) = \cdots = 9$;

Then (X, g_{pq}) is a GPQ-metric space,but Clearly (X, g_{pq}) is not a GP- metric space and a G-metric space and a GQ-metric space.

3.6. Lemma

a) Each generalized quasi-metric g_{pq} on X is a generalized partial quasi-metric on X with $g_{pq}(x, x, x) = 0$ whenever $x \in X$.

b) If g_{pq} is a generalized (partial) quasi-metric on X, then its conjugate $g_{pq}^{-1}(x, y, z) = g_{pq}(z, y, x)$ whenever $x, y \in X$ is a generalized (partial) quasi-metric on X.

c) If g_{pq} is a generalized (partial) quasi-metric on X, then g_{pq}^{+} defined by $g_{pq}^{+}(x, y, z) = g_{pq}(x, y, z) + g_{pq}^{-1}(x, y, z)$ whenever $x, y, z \in X$ is a generalized (partial) metric on X.

3.7. Proposition

Every GPQ-metric space (X, g_{pq}) defines a metric space (X, dg_{pq}) as follows (Abdeljawad *et al.*, 2012; Dehghan and Mazaheri, 2012):

$d_{gpq}(x, y) = g_{pq}(x, y, y) + g_{pq}(y, x, x) - g_{pq}(x, x, x) - g_{pq}(y, y, y)$ for all $x, y \in X$.

Proof. 1)$d_{gpq}(x, y) = 0$ iff $(g_{pq}(x, y, y) - g_{pq}(x, x, x)) + (g_{pq}(y, x, x) - g_{pq}(y, y, y)) = 0$ iff $g_{pq}(x, y, y) - g_{pq}(x, x, x) = 0$ and $g_{pq}(y, x, x) - g_{pq}(y, y, y) = 0$, since $0 \leq g_{pq}(x, y, y) - g_{pq}(x, x, x))$ and $0 \leq g_{pq}(y, x, x) - g_{pq}(y, y, y) = 0$. so the statements $d_{gpq}(x, y) = 0$ and $x = y$ are equivalent.

2) $d_{gpq}(y, x) = g_{pq}(y, x, x) + g_{pq}(x, y, y) - g_{pq}(y, y, y) - g_{pq}(x, x, x) = d_{gpq}(x, y)$ for all $x, y \in X$.

similary other conditions of a metric space are hold.

3.8. Proposition

Every GPQ-metric space (X, g_{pq}) defines a quasi- metric space (X, q_{gpq}) as follows:

$q_{gpq}(x, y) = g_{pq}(x, y, y) + g_{pq}(y, x, x) - 2g_{pq}(x, x, x)$ for all $x, y \in X$.

Proof. (1) $q_{gpq}(x, y) = q_{gpq}(y, x) = \circ$
$\Leftrightarrow g_{pq}(x, y, y) + g_{pq}(y, x, x) - 2g_{pq}(x, x, x)$
$= g_{pq}(y, x, x) + g_{pq}(x, y, y) - 2g_{pq}(y, y, y) = \circ$
$\Leftrightarrow x = y$.

(2) $q_{gpq}(x, y) + q_{gpq}(y, z) = g_{pq}(x, y, y) + g_{pq}(y, x, x) - 2g_{pq}(x, x, x)$
$+ g_{pq}(y, z, z) + g_{pq}(z, y, y) - 2g_{pq}(y, y, y)$
$= (g_{pq}(x, y, y) + g_{pq}(y, z, z) - g_{pq}(y, y, y))$
$+ (g_{pq}(z, y, y) + g_{pq}(y, x, x) - g_{pq}(y, y, y))$
$- 2g_{pq}(x, x, x)$

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$\geq g_{pq}(x, z, z) + g_{pq}(z, x, x) - 2g_{pq}(x, x, x)$
$= q_{gpq}(x, z)$.

3.9. Definition

Let (X, g_{pq}) be a GPQ-metric space and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n;m\to\infty} g_{pq}(x, x_n, x_n) = 0$, and one says that the sequence $\{x_n\}$ is GPQ-convergent to x.

3.10. Definition

Let (X, g_{pq}) be a GPQ-metric space. A sequence $\{x_n\}$ is called GPQ-Cauchy if, for every $\epsilon > 0$, there is a positive integer N such that $g_{pq}(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$, that is, if $g_{pq}(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.

3.11. Definition

A generalized partial metric space (X, g_{pq}) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges.

3.12. Lemma

If (X, g_{pq}) is a GPQ-metric space, then the following are equivalent.

1) $\{x_n\}$ is GPQ-convergent to x.
2) $g_{pq}(x_n, x_n, x) \to 0$, as $n \to \infty$.
3) $g_{pq}(x_n, x, x) \to 0$, as $n \to \infty$.
4) $g_{pq}(x_m, x_n, x) \to 0$, as $n,m \to \infty$.

Proof. The proof is straightforward.

3.13. Definition

Let (X, g_{pq}) be a GPQ-metric space. Then the following are equivalent

1) the sequence $\{x_n\}$ is GPQ-Cauchy
2) for an y $\epsilon > 0$,there exists $N \in N$ such that $g_{pq}(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$.

3.14. Definition

Let (X_1, g_{pq1}) and (X_2, g_{pq2}) be two GPQ-metric spaces and let $f : (X_1, g_{pq1}) \to (X_2, g_{pq2})$ be a function, then f is said to be GPQ- continuous at a point $a \in X_1$ iff for a given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X_1$ and the inequality $gpq1(a, x, y) < \delta + g_{pq1}(a, a, a)$ implies that $g_{pq2}(f(a), f(x), f(y)) < \epsilon + g_{pq2}(f(a), f(a), f(a))$.

A function f is GPQ-continuous on X_1 iff it is GPQ-continuous at all $a \epsilon X_1$.

3.15. Proposition

Let (X_1, g_{pq1}), (X_2, g_{pq2}) be GPQ-metric spaces.

Then a function $f : X_1 \to X_2$ is GPQ-continuous at a point $x \in X$ iff it is GPQ-sequentially continuous at x; that is, whenever $\{x_n\}$ is GPQ-convergent to x one has $\{f(x_n)\}$ is GPQ-convergent to f(x).

Proof. The proof is straightforward.

3.16. Definition

Let (X, g_{pq}) be a GPQ-metric space, $A \subseteq X$. The set A is GPQ-compact, if for every GPQ-sequence $\{x_n\}$ in A there exists subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that GPQ-converges to $x_0 \in A$.

3.17. Theorem

Let (X_1, g_{pq1}) and (X_2, g_{pq2}) be GPQ-metric spaces and $f : X_1 \to X_2$ a GPQ-continuous function on X_1. If X is GPQ- compact, then f(X) is GPQ-compact.

Proof. It is clear, since f is GPQ-sequentially continuous on X.

4. Gq-metric Spaces with Weight

4.1. Definition

An arbitrary generalized quasi-metric space (X, g_q) equipped with an arbitrary (so-called weight) function $w : X \to [0,\infty)$ will be called a generalized quasi-metric space with weight.(It should be stressed that no condition of compatibility is assumed at this stage.) Next we define a compatibility condition between generalized quasimetric and weight that will be crucial for the following investigations.

4.2. Definition

A generalized quasi-metric space with compatible weight on a set X is a triple (X, g_q, w) where $g_q : X \times X \times X \to [0,\infty)$ is a generalized quasi-metric on X and $w : X \to [0,\infty)$ is a function satisfying $2w(x) \le g_q(x, y, z) + w(y) + w(z)$ whenever $x, y, z \in X$.

4.3. Lemma

Let (X, g_q, w) be a generalized quasi-metric space with weight.Then clearly w is a compatible weight on X iff $\hat{g}_q$ defined by $\hat{g}_q(x, y, z) = g_q(x, y, z) + w(y) + w(z) - 2w(x)$ whenever $x, y, z \in X$ is a generalized quasi-metric on X.

Proof(1) . $\Rightarrow \hat{g}_q(x, x, x) = \hat{g}_q(y, y, y) = \hat{g}_q(y, x, z) = \hat{g}_q(z, y, x) = \hat{g}_q(z, z, z) = \hat{g}_q(x, z, y)$
$= \hat{g}_q(x, y, z) = g_q(x, y, z) + \omega(y) + \omega(z) - 2\omega(x) = \circ$
$\Leftrightarrow g_q(x, x, x) + \omega(x) + \omega(x) - 2\omega(x) = \circ$
$\Leftrightarrow g_q(y, y, y) + \omega(y) + \omega(y) - 2\omega(y) = \circ$
$\Leftrightarrow g_q(z, z, z) + \omega(z) + \omega(z) - 2\omega(z) = \circ$
$\Leftrightarrow x = y = z.$

(2) $\hat{g}_q(x, x, y) = g_q(x, x, y) + \omega(x) + \omega(y) - 2\omega(x)$
$\le g_q(x, y, z) + \omega(y) + \omega(z) - 2\omega(x)$
$= \hat{g}_q(x, y, z),$
in the same way
$\hat{g}_q(x, y, x) \le \hat{g}_q(x, z, y),$
$\hat{g}_q(y, x, x) \le \hat{g}_q(x, y, z),$
$\hat{g}_q(y, x, x) \le \hat{g}_q(x, z, y).$

(3) $\hat{g}_q(x, y, z) = g_q(x, y, z) + \omega(y) + \omega(z) - 2\omega(x)$
$\le g_q(x, a, a) + g_q(a, y, z) + \omega(y) + \omega(z) - 2\omega(a)$
$= (g_q(x, a, a) + \omega(a) + \omega(a) - 2\omega(x))$
$+ (g_q(a, y, z) + \omega(z) + \omega(y) - 2\omega(a))$
$= \hat{g}_q(x, a, a) + \hat{g}_q(a, y, z).$
then $\hat{g}_q$ is a generalized quasi-metric on X.
$\Leftarrow$ let $\hat{g}_q(x, y, z) = g_q(x, y, z)+\omega(x)+\omega(y)-2\omega(z)$. since $\hat{g}_q(x, y, z) \ge \circ$,
then $2\omega(z) \le g_q(x, y, z) + \omega(x) + \omega(y)$.

4.4. Lemma

Let (X, g_q) be a generalized quasi-metric space with weight $w : X \to [0,\infty)$. Then w is a compatible weight on the generalized quasi-metric space (X, g'_q) where g'_q denotes the generalized quasi- metric on X defined by $g'_q(x, y, z) = g_q(x, y, z) + g_{qw}(x, y, z)$ whenever
$x, y, z \in X$.

Proof. $g'_q(x, y, z) = g_q(x, y, z) + g_{qw}(x, y, z) \le g_q(x, a, a) + g_q(a, y, z) +$
$g_{qw}(x, a, a) + g_{qw}(a, y, z) = g'_q(x, a, a) + g'_q(a, y, z)$.

4.5. Proposition

If (X, g_q, w) is a generalized quasi-metric space with compatible weight, then $(X, \hat{g}_q, \widehat{w})$ where $\hat{g}_q(x, y, z) = \min\{g_q(x, y, z), 1\}$ and $\widehat{w}(x) = \min\{\widehat{w}(x), 1\}$ whenever $x, y, z \in X$.

Proof. For either case it is well known and easy to see that $\hat{g}_q$ is a generalized quasi-metric on X. Related arguments show that given $x, y, z \in X$, $2w(x) \le g_q(x, y, z) + w(y) + w(z)$ implies that $\min\{2w(x), 1\} \le \min\{g_q(x, y, z), 1\} + \min\{w(y), 1\} + \min\{w(z), 1\}$.

4.6. Lemma

Note that if (X, g_q, w) is a generalized quasi-metric space with compatible weight, then for any generalized quasi-metric g'_q on X such that $g'_q(x, y, z) \ge g_q(x, y, z)$ whenever $x, y, z \in X$, (X, g'_q, w) is also a generalized quasi-metric space with compatible weight.

Proof. The assertion is obvious, since $2w(x) - w(y) - w(z) \le g_q(x, y, z) \le g'_q(x, y, z)$ whenever $x, y, z \in X$.

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