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Research Article

APPROXIMATING THE FORTH ORDER STRUM-LIOUVILLE EIGENVALUE PROBLEMS BY HOMOTOPY ANALYSIS METHOD

***Nader Rafatimaleki**
Department of Mathematics, College of Science, Islamic Azad University, Tabriz Branch, Tabriz, Iran
**Author for Correspondence*

ABSTRACT
In this paper, the homotopy analysis method (HPM) is applied to approximate the eigenvalues of the forth order Sturm-Liouville problems. The approximations start with an appropriate initial guesses. The approximate solution for eigenfunction obtains as a series with $\hbar$, λ and x. Substituting the initial guess of the main problem gives an equation with two parameters $\hbar$ which controls the convergence rate and λ which is the unknown eigenvalue. The obtained equation is solved using Newton's method with a special search algorithm to approximate the eigenvalues. The numerical experiments show that, this method is more accurate than the other methods.

***Keywords**: Homotopy, Strum-Liouville, Homotopy Analysis Method*

INTRODUCTION
The homotopy analysis method (HAM), which first was proposed by Liao in 1992 is a method for solving nonlinear problems. This method has been successfully applied to many nonlinear problems, such as nonlinear differential equations, nonlinear integral equations, partial differential equations, fractional differential equations and so on (Jafari and Seifi, 2009; Abbasbandy *et al.*, 2009)
The HAM does not depend on any small or large parameter. Besides, it logically contains other non-perturbation techniques, such as Adomian decomposition method, Lyapanov artificial small parameter method and δ-expansion method, as proved by (Liao, 2003). Thus, the HAM is valid for much more nonlinear problems in science and engineering. Moreover, there is a qualitative difference between HAM and other methods that HAM includes an auxiliary parameter $\hbar$ which controls the convergence rate of the HAM series. (Abbasbandy *et al.*, 2009), have shown that auxiliary parameter $\hbar$ plays a basic role in the convergence rate control and prediction of multiple solutions of the equation. They use the auxiliary parameter $\hbar$, for calculating multiple solutions of the Sturm-Liouville problems (Abbasbandy and Shirzadi, 2011). We show that the method proposed by Abbasbandy *et al.*, (2013), is useful for finding just some first eigenvalues of the Sturm-Liouville problems. We use Newton's method to find arbitrary large eigenvalues of the Sturm-Liouville problems accurately.

MATERIALS AND METHODS
In this paper we consider the following class of two point eigenvalue problem of the form

$$y^{(4)}(x) - \lambda y(x) = 0, \qquad x \in (0,1),$$

Subject to

$$y(0) = y'(0) = 0, \quad y(1) = y''(1) = 0.$$

Homotopy analysis method
Consider the following nonlinear equation

$$N[u(x)] = 0,$$

Where, N is a nonlinear operator and $u(x)$ is an unknown function. For simplicity, we ignore all initial and boundary conditions, which can be treated in a similar way. By means of the traditional homotopy method, Liao constructs the so-called zero-order deformation equation as follows

$$(1-p)L[\phi(x;p) - u_0(x)] = p\hbar N[\phi(x)],$$

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Where, $p \in [0,1]$ is embedding parameter, $\hbar \neq 0$ is a nonzero auxiliary parameter, L is an auxiliary linear operator, $u_0(x)$ is an initial guess of $u(x)$ and $\phi(x;p)$ is an unknown function. It is important that we have great freedom to choose auxiliary operator in the HAM. Obviously when $p = 0$ and $p = 1$, the following relations hold respectively

$$\phi(x;0) = u_0(x), \qquad \phi(x;1) = u(x).$$

Thus, as p increases from 0 to 1, the the solution $\phi(x;p)$ varies from the initial guess $u_0(x)$ to the solution $u(x)$. Expanding $\phi(x;p)$ in Taylor series with respect to p, we have

$$\phi(x;p) = u_0(x) + \sum_{m=1}^{\infty} u_m(x)p^m, \tag{3}$$

Where

$$u_m(x) = \frac{1}{m!}\frac{\partial^m \phi(x;p)}{\partial p^m}\bigg|_{p=0}.$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function are chosen properly, the series (3) converges at $p = 1$, and we have

$$u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x),$$

This must be one of the solutions of the original nonlinear equation as proved by Liao. The governing equation can be deduced from the zero-order deformation equation. Define the vector

$$\vec{u}_n = \{u_0(x), u_1(x), \dots, u_n(x)\}.$$

Differentiating (2), m times with respect to the embedding parameter $p$and setting $p = 0$ and finally dividing by $m!$. We have the so-called mth-order deformation equation

$$L[u_m(x) - \mathcal{X}_m u_{m-1}(x)] = \hbar R_m(\vec{u}_{m-1}(x)),$$

Where

$$R_m(\vec{u}_{m-1}(x)) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x;p)]}{\partial p^{m-1}}\bigg|_{p=0},$$

And

$$\mathcal{X}_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

The Method of Abbasbandy *et al.*, (2011)

Consider the following eigenvalue problem

$$y^{(4)}(x) - \lambda y(x) = 0, \qquad x \in (0,1), \tag{6}$$

Subject to

$$y(0) = y'(0) = 0, \quad y(1) = y''(1) = 0. \tag{7}$$

We follow the method presented by Abbasbandy *et al.*, (2009) t solve (6) and (7). They assumed that the solution of (6) can be expressed by a set of base functions

$$\{x^2, x^3, x^6, x^7, \dots\}$$

In the form

$$u(x) = \sum_{i=0}^{\infty} d_i x^{4i-2} + \sum_{i=0}^{\infty} c_i x^{4i+3}, \tag{8}$$

Where c_i and d_is are coefficients to be determined. Under the rule of solution expression denoted by (8), it is obvious that one must choose the auxiliary linear operator

$$L[\phi(x,p)] = \phi^{(4)}(x,p).$$

From (6), the nonlinear operator is chosen as

$$N[\phi(x,p)] = \phi^{(4)} - \lambda\phi(x,p).$$

According to boundary conditions (7) and the rule of solution expression (8), the initial approximation is chosen in the form

$$u_0(x) = \frac{cx^2}{2} + \frac{dx^3}{6}.$$

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Then we have

$$R_m(\vec{u}_{m-1}(x)) = u_m^{(4)} - 1\,(x) - \lambda u_{m-1}(x).$$

Now the solution of the mth-order deformation equation for $m \geq 1$ becomes

$$u_m(x) = \chi_m u_{m-1}(x) + \hbar \int_0^x \int_0^x \int_0^x \int_0^x R_m(\vec{u}_{m-1}(x))dx\,dx\,dx\,dx.$$

Consequently, the first few terms of the HAM series solution are as follows

$$u_1(x) = -\frac{1}{720}\lambda c\hbar\, x^6 - \frac{1}{5040}\lambda d\hbar\, x^7,$$

$$u_2(x) = -\left(\frac{\lambda c\hbar+\lambda c\hbar^2}{720}\right)x^6 - \left(\frac{\lambda d\hbar+\lambda d\hbar^2}{5040}\right)x^7 + \frac{\lambda^2 c\hbar^2}{5040}x^{10} + \frac{\lambda^2 d\hbar^2}{39916800}x^{11},$$

$$\vdots$$

Accordingly, the mth order approximate solution of the HAM, $w_m(x)$, is in the form

$$w_m(x) = \sum_{i=0}^{m} u_i(x) = c\,p_m(x) + d\,q_m(x). \tag{9}$$

To the mt order approximate solution (9), which still depends on the eigenvalues λ, c, d and parameter $\hbar$, conditions (7) reads

$$c\,p_m(1) + d\,q_m(1) = 0,$$
$$c\,p''_m(1) + d\,q''_m(1) = 0.$$

This means that to obtain the eigenvalues λ, and in order to have a nontrivial eigenfunction solution, we must have

$$\mathrm{w}(\hbar,\lambda) = \det\begin{pmatrix} p_m(1) & q_m(1) \\ p''_m(1) & q''_m(1) \end{pmatrix} = 0. \tag{10}$$

Equation (10), which depends on the eigenvalue λ and the auxiliary parameter $\hbar$, is used in plotting $\hbar$-curve and consequently, in finding eigenvalues.

RESULTS AND DISCUSSION

The first six eigenvalues which were approximated by Abbasbandy *et al*., (2011) are as follows

Table 1: First six eigenvalues approximated by Abbasbandy *et al*., (2011).

k	**Approximated λ_k**
1	237.72106753
2	2496.48743785
3	10867.58221697
4	31780.09645427
5	74000.84934655
6	148634.47747229

Newton's Approximation Method

To try for approximating some grater eigenvalues of equation (6), it is difficult to identify the horizontal plateaus of λ. Then we conclude that the method proposed by Abbasbandy *et al*., (2009) is not convenient for approximating of large eigenvalues.

In this paper we solve the equation (10) substituting $\hbar = -1$ using Newton's iteration method. We plot $w(h = -1, \lambda)$ if figure 1 and 2 for $\lambda \in [0,10000]$ and $\lambda \in [10000,100000]$ to choose initial guess for Newton's iteration method. Table 2 shows approximated eigenvalues and their initial guess for 10 first eigenvalues of equation (6). We use 40 digits for computations and take $m = 100$.

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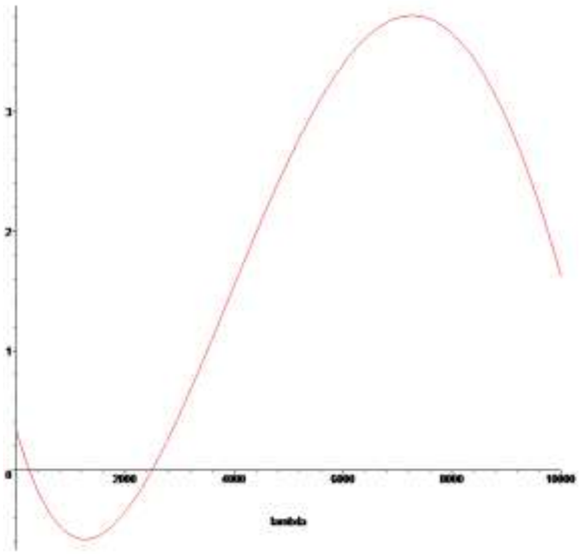

Figure 1: plot of $w(-1,\lambda)$ in Eq. (10) for $\lambda \in [0,10000]$.

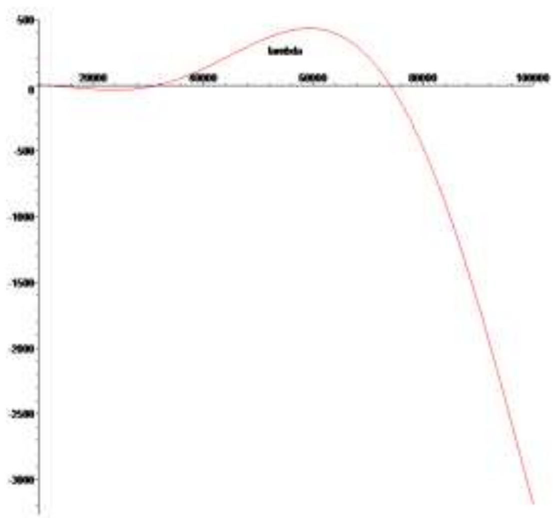

Figure 2: plot of $w(-1,\lambda)$ in Eq. (10) for $\lambda \in [10000,100000]$.

Table 2: First 15 eigen values and corresponding absolute errors of equation (Abbasbandy *et al.*, 2099) approximated in this paper

k	Initial guess	Approximated λ_k
1	200	237.72106753111664659000227147117572433 48
2	2000	2496.4874378568316694407338614891519844101
3	10000	10867.582216978888875770448686900452590 95
4	30000	31780.096454081076648268326207682572453 75
5	70000	74000.849349155493380352270105294715619 71
6	140000	148634.47728577032022321443466213398912 74
7	250000	269123.43482664171019854510580693028996 33
8	450000	45127.994719278142069828154798223646335 5
9	700000	713126.24789600406942767284788291768710 73
10	1500000	1560305.2887551025163776488543733781943 66

Conclusion

In this paper, the HAM has been applied to numerically approximate the eigenvalues of the Sturm-Liouville problems. In the plot of λ as a function of $\hbar$, several horizontal plateaus occur, which indicates the existence of multiple solutions. Indeed, every horizontal plateaus corresponds to an eigenvalue of the Sturm-Liouville problem. We show that this method is just suitable for approximating some first eigenvalues of the Sturm-Liouville problem, because it is difficult to identify the horizontal plateaus of $\hbar$-curve for large values of $\hbar$. Instead, we use Newton's iteration method with suitable initial values to solve the proposed nonlinear equation. The illustrated example shows the efficiency of this method compared with classic HAM.

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