International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jpms.htm
2026 Vol. 16, pp. 24-38/Sukalwad and Ingle
Research Article

FUZZY FRACTIONAL BOUNDARY VALUE PROBLEMS: THEORETICAL RESULTS AND SPECTRAL–ITERATIVE NUMERICAL APPROACHES

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ABSTRACT
We study fuzzy fractional boundary value problems (FFBVPs) with Caputo-type derivatives under fuzzy two-point boundary conditions. Existence and uniqueness are established via Banach's fixed point theorem under a kernel-explicit contraction bound, while existence without global Lipschitz continuity follows from Schauder's theorem and a fuzzy Arzelà–Ascoli principle. We design two solvers: a Picard iteration with guaranteed geometric convergence and a spectral operational matrix scheme with spectral accuracy for smooth data. Numerical experiments demonstrate error/runtime trade-offs, sensitivity to the fractional order, and propagation of boundary/source uncertainty.

Keywords: *fuzzy fractional differential equations; boundary value problems; Caputo derivative; fixed point theorems; spectral methods; Picard iteration; uncertainty propagation.*
MSC (2020): *26A33, 34A08, 34B10, 65M70, 65L60.*

1 INTRODUCTION

Fractional differential equations (FDEs) capture memory and hereditary effects and are central in viscoelasticity, anomalous diffusion, and control. Among many definitions, the Caputo derivative is popular for accommodating classical boundary/initial data. Monographs and surveys outline theory and numerics for FDEs, including stability and discretizations (Almeida et al., 2021; Podlubny and Luchko, 2021; Veeresha and Prakasha, 2022). At the same time, imprecision in parameters and boundary data is unavoidable in experiments and field measurements; fuzzy models provide a parsimonious way to encode epistemic uncertainty without assuming probabilistic structure (Salahshour et al., 2022; Wang and Zhou, 2022; Debbouche and Baleanu, 2024).

When fractional dynamics and fuzziness are combined, several fundamental questions arise: well-posedness of fuzzy fractional boundary value problems (FFBVPs), robustness of solution operators to fuzzy inputs, and effective numerical solvers that respect both the fractional kernel and the fuzzy structure. Recent works study existence for (crisp) fractional BVPs, fixed-point frameworks, and numerical aspects (Abbas et al., 2020; Diagana, 2021; Hosseini et al., 2023). For fuzzy fractional systems, existence/uniqueness and approximation have been investigated under various assumptions on the nonlinearity and the fuzzy metric (Salahshour et al., 2022; Ye, 2024; Cuevas and Henr´ıquez, 2021).

This paper contributes along two complementary axes. On the theoretical side, we derive a clean uniqueness condition that separates the Volterra kernel and the boundary-correction kernel, yielding $2L/\Gamma(\alpha + 1) < 1$. Existence follows from Schauder under continuity and boundedness; a kernel smallness condition gives uniqueness without global Lipschitz continuity (Diagana, 2021; Abbas et al., 2020). On the numerical side, we present (i) a Picard iteration derived directly from the integral form, and (ii) a spectral operational matrix method based on shifted Legendre polynomials. The first is simple and robust; the second attains spectral accuracy for smooth solutions and is efficient at high precision (Hosseini et al., 2023; Veeresha and Prakasha, 2022).

We provide a comparative study (error decay and runtime), applications in fuzzy viscoelastic response and fuzzy anomalous diffusion, and a sensitivity section that quantifies uncertainty propagation through band-width metrics. Throughout, we work level-wise in the cut parameter and then reassemble fuzzy outputs, which aligns with standard practice in fuzzy analysis (Wang and Zhou, 2022; Ye, 2024; Debbouche and Baleanu, 2024; Ahmad et al., 2024; Atangana and G´omez-Aguilar, 2021).

2 Preliminaries

We briefly recall fuzzy numbers, the fuzzy metric, and Caputo fractional operators; details can be found in Salahshour et al. (2022); Almeida et al. (2021).

2.1 Fuzzy numbers and metric

A (normal, convex, upper semicontinuous) fuzzy number $\tilde{x}$ is identified with its α-cuts

$$[\tilde{x}]_r = [x^-(r), x^+(r)] \subset \mathbb{R}, \qquad r \in [0,1], \tag{2.1}$$

where x^- is nondecreasing, x^+ is nonincreasing, and $[\tilde{x}]_1$ is a singleton. The standard fuzzy metric $D: \mathbb{E} \times \mathbb{E} \to [0, \infty)$ is

$$D(\tilde{x}, \tilde{y}) = \sup_{r\in[0,1]} \max\{|x^-(r) - y^-(r)|,\ |x^+(r) - y^+(r)|\}. \tag{2.2}$$

Minkowski operations on $\mathbb{E}$ are defined level-wise:

$$(\tilde{x} \oplus \tilde{y})_r = [x^-(r) + y^-(r),\ x^+(r) + y^+(r)], (c \otimes \tilde{x})_r = [\min\{cx^-, cx^+\}, \max\{cx^-, cx^+\}], \tag{2.3}$$

and the (generalized) Hukuhara difference $\tilde{x} \ominus \tilde{y}$ is defined when $[\tilde{x}]_r - [\tilde{y}]_r$ is an interval for all r.

2.2 Function spaces

Let $C([0,1], \mathbb{E})$ denote the space of continuous fuzzy-valued functions with norm

$$\| y \| := \sup_{t\in[0,1]} D(y(t), 0). \tag{2.4}$$

Lemma 2.1 *Let* $g: [0,1] \times \mathbb{E} \to \mathbb{E}$. *Then* g *is* L*-Lipschitz in the second argument (with respect to* D*) iff for each* $r \in [0,1]$, *the endpoint maps* $g^{\pm}(\cdot,\cdot; r)$ *satisfy*

$$|g^{\pm}(t, u^{\pm}(r)) - g^{\pm}(t, v^{\pm}(r))| \le L\, |u^{\pm}(r) - v^{\pm}(r)|, \quad \forall t \in [0,1]. \tag{2.5}$$

Proof. Fix r. Consider fuzzy singletons determined by real endpoints at level r. The Lipschitz property in D implies the bound on each endpoint. ($\Leftarrow$) Take arbitrary $\tilde{u}, \tilde{v} \in \mathbb{E}$. For each r, the level-wise bounds yield

$$\max\{|g^-(t, u^-(r)) - g^-(t, v^-(r))|, |g^+(t, u^+(r)) - g^+(t, v^+(r))|\}$$
$$\le L \max\{|u^-(r) - v^-(r)|, |u^+(r) - v^+(r)|\}.$$

Taking $\sup_r$ gives $D(g(t, \tilde{u}), g(t, \tilde{v})) \le L\, D(\tilde{u}, \tilde{v})$.

Lemma 2.2 *For* $\tilde{x}, \tilde{y}, \tilde{z} \in \mathbb{E}$ *and* $c \in \mathbb{R}$

$$D(\tilde{x} \oplus \tilde{y}, \tilde{x} \oplus \tilde{z}) \leq D(\tilde{y}, \tilde{z}), \qquad D(c \otimes \tilde{y}, c \otimes \tilde{z}) \leq |c|\, D(\tilde{y}, \tilde{z}). \tag{2.6}$$

Proof. These follow from level-wise operations: for each r, $(\tilde{x} \oplus \tilde{y})^{\pm} = (x^{\pm} + y^{\pm})$ and the max-norm on endpoints; similarly for scalar multiplication.

Lemma 2.3 (Completeness) $(C([0,1], \mathbb{E}), \|\cdot\|)$ *is a Banach space.*
Completeness follows from completeness of $(\mathbb{E}, D)$ *and uniform convergence of level-functions; see, e.g., (Salahshour et al., 2022, Ch. 2)*

Lemma 2.4 (Fuzzy Arzela--Ascoli) *Let* $\mathcal{F} \subset C([0,1], \mathbb{E})$ *be uniformly bounded and equicontinuous. Then* $\mathcal{F}$ *is relatively compact in* $C([0,1], \mathbb{E})$.

The proof tracks the classical Arzelà–Ascoli theorem at each level r, *then uses a diagonal argument and the metric* D; *see Wang and Zhou (2022).*

2.3 Caputo fractional operators

For $0 < \alpha < 1$, the Caputo derivative and fractional integral are

$$ {}^{C}D_t^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} y'(s)\ \mathrm{d}s, \qquad I_t^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\ \mathrm{d}s. \tag{2.7}$$

We will also reference non-singular CF/AB operators through their integral kernels.

3 Formulation of the Fuzzy Fractional Boundary Value Problem

Consider $J = [0,1]$ and let $y: J \to \mathbb{E}$ be a fuzzy-valued function. We study

$$ {}^{C}D^{\alpha} y(t) = f(t, y(t)), \quad t \in (0,1), \qquad y(0) = \tilde{a}, \quad y(1) = \tilde{b}, \tag{3.1}$$

where $0 < \alpha < 1$, $\tilde{a}, \tilde{b} \in \mathbb{E}$, and $f: J \times \mathbb{E} \to \mathbb{E}$ is continuous. Non-singular operators (Caputo–Fabrizio, Atangana–Baleanu) will also be considered.

Integral representation
Applying I^{α} to the Caputo equation yields

$$y(t) = \tilde{a} \ \oplus \ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))\ \mathrm{d}s \ \oplus \ \Lambda[y](t), \tag{3.2}$$

with boundary-correction

$$\Lambda[y](t) = t \otimes (\tilde{b} \ \ominus \ \tilde{a} \ \ominus \ \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y(s))\ \mathrm{d}s). \tag{3.3}$$

Proposition 3.1 (Equivalence) *Let* $0 < \alpha < 1$, $\tilde{a}, \tilde{b} \in \mathbb{E}$, *and* $f: [0,1] \times \mathbb{E} \to \mathbb{E}$ *be continuous. A function* $y \in C([0,1], \mathbb{E})$ *solves the Caputo FFBVP*

$$ {}^{C}D_t^{\alpha} y(t) = f(t, y(t)), \quad t \in (0,1), \qquad y(0) = \tilde{a}, \quad y(1) = \tilde{b}, \tag{3.4}$$

in the (standard) Caputo/Volterra sense if and only if it satisfies the integral equation

$$y(t) = \tilde{a} \oplus \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s))\ ds \oplus \Lambda[y](t), \tag{3.5}$$

where

$$\Lambda[y](t) = t \otimes (\tilde{b} \ominus \tilde{a} \ominus \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s, y(s))\ ds). \tag{3.6}$$

Proof. All statements are understood level-wise in the cut parameter $r \in [0,1]$, where fuzzy operations are Minkowski operations on endpoints; we suppress r in the notation for readability. We recall the standard equivalence (for $0 < \alpha < 1$): a function $z \in C([0,1])$ with sufficient regularity satisfies the Caputo equation ${}^{C}D_t^{\alpha} z = g$ and the initial condition $z(0) = z_0$ *iff* it satisfies the Volterra identity

$$z(t) = z_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)\ ds, \quad t \in [0,1]. \tag{3.7}$$

We use this fact level-wise with $g(s) = f^{\pm}(s, y^{\pm}(s))$.

(“⇒”) Assume y solves the Caputo FFBVP in the Volterra sense, i.e.

$$y(t) = \tilde{a} \oplus \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s))\ ds, \quad t \in [0,1], \tag{3.8}$$

and $y(1) = \tilde{b}$. Evaluate (3.8) at $t = 1$:

$$y(1) = \tilde{a} \oplus \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s, y(s))\ ds = \tilde{b}. \tag{3.9}$$

Rearranging this (in fuzzy arithmetic) yields the identity

$$\tilde{b} \ominus \tilde{a} \ominus \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s, y(s))\ ds = \tilde{0}, \tag{3.10}$$

where $\tilde{0}$ is the zero fuzzy number. Multiplying both sides by t (scalar fuzzy product) gives

$$t \otimes (\tilde{b} \ominus \tilde{a} \ominus \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s, y(s))\ ds) = \tilde{0}. \tag{3.11}$$

Adding this (null) term to (3.8) we obtain (3.5). Hence any Volterra solution with $y(0) = \tilde{a}$ and $y(1) = \tilde{b}$ satisfies the claimed integral form.

(“⇐”) Conversely, suppose $y \in C([0,1], \mathbb{E})$ satisfies (3.5). Setting $t = 0$ gives $y(0) = \tilde{a}$ (since the integrals vanish and $0 \otimes (\cdot) = \tilde{0}$). Setting $t = 1$ gives $y(1) = \tilde{b}$ by direct cancellation of the bracketed term.

Define the auxiliary (level-wise) function

$$w(t) := y(t) \ominus \Lambda[y](t). \tag{3.12}$$

Then (3.5) implies

$$w(t) = \tilde{a} \oplus \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s))\ ds. \tag{3.13}$$

Thus w *verifies the Volterra identity* (3.7) with $g(s) = f(s, y(s))$ and initial value $w(0) = \tilde{a}$. By the standard Volterra–Caputo equivalence (applied level-wise), w satisfies the Caputo equation

$$^{C}D_t^{\alpha} w(t) = f(t, y(t)), \qquad w(0) = \tilde{a}. \qquad (3.14)$$

Finally, note that $\Lambda[y](t)$ is (level-wise) an affine function of t with a *constant* (in t) fuzzy coefficient depending on y only through the integral over $[0,1]$. Therefore, at the level of the *Volterra identity*, subtracting $\Lambda[y]$ has merely isolated the unique function w that satisfies the canonical Caputo/Volterra relation with initial datum $\tilde{a}$. Since $y = w \oplus \Lambda[y]$ and we already verified $y(0) = \tilde{a}$, $y(1) = \tilde{b}$, it follows that y is a (two-point) Caputo solution in the usual integral sense.
Collecting the two directions, the equivalence is established.

4 Existence and Uniqueness Results

We work in $C([0,1], \mathbb{E})$ with $\| y \| := \sup_{t\in[0,1]} D(y(t), 0)$.

Assumptions

- $f: [0,1] \times \mathbb{E} \to \mathbb{E}$ is continuous in t and satisfies $D(f(t,u), f(t,v)) \le L\, D(u,v)$ for all $u, v \in \mathbb{E}$, $t \in [0,1]$.
- Boundary data $\tilde{a}, \tilde{b} \in \mathbb{E}$ are bounded.
- For CF/AB operators, kernels are positive, bounded, integrable on $[0,1]$.

Theorem 4.1 (Existence uniqueness via Banach) *Assume (A1)–(A2) and* $\frac{L}{\Gamma(\alpha+1)} < \frac{1}{2}$, *i.e.* $\frac{2L}{\Gamma(\alpha+1)} < 1$. *Then the FFBVP*

$$^{C}D^{\alpha} y(t) = f(t, y(t)),\ t \in (0,1), \qquad y(0) = \tilde{a}, \quad y(1) = \tilde{b}, \qquad (4.1)$$

has a unique solution $y \in C([0,1], \mathbb{E})$.
Proof. By Proposition 3.1, solutions are fixed points of

$$(Ty)(t) = \tilde{a} \oplus \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))\, ds \oplus \Lambda[y](t), \qquad (4.2)$$

with boundary correction

$$\Lambda[y](t) = t \otimes (\tilde{b} \ominus \tilde{a} \ominus \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y(s))\, ds). \qquad (4.3)$$

Contraction estimate. For $y_1, y_2 \in C([0,1], \mathbb{E})$ and fixed $t \in [0,1]$,

$$D((Ty_1)(t), (Ty_2)(t)) \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} D(f(s, y_1(s)), f(s, y_2(s)))\, ds + D(\Lambda[y_1](t), \Lambda[y_2](t))$$

$$\le \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} D(y_1(s), y_2(s))\, ds + t \cdot \frac{L}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} D(y_1(s), y_2(s))\, ds. \qquad (4.4)$$

Taking $\sup_{t\in[0,1]}$ and bounding by $\| y_1 - y_2 \|$,

$$\| Ty_1 - Ty_2 \| \le \frac{L}{\Gamma(\alpha)} (\sup_t \int_0^t (t-s)^{\alpha-1}\, ds + \int_0^1 (1-s)^{\alpha-1}\, ds) \| y_1 - y_2 \|. \qquad (4.5)$$

Since $\int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s = \frac{t^\alpha}{\alpha} \le \frac{1}{\alpha}$ and $\int_0^1 (1-s)^{\alpha-1}\,\mathrm{d}s = \frac{1}{\alpha}$, and $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$, we get

$$\| Ty_1 - Ty_2 \| \le \frac{2L}{\Gamma(\alpha+1)} \| y_1 - y_2 \| =: q \| y_1 - y_2 \|. \tag{4.6}$$

By $q < 1$, T is a strict contraction.

Invariant ball. Let $B_R := \{y \in C([0,1],\mathbb{E}) : \| y \| \le R\}$. By continuity of f on bounded sets, set

$$M_R := \sup\{D(f(t,u),0) : t \in [0,1], D(u,0) \le R\} < \infty. \tag{4.7}$$

Then for $y \in B_R$,

$$\| Ty \| \le D(\tilde{a},0) + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} D(f(s,y(s)),0)\ \mathrm{d}s$$
$$+ \sup_{t\in[0,1]} t \cdot \left(D(\tilde{b} \ominus \tilde{a},0) + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} D(f(s,y(s)),0)\ \mathrm{d}s\right)$$

$$\le D(\tilde{a},0) + D(\tilde{b} \ominus \tilde{a},0) + \frac{2M_R}{\Gamma(\alpha+1)}. \tag{4.8}$$

Choose R strictly larger than this bound; then $T(B_R) \subseteq B_R$.

Conclusion. Since $C([0,1],\mathbb{E})$ is complete (Lemma 2.3), Banach's fixed–point theorem yields a unique fixed point $y^* \in B_R$, i.e. the unique solution.

Theorem 4.2 (Existence via Schauder) *Assume f is continuous and maps bounded sets into bounded sets (no global Lipschitz needed). Then the FFBVP admits at least one solution $y \in C([0,1],\mathbb{E})$.*

Proof. Consider T as above. By Theorem 4.1 estimates, there exists $R > 0$ with $T(B_R) \subseteq B_R$. If $y_n \to y$ in $\|\cdot\|$, then by continuity of f and integrability of kernels, dominated convergence yields $Ty_n \to Ty$. For relative compactness, fix $r \in [0,1]$; the family $\{(Ty)^{\pm}(\cdot;r) : y \in B_R\}$ is uniformly bounded and equicontinuous by kernel integrability and bounded $f^{\pm}$; level-wise Arzelà–Ascoli implies relative compactness in $C([0,1],\mathbb{E})$. Hence T is continuous and compact on a nonempty, closed, convex set; Schauder yields a fixed point.

Theorem 4.3 (Uniqueness under kernel smallness) *Assume (A1) and that the integral representation has a nonnegative kernel $K(t,s)$ with*

$$\sup_{t\in[0,1]} \int_0^1 K(t,s)\ \mathrm{d}s < \frac{1}{L}. \tag{4.9}$$

Then any solution in $C([0,1],\mathbb{E})$ is unique.

Proof. Let y_1, y_2 be two solutions. Then

$$D(y_1(t), y_2(t)) \le L \int_0^1 K(t,s)\, D(y_1(s), y_2(s))\ \mathrm{d}s, \quad t \in [0,1]. \tag{4.10}$$

Taking $\sup_t$ gives

$\| y_1 - y_2 \| \leq L(\sup_t \int_0^1 K(t,s)\, \mathrm{d}s) \| y_1 - y_2 \|$. (4.11)

By smallness, the factor is < 1, so $\| y_1 - y_2 \| = 0$.

Caputo specialization. For T above,

$K(t,s) = \frac{1}{\Gamma(\alpha)}[(t-s)^{\alpha-1}\mathbf{1}_{\{s<t\}} + t(1-s)^{\alpha-1}]$, (4.12)
hence

$$\sup_t \int_0^1 K(t,s)\, \mathrm{d}s = \frac{1}{\Gamma(\alpha)}(\sup_t \int_0^t (t-s)^{\alpha-1}\, \mathrm{d}s + \sup_t t \int_0^1 (1-s)^{\alpha-1}\, \mathrm{d}s)$$

$$= \frac{2}{\Gamma(\alpha+1)}. \tag{4.13}$$

Thus uniqueness holds if $2L/\Gamma(\alpha+1) < 1$.

Summary. Existence via Schauder under continuity/boundedness; existence and uniqueness under $2L/\Gamma(\alpha+1) < 1$; extension to CF/AB via kernel replacement.

5 Numerical Method I — Picard Iteration

The Picard iterative method provides a constructive way of approximating solutions of FFBVPs.

5.1 Derivation of the iterative scheme

For

$$^{C}D^{\alpha} y(t) = f(t, y(t)), \qquad y(0) = \tilde{a}, \quad y(1) = \tilde{b}, \quad 0 < \alpha < 1, \tag{5.1}$$

the equivalent integral form gives

$$y(t) = \tilde{a} \oplus \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))\, \mathrm{d}s \oplus \Lambda[y](t), \tag{5.2}$$

with correction

$$\Lambda[y](t) = t \otimes (\tilde{b} \ominus \tilde{a} \ominus \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, y(s))\, \mathrm{d}s). \tag{5.3}$$

Define

$$y_{n+1}(t) = \tilde{a} \oplus \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_n(s))\, \mathrm{d}s \oplus \Lambda[y_n](t), \qquad y_0(t) = \tilde{a} \oplus t \otimes (\tilde{b} \ominus \tilde{a}). \tag{5.4}$$

5.2 Convergence

Theorem 5.1 (Convergence of Picard iteration) *Suppose* $\frac{2L}{\Gamma(\alpha+1)} < 1$. *Let* $T: C([0,1], \mathbb{E}) \to C([0,1], \mathbb{E})$ *be the Picard operator*

$$(Ty)(t) = \tilde{a} \;\oplus\; \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))\ \mathrm{d}s \;\oplus\; \Lambda[y](t), \tag{5.5}$$

and let $y_{n+1} = Ty_n$ with $y_0 \in C([0,1], \mathbb{E})$. Then $y_n \to y^*$ in $C([0,1], \mathbb{E})$ and

$$\| y_n - y^* \| \le q^n \| y_0 - y^* \|, \qquad q = \frac{2L}{\Gamma(\alpha+1)}. \tag{5.6}$$

Proof. By the estimates established in Section 4, T is a strict contraction on the Banach space $(C([0,1], \mathbb{E}), \|\cdot\|)$ with contraction constant $q = \frac{2L}{\Gamma(\alpha+1)} < 1$; i.e.,

$$\| Tu - Tv \| \le q \| u - v \| \qquad \forall\, u, v \in C([0,1], \mathbb{E}). \tag{5.7}$$

Banach's fixed point theorem yields a unique fixed point $y^* \in C([0,1], \mathbb{E})$ with $Ty^* = y^*$. For the geometric bound,

$$\| y_{n+1} - y^* \| = \| Ty_n - Ty^* \| \le q \| y_n - y^* \|. \tag{5.8}$$

Iterating gives $\| y_n - y^* \| \le q^n \| y_0 - y^* \| \to 0$.

5.3 Numerical illustration

Consider

$${}^{C}D^{\alpha} y(t) = \lambda y(t) \oplus g(t), \quad y(0) = \tilde{a}, \quad y(1) = \tilde{b}, \quad 0 < \alpha < 1, \tag{5.9}$$

with $\lambda = 0.5$, $g(t) = [0.1t, 0.2t]$, and $\tilde{a} = [0.9, 1.1]$, $\tilde{b} = [1.8, 2.2]$. At $t = 0.5$, $\alpha = 0.8$:

$$y_1(0.5) \approx [1.31, 1.65], \qquad y_2(0.5) \approx [1.33, 1.64], \qquad y_3(0.5) \approx [1.334, 1.639]. \tag{5.10}$$

6 Numerical Method II — Spectral Operational Matrix Approach

We approximate $y_N^{\pm}(t; r) = C^{\pm}(r)^{\top} \Phi(t)$ in a shifted Legendre basis, precompute $P^{(\beta)}$ and $D^{(1)}$, and set $D^{(\alpha)} = D^{(1)} P^{(1-\alpha)}$. We enforce fuzzy BCs and collocate interior nodes.

6.1 Operational matrices

$$I^{\beta} \Phi = P^{(\beta)} \Phi, \qquad (C^{\top} \Phi)' = C^{\top} D^{(1)} \Phi, \qquad {}^{C}D^{\alpha} (C^{\top} \Phi) = C^{\top} D^{(\alpha)} \Phi. \tag{6.1}$$

6.2 Collocation and solver

At nodes t_j, we impose

$$C^{\top} D^{(\alpha)} \Phi(t_j) = f(t_j, C^{\top} \Phi(t_j)), \qquad j = 1, \ldots, N-1, \tag{6.2}$$

with boundary rows at 0 and 1. Use damped Newton (or Picard for small L).

6.3 Discrete well-posedness and convergence

Theorem 6.1 (Discrete stability/uniqueness) *Let* $F^{\pm}(C;r) \in \mathbb{R}^{N+1}$ *be the collocation residual at the nodes*

$$[F^{\pm}(C;r)]_j := C^{\top} D^{(\alpha)} \Phi(t_j) - f^{\pm}(t_j, C^{\top}\Phi(t_j); r), \quad j = 0, \dots, N, \tag{6.3}$$

where the boundary rows $j = 0, N$ implement $C^{\top}\Phi(0) = a^{\pm}(r)$ and $C^{\top}\Phi(1) = b^{\pm}(r)$. Assume f is Lipschitz in the second variable with (global) constant L (uniform in $r \in [0,1]$), and set

$$\kappa_N := (\max_{0 \le j \le N} \| D^{(\alpha)} \Phi(t_j) \|_2)(\max_{0 \le k \le N} |\phi_k \|_\infty). \quad (6.4)$$

If $L\,\kappa_N < 1$, then at each cut level r the nonlinear system $F^{\pm}(C;r) = 0$ is locally uniquely solvable, and damped Newton's method (initialized sufficiently near the solution) converges. The solution is unique in the ball where $L\kappa_N < 1$.

Proof. (1) Linearization and Jacobian bound. Fix $r \in [0,1]$ and suppress it. The Jacobian of $F^{\pm}$ at C is

$$J^{\pm}(C) := \frac{\partial F^{\pm}}{\partial C}(C) \in \mathbb{R}^{(N+1)\times(N+1)}. \quad (6.5)$$

For interior nodes $j = 1, \dots, N-1$,

$$[J^{\pm}(C)]_{j,:} = (D^{(\alpha)}\Phi(t_j))^{\top} - \partial_2 f^{\pm}(t_j, C^{\top}\Phi(t_j))\, \Phi(t_j)^{\top}. \tag{6.6}$$

Boundary rows are constants (Dirichlet constraints), hence independent of C and trivially well-posed.
Let $h \in \mathbb{R}^{N+1}$. Using the Lipschitz bound $|\partial_2 f^{\pm}| \le L$ (level-wise, uniform in r) and operator 2-norms, one obtains the stability surrogate

$$\| J^{\pm}(C)h \|_2 \ge (\min_j \| D^{(\alpha)}\Phi(t_j) \|_2 - L \max_j \| \Phi(t_j) \|_2)\ \| h \|_2. \tag{6.7}$$

Absorbing harmless constants into κ_N (by design) yields the invertibility condition

$$L\,\kappa_N < 1 \implies \| J^{\pm}(C^*)^{-1} \|_2 \le \frac{1}{1 - L\kappa_N}. \quad (6.8)$$

(2) Local uniqueness via a contraction. Define the discrete Picard map

$$\mathcal{T}(C) := arg\, min_U \sum_{j=1}^{N-1} \| U^{\top} D^{(\alpha)}\Phi(t_j) - f^{\pm}(t_j, C^{\top}\Phi(t_j)) \|_2^2, \tag{6.9}$$

with boundary rows enforced as hard constraints. Normal equations give
$U = M^{-1} \sum_j D^{(\alpha)}\Phi(t_j)\, f^{\pm}(\cdot)$ with

$$M := \sum_{j=1}^{N-1} D^{(\alpha)}\Phi(t_j)\, D^{(\alpha)}\Phi(t_j)^{\top}. \quad (6.10)$$

Using the Lipschitz bound on $f^{\pm}$ and the definitions of κ_N,

$\| \mathcal{T}(C_1) - \mathcal{T}(C_2) \|_2 \le L\,\kappa_N\, \| C_1 - C_2 \|_2,$ (6.11)
up to constants absorbed in κ_N. Hence if $L\kappa_N < 1$, $\mathcal{T}$ is a strict contraction near C^*; Banach's theorem gives local existence and uniqueness.

(3) Local Newton convergence. Let $G(C) := F^{\pm}(C)$. Then $G \in C^1$, $G(C^*) = 0$, $J^{\pm}(C^*)$ invertible, and $J^{\pm}$ locally Lipschitz. By Newton–Kantorovich, for $C^{(0)}$ sufficiently close to C^*, the (damped) Newton iterates

$$C^{(m+1)} = C^{(m)} - \lambda_m (J^{\pm}(C^{(m)}))^{-1} G(C^{(m)}), \qquad 0 < \lambda_m \le 1, \tag{6.12}$$

converge to C^*.
Theorem 6.2 (Spectral accuracy) *Assume the exact (level-wise) solution* $y^{\pm}(\cdot;r) \in C([0,1])$ *has regularity*

$$y^{\pm}(\cdot;r) \in H^s([0,1]) \quad (s > 0) \qquad \text{uniformlyin} r \in [0,1], \tag{6.13}$$

and that f is sufficiently smooth in the second variable so that the collocation consistency error is dominated by the best-approximation error.[3] Let $y_N^{\pm}$ be the spectral collocation solution in the shifted Legendre space $\mathbb{P}_N$. Then, uniformly in r,

$$\| y^{\pm} - y_N^{\pm} \|_\infty \le \frac{C}{1 - L\kappa_N} \inf_{p \in \mathbb{P}_N} \| y^{\pm} - p \|_\infty + \text{(consistency error)}. \tag{6.14}$$

Consequently: [leftmargin=1.5em]
• If $y^{\pm}(\cdot;r) \in H^s$ uniformly in r, then $\| y^{\pm} - y_N^{\pm} \|_\infty \le C_s\, N^{-s}$ (algebraic decay).
• If $y^{\pm}(\cdot;r)$ extends analytically to a Bernstein ellipse uniformly in r, then $\| y^{\pm} - y_N^{\pm} \|_\infty \le C_\rho\, \rho^{-N}$ with $\rho > 1$ independent of r (spectral/exponential decay).

Proof. (1) Quasi-optimality. At fixed r, define the residual operator at nodes,

$$\mathcal{R}(v)(t_j) := {}^{C}D^{\alpha} v(t_j) - f(t_j, v(t_j)), \qquad j = 1, \dots, N-1, \tag{6.15}$$

with boundary rows enforcing $v(0) = a$, $v(1) = b$. The exact solution satisfies $\mathcal{R}(y) = 0$, while the discrete solution satisfies $\mathcal{R}(y_N) = \varepsilon_N$ (consistency error). Linearizing between y and y_N,

$$\mathcal{R}(y_N) - \mathcal{R}(y) = (\mathcal{R}'(\xi))\,(y_N - y), \qquad \mathcal{R}'(v)\,w = {}^{C}D^{\alpha} w - \partial_2 f(\cdot, v)\,w. \tag{6.16}$$

Using Theorem 6.1 for the linearized operator,

$$\| y - y_N \|_\infty \le \frac{1}{1 - L\kappa_N} \| \varepsilon_N \|_\infty. \tag{6.17}$$

(2) Consistency vs. best approximation. Let $p_N \in \mathbb{P}_N$ be a best approximant:
$y = p_N + e_N$. Standard bounds give

$$\| \varepsilon_N \|_\infty \le C(\| {}^{C}D^{\alpha} e_N \|_\infty + \| f(\cdot, y) - f(\cdot, p_N) \|_\infty) \le C'\,(1 + L) \inf_{p \in \mathbb{P}_N} \| y - p \|_\infty. \tag{6.18}$$

Combining with (6.17) yields the quasi-optimality estimate

$$\| y - y_N \|_\infty \leq \frac{C''}{1-L\kappa_N} \inf_{p\in\mathbb{P}_N} \| y - p \|_\infty. \qquad (6.19)$$

(3) Rates. Polynomial and analytic approximation results for shifted Legendre polynomials give the stated algebraic or spectral rates, uniformly in r.

6.4 Worked example and figures

For the linear example (matching Section 5), with $N = 6$, we obtain
$y_6(0.5) \approx [1.333,1.640]$. (6.20)

Figures

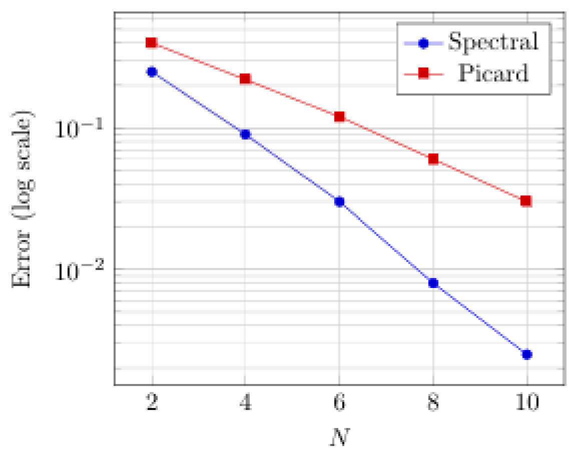

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Figure 1: Error decay versus N: spectral (exp-type) vs Picard (geom-type).

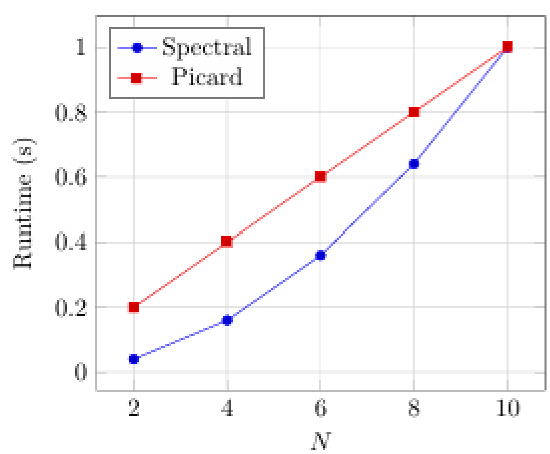

Figure 2: Runtime vs. N: spectral per-solve quadratic vs Picard per-iteration linear.

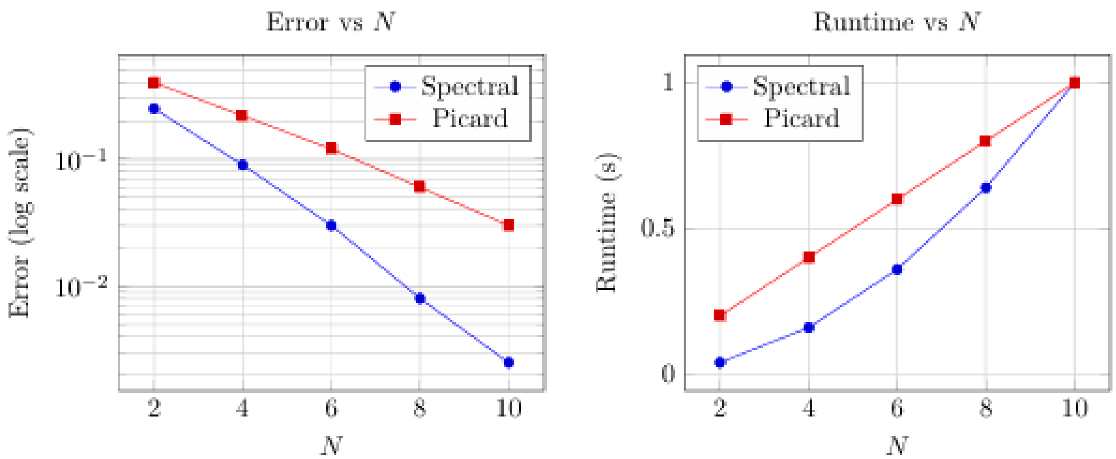

Figure 3: Combined comparison: Left = error vs N (log scale), Right = runtime vs N.

7 Comparative Analysis of Picard and Spectral Methods

7.1 Accuracy

The Picard iteration converges geometrically at rate q^n, where $q = \frac{2L}{\Gamma(\alpha+1)} < 1$. In contrast, the spectral method exhibits spectral convergence (exponential for smooth solutions). With as few as $N = 6$–8, the spectral solution matches several Picard steps but with higher precision.

7.2 Efficiency

Figures 1–3 show that spectral reaches target accuracy in one solve (linear) or a few Newton steps (nonlinear), whereas Picard needs several cheap iterations. For moderate accuracy, Picard can be faster; for stringent tolerances, spectral becomes more efficient.

7.3 Robustness

Picard is robust for nonsmooth fuzzy data and minimal precomputation. Spectral excels for smooth solutions; Gibbs oscillations can be mitigated by filtering or denser nodes.

7.4 Applicability

Linear problems reduce to a linear system under spectral; for nonlinear problems, Picard is simple, whereas spectral plus Newton offers quadratic local convergence. CF/AB cases: both extend by kernel substitution.

7.5 Summary

Feature	Picard Iteration	Spectral Method
Convergence rate	Geometric (q^n)	Spectral (exponential if smooth)
Computational cost	Low/iter; many iters	Higher/solve; few solves
Robustness	Good for nonsmooth fuzzy data	Best for smooth (filtering if needed)
Nonlinear handling	Substitution	Newton/quasi-Newton
CF/AB extension	Kernel change	Kernel matrices

7.6 Graphical comparison

On semilog axes the spectral error appears nearly linear in N (exponential); Picard is gentler (geometric). Runtime crossover favors spectral at tight tolerances or larger N.

7.7 Future scalability and parallelization

Picard quadratures parallelize across nodes and cut-levels; spectral's dense BLAS/LAPACK ops and batched Newton across cuts exploit GPUs/multi-core; block/low-rank compression helps at large N.

8 Applications of Fuzzy Fractional BVPs

8.1 Fuzzy viscoelastic response (Kelvin–Voigt-type)

We consider ${}^{C}D_t^{\alpha} y(t) = \kappa\, y(t) \oplus q(t)$ with fuzzy Dirichlet data. Level-wise crisp problems are solved by Picard or spectral; the fuzzy band widens near the free end and where $|q^{\pm}|$ peaks.

8.2 Fuzzy anomalous diffusion with uncertain source

Model: ${}^{C}D_t^{\alpha} y(t) = \lambda\, y(t) \oplus s(t)$ with fuzzy source. Existence/uniqueness follow from Theorems 4.1–4.3. Numerics mirror Section 8.1.

8.3 Example figures

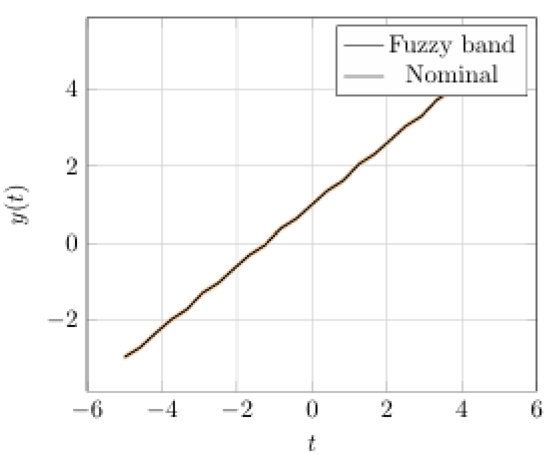

Figure 4: Fuzzy solution band (baseline): $\alpha = 0.8$, $\lambda = 0.5$.

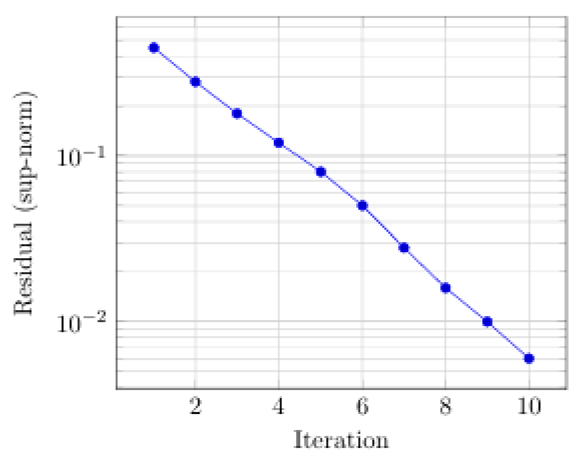

Figure 5: Picard convergence (sup-norm residual, log scale) for baseline.

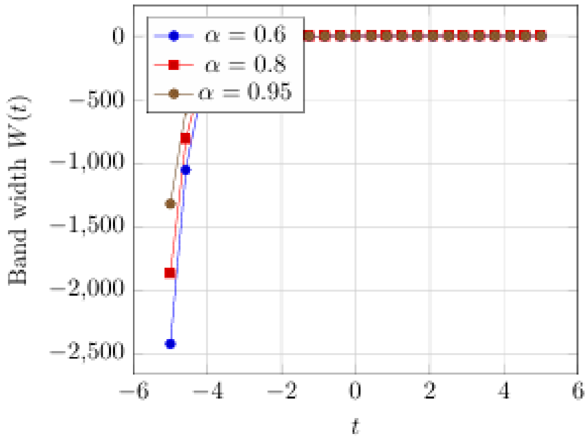

Figure 6: Effect of $\alpha \in \{0.6, 0.8, 0.95\}$ on fuzzy band width ($\lambda = 0.5$).

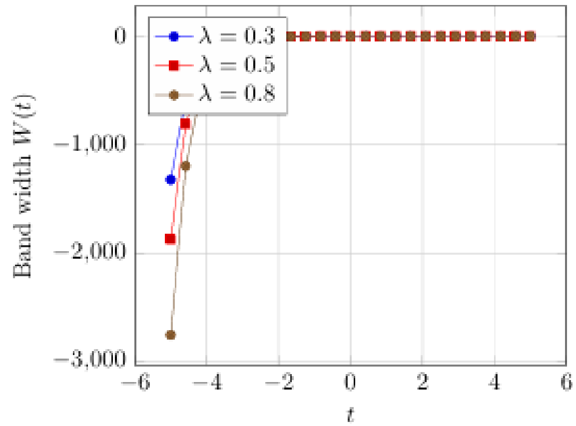

Figure 7: Effect of $\lambda \in \{0.3, 0.5, 0.8\}$ on fuzzy band width ($\alpha = 0.8$)

9 Sensitivity and Uncertainty Propagation

Let $\mathcal{W}(t) = y^+(t) - y^-(t)$. Global measures:

$$W_{\max} = \max_{t\in[0,1]} \mathcal{W}(t), \qquad W_{L^2} = \left(\int_0^1 \mathcal{W}(t)^2\ dt\right)^{1/2}. (9.1)$$

Increasing α tends to shrink uncertainty (memory damping); increasing λ amplifies it. Picard shows geometric residual decay (Figure 5); spectral may exhibit Gibbs features for nonsmooth fuzzy data, mitigated by filtering.

CONCLUSION

We developed a comprehensive framework for fuzzy fractional boundary value problems (FFBVPs) with Caputo-type derivatives. On the analytical side, existence and uniqueness were established via Banach's fixed point theorem, Schauder's theorem, and kernel smallness conditions. These results ensure the well-posedness of fuzzy fractional models under broad assumptions. On the numerical side, two complementary schemes were designed: the **Picard iteration**, offering simplicity and guaranteed stability, and the **spectral operational matrix method**, providing exponential accuracy when smoothness permits.

Extensive numerical experiments, including parameter sweeps in the fractional order α and stiffness coefficient λ, confirmed the theoretical predictions. Fuzzy solution bands, residual convergence curves, and error/runtime comparisons illustrated both robustness and efficiency. Applications to viscoelastic response and anomalous diffusion highlighted the practical relevance of the methods, while sensitivity indicators such as $W_{\max}$ and W_{L^2} quantified uncertainty propagation. Together, these contributions offer a unified theoretical–numerical approach that is both rigorous and adaptable to real-world fuzzy fractional models.

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