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Research Article

ENHANCED ANALYTICAL PERSPECTIVES ON HYPERBOLIC GEOMETRY AND ITS PROPERTIES

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ABSTRACT

A hyperbola is defined as the locus of points for which the absolute difference of the distances to two fixed points, known as the foci, remains constant. Its fundamental components include the center, transverse and conjugate axes, vertices, foci, and asymptotes. The geometric properties of a hyperbola establish intrinsic relationships among its axes, focal distances, tangent, normal, directrix, and vertex.

This research article examines the geometric and mathematical significance of the hyperbola as a fundamental conic section. Extending beyond its classical definition, the study presents seven newly derived properties and three theorems that characterize the relationships between the angle formed between elements such as tangent, normal & focal distances. Each property and theorem is rigorously formulated and supported with detailed proofs, step-by-step derivations, and illustrative diagrams to enhance conceptual clarity.

These contributions collectively enrich the theoretical framework of hyperbola geometry and provide a valuable reference for researchers engaged in advanced studies of conic sections. Furthermore, the newly identified properties offer potential applications in the development of Artificial Intelligence techniques for analysing hyperbolic structures and their role in mathematical modelling.

Keywords: *Hyperbola, Conic sections, Tangent, Normal & Focal distances*

INTRODUTION

A hyperbola *[Borowski E.J & Borwein J.M (1991)]* can be defined in two equivalent ways: as the locus of a point whose distance from a fixed point (focus) *[Christopher Clapham & James Nicolson (2009)]*, and a fixed line (directrix) *[Borowski E.J & Borwein J.M (1991)]* are in a constant ratio, or as the set of points for which the absolute difference of the distances from two fixed points, known as the foci, remains constant. Fundamental concepts such as the foci, directrix, latus rectum, and eccentricity *[Borowski E.J & Borwein J.M (1991)]* play a central role in its geometric characterization. Hyperbolas appear naturally in various phenomena, including the path traced by the tip of a sundial's shadow and the scattering trajectories of subatomic particles. Owing to their unique structure, hyperbolas hold significant importance in astronomy, physics, engineering, and several branches of applied mathematics.

This research article extends the traditional study of the hyperbola by introducing newly derived properties and theorems concerning the angle in between its tangent *[Borowski E.J & Borwein J.M (1991)]*, normal *[Borowski E.J & Borwein J.M (1991)]*, and focal distances *[Borowski E.J & Borwein J.M (1991)]*. Each result is rigorously formulated and supported by comprehensive proofs, detailed step-by-step derivations, and analytically constructed diagrams that enhance conceptual clarity. In addition to these contributions, the study establishes refined mathematical relationships with respect to angle between the elements involving the tangent, normal, focal distances, and other essential geometric elements of the hyperbola.

To support the development of these results, a separate Analysis section presents the fundamental formulas derived from the parametric equations *[Borowski E.J & Borwein J.M (1991)]* of the hyperbola. These serve as the foundational framework for the logical progression of proofs and derivations. Collectively, the findings enrich the theoretical landscape of hyperbola geometry and offer new insights for advanced studies in conic sections and related mathematical disciplines.

ANALYSIS & DERIVATIONS OF EQUATIONS FOR ELEMENTS

Referring fig.1,

This figure shows the basic elements of an ellipse that should be understood before proceeding to the analysis and derivations.

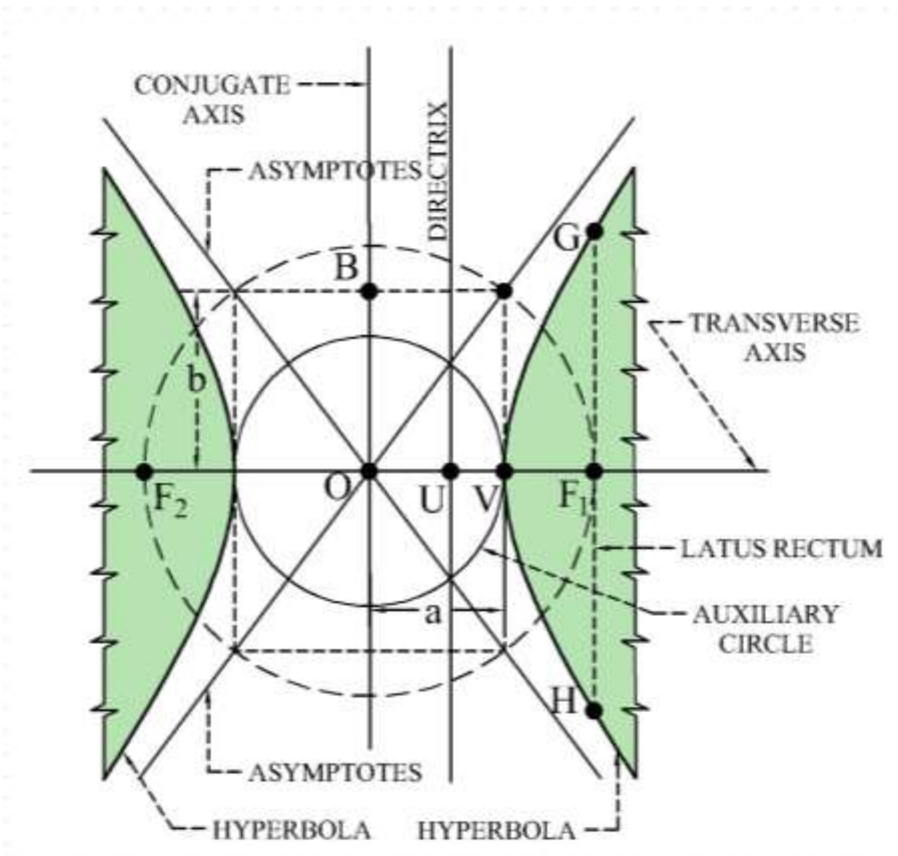

Fig. 1: Basic elements of an Hyperbola

Referring fig. 2,

$$OQ = \frac{a}{\cos(\theta°)} \quad \ldots(A.1)$$

$$PQ = OR = \frac{b\sin(\theta°)}{\cos(\theta°)} \quad \ldots(A.2)$$

$$OF_1 = OF_2 = \sqrt{a^2 + b^2} \quad \ldots(A.3)$$

Referring fig. 2, $F_1Q = OQ - OF_1$

Substituting eqns. (A.2) & (A.1),

$$\therefore \mathrm{F_1Q} = \left(\frac{\mathrm{a}}{\cos(\theta°)}\right) - \sqrt{\mathrm{a}^2 + \mathrm{b}^2}$$

$$\therefore \mathrm{F_1Q} = \frac{\mathrm{a} - \sqrt{\mathrm{a}^2 + \mathrm{b}^2}\cos(\theta°)}{\cos(\theta°)} \qquad \ldots(\mathrm{A.4})$$

Referring fig. 2, in right-triangle $\mathrm{F_1QP}$,

$\mathrm{PF_1}^2 = \mathrm{F_1Q}^2 + \mathrm{PQ}^2$

Referring fig. 2,

We know that parametric equation is $\mathrm{x} = \mathrm{a}\sec(\theta°)$ & $\mathrm{y} = \mathrm{b}\tan(\theta°)$.

In this figure let $(\mathrm{x}, \mathrm{y}) = (\mathrm{OQ}, \mathrm{PQ})$. Therefore,

Substituting eqns. (A.4) & (A.3),

$$\therefore \mathrm{PF_1}^2 = \left(\frac{\mathrm{a} - \sqrt{\mathrm{a}^2 + \mathrm{b}^2}\cos(\theta°)}{\cos(\theta°)}\right)^2 + \left(\frac{\mathrm{b}\sin(\theta°)}{\cos(\theta°)}\right)^2$$

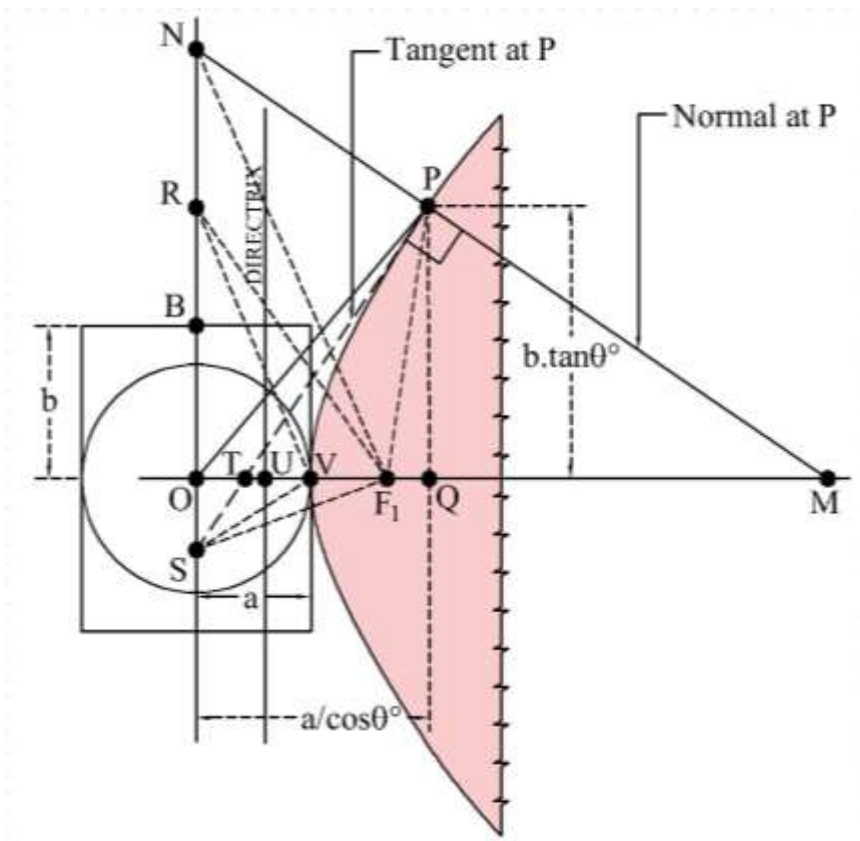

Fig.2

In this figure. 2,

Point O — *is the centre of the hyperbola, otherwise it is intersection of transverse axis [Christopher Clapham & James Nicolson (2009)] & conjugate axis [Borowski E.J & Borwein J.M (1991)].*

Point F_1 — *are foci of the hyperbola.*

Point P	*is a point considered anywhere on hyperbola.*
Point Q	*is projection of point P on transverse axis.*
OQ	*is abscissa [Borowski E.J & Borwein J.M (1991)] of point P.*
PQ or OR	*is ordinate [Borowski E.J & Borwein J.M (1991)] of point P.*
Point R	*is projection of point P on conjugate axis.*
Point T	*is intersection of the tangent and transverse axis, where PT is a tangent drawn at point P on hyperbola.*
QT	*is sub-tangent [Borowski E.J & Borwein J.M (1991)] about transverse axis.*
Point S	*PS is a tangent drawn at point P on hyperbola and S is intersection of tangent and conjugate axis.*
RS	*is sub-tangent about conjugate axis.*
Point M	*is intersection of normal and transverse axis, where MN is a normal drawn at point P on hyperbola.*
QM	*is sub-normal [Mark D Licker (2003)] about transverse axis.*
Point N	*is intersection of normal & conjugate axis.*
RN	*is sub-normal about conjugate axis.*

$$\therefore \mathrm{PF_1}^2 = \frac{\mathrm{a}^2 + (\mathrm{a}^2 + \mathrm{b}^2)\cos^2(\theta°) - 2\mathrm{a}\sqrt{\mathrm{a}^2 + \mathrm{b}^2}\cos(\theta°)}{\cos^2(\theta°)} + \frac{\mathrm{b}^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{PF_1}^2 = \frac{\mathrm{a}^2 + \mathrm{a}^2\cos^2(\theta°) + \mathrm{b}^2\cos^2(\theta°) - 2\mathrm{a}\sqrt{\mathrm{a}^2 + \mathrm{b}^2}\cos(\theta°) + \mathrm{b}^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{PF_1}^2 = \frac{\mathrm{a}^2 + \mathrm{b}^2[\cos^2(\theta°) + \sin^2(\theta°)] + \mathrm{a}^2\cos^2(\theta°) - 2\mathrm{a}\sqrt{\mathrm{a}^2 + \mathrm{b}^2}\cos(\theta°)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{PF_1}^2 = \frac{\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{a}^2\cos^2(\theta°) - 2\mathrm{a}\sqrt{\mathrm{a}^2 + \mathrm{b}^2}\cos(\theta°)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{PF_1}^2 = \frac{\left(\sqrt{\mathrm{a}^2 + \mathrm{b}^2}\right)^2 + [\mathrm{a}\cos(\theta°)]^2 - 2\sqrt{\mathrm{a}^2 + \mathrm{b}^2}\,\mathrm{a}\cos(\theta°)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{PF_1}^2 = \left(\frac{\sqrt{\mathrm{a}^2 + \mathrm{b}^2} - \mathrm{a}\cos(\theta°)}{\cos(\theta°)}\right)^2$$

$$\therefore \mathrm{PF_1} = \frac{\sqrt{\mathrm{a}^2 + \mathrm{b}^2} - \mathrm{a}\cos(\theta°)}{\cos(\theta°)} \qquad \ldots(\mathrm{A.5})$$

Referring fig. 2,

We know that $\mathrm{OT} \times \mathrm{OQ} = \mathrm{a}^2$

$$\therefore \mathrm{OT} = \frac{\mathrm{a}^2}{\mathrm{OQ}}$$

Substituting eqns. (A.2) in above eqn.,

$$\therefore \mathrm{OT} = \frac{\mathrm{a}^2}{\left(\frac{\mathrm{a}}{\cos(\theta°)}\right)}$$

$$\therefore \mathrm{OT} = \mathrm{a}^{\cancel{2}} \times \frac{\cos(\theta°)}{\cancel{\mathrm{a}}}$$

$$\therefore \mathrm{OT} = \mathrm{a}\cos(\theta°) \qquad \ldots(\mathrm{A.6})$$

We know that $\mathrm{OS} \times \mathrm{OR} = \mathrm{b}^2$

$$\therefore \mathrm{OS} = \frac{\mathrm{b}^2}{\mathrm{OR}}$$

Substituting eqns. (A.3) in above eqn.,

$$\therefore \mathrm{OS} = \frac{\mathrm{b}^2}{\left(\frac{\mathrm{b}\sin(\theta°)}{\cos(\theta°)}\right)}$$

$$\therefore \mathrm{OS} = \mathrm{b}^{\cancel{2}} \times \frac{\cos(\theta°)}{\cancel{\mathrm{b}}\sin(\theta°)}$$

$$\therefore \mathrm{OS} = \frac{\mathrm{b}\cos(\theta°)}{\sin(\theta°)} \qquad \ldots(\mathrm{A.7})$$

Referring fig. 2, $\mathrm{TQ} = \mathrm{OQ} - \mathrm{OT}$

Substituting eqns. (A.2) & (A.6) in above eqn.,

$$\therefore \mathrm{TQ} = \left(\frac{\mathrm{a}}{\cos(\theta°)}\right) - \mathrm{a}\cos(\theta°)$$

$$\therefore \mathrm{TQ} = \frac{\mathrm{a} - \mathrm{a}\cos^2(\theta°)}{\cos(\theta°)}$$

$$\therefore \mathrm{TQ} = \frac{\mathrm{a}[1 - \cos^2(\theta°)]}{\cos(\theta°)}$$

$$\therefore \mathrm{TQ} = \frac{\mathrm{a}\sin^2(\theta°)}{\cos(\theta°)} \qquad \ldots(\mathrm{A.8})$$

Referring fig. 2, Δ MQP & Δ PQT are similar.

$$\therefore \frac{\mathrm{PQ}}{\mathrm{QM}} = \frac{\mathrm{TQ}}{\mathrm{PQ}}$$

$$\therefore \mathrm{QM} = \frac{\mathrm{PQ}}{\mathrm{TQ}} \times \mathrm{PQ}$$

$$\therefore \mathrm{QM} = \frac{\mathrm{PQ}^2}{\mathrm{TQ}}$$

Substituting eqns. (A.3) & (A.8),

$$\therefore \mathrm{QM} = \left(\frac{\mathrm{b}\sin(\theta°)}{\cos(\theta°)}\right)^2 \div \left(\frac{\mathrm{a}\sin^2(\theta°)}{\cos(\theta°)}\right)$$

$$\therefore \mathrm{QM} = \left(\frac{\mathrm{bsin}(\theta°)}{\cos(\theta°)}\right)^2 \times \left(\frac{\cos(\theta°)}{\mathrm{asin}^2(\theta°)}\right)$$

$$\therefore \mathrm{QM} = \frac{\mathrm{b}^2\cancel{\sin^2(\theta°)}}{\cos^{\cancel{2}}(\theta°)} \times \frac{\cancel{\cos(\theta°)}}{\mathrm{a}\cancel{\sin^2(\theta°)}}$$

$$\therefore \mathrm{QM} = \frac{\mathrm{b}^2}{\mathrm{acos}(\theta°)} \qquad \ldots(\mathrm{A.9})$$

Referring fig. 2, $\mathrm{OM} = \mathrm{OQ} + \mathrm{QM}$

Substituting eqns. (A.2) & (A.9) in above eqn.,

$$\therefore \mathrm{OM} = \left(\frac{\mathrm{a}}{\cos(\theta°)}\right) + \left(\frac{\mathrm{b}^2}{\mathrm{acos}(\theta°)}\right)$$

$$\therefore \mathrm{OM} = \frac{\mathrm{a}^2 + \mathrm{b}^2}{\mathrm{acos}(\theta°)} \qquad \ldots(\mathrm{A.10})$$

Referring fig. 2, Δ MON & Δ MQP are similar.

$$\therefore \frac{\mathrm{ON}}{\mathrm{OM}} = \frac{\mathrm{PQ}}{\mathrm{QM}}$$

$$\therefore \mathrm{ON} = \frac{\mathrm{PQ} \times \mathrm{OM}}{\mathrm{QM}}$$

Substituting eqns. (A.3), (A.10) & (A.9),

$$\therefore \mathrm{ON} = \frac{\left(\frac{\mathrm{bsin}(\theta°)}{\cos(\theta°)}\right) \times \left(\frac{\mathrm{a}^2 + \mathrm{b}^2}{\mathrm{acos}(\theta°)}\right)}{\left(\frac{\mathrm{b}^2}{\mathrm{acos}(\theta°)}\right)}$$

$$\therefore \mathrm{ON} = \frac{\cancel{\mathrm{b}}\sin(\theta°)}{\cos(\theta°)} \times \frac{\mathrm{a}^2 + \mathrm{b}^2}{\cancel{\mathrm{acos}(\theta°)}} \times \frac{\cancel{\mathrm{acos}(\theta°)}}{\mathrm{b}^{\cancel{2}}}$$

$$\therefore \mathrm{ON} = \frac{(\mathrm{a}^2 + \mathrm{b}^2)\sin(\theta°)}{\mathrm{bcos}(\theta°)} \qquad \ldots(\mathrm{A.11})$$

Referring fig. 2, $\mathrm{TM} = \mathrm{OM} - \mathrm{OT}$

Substituting eqns. (A.10) & (A.6) in above eqn.,

$$\therefore \mathrm{TM} = \left(\frac{\mathrm{a}^2 + \mathrm{b}^2}{\mathrm{acos}(\theta°)}\right) - \mathrm{acos}(\theta°)$$

$$\therefore \mathrm{TM} = \frac{\mathrm{a}^2 + \mathrm{b}^2 - \mathrm{a}^2\cos^2(\theta°)}{\mathrm{acos}(\theta°)}$$

$$\therefore \mathrm{TM} = \frac{\mathrm{a}^2[1 - \cos^2(\theta°)] + \mathrm{b}^2}{\mathrm{acos}(\theta°)}$$

$$\therefore \mathrm{TM} = \frac{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}{\mathrm{acos}(\theta°)} \qquad \ldots(\mathrm{A.12})$$

Referring fig. 2, $F_1T = OF_1 - OT$

Substituting eqns. (A.1) & (A.6),

$$F_1T = \sqrt{a^2 + b^2} - a\cos(\theta°) \qquad \text{...(A.13)}$$

Referring fig. 2, in right-triangle MQP,

$$PM^2 = QM^2 + PQ^2$$

Substituting eqns. (A.9) & (A.3) in above eqn.,

$$\therefore PM^2 = \left(\frac{b^2}{a\cos(\theta°)}\right)^2 + \left(\frac{b\sin(\theta°)}{\cos(\theta°)}\right)^2$$

$$\therefore PM^2 = \frac{b^4}{a^2\cos^2(\theta°)} + \frac{b^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore PM^2 = \frac{b^4 + a^2b^2\sin^2(\theta°)}{a^2\cos^2(\theta°)}$$

$$\therefore PM^2 = \frac{b^2[b^2 + a^2\sin^2(\theta°)]}{a^2\cos^2(\theta°)}$$

$$\therefore PM = \frac{b\sqrt{b^2 + a^2\sin^2(\theta°)}}{a\cos(\theta°)} \qquad \text{...(A.14)}$$

Referring fig. 2, in right-triangle MPT,

$$PT^2 + PM^2 = TM^2$$

$$\therefore PT^2 = TM^2 - PM^2$$

Substituting eqns. (A.12) & (A.14),

$$\therefore PT^2 = \left(\frac{a^2\sin^2(\theta°) + b^2}{a\cos(\theta°)}\right)^2 - \left(\frac{b\sqrt{b^2 + a^2\sin^2(\theta°)}}{a\cos(\theta°)}\right)^2$$

$$\therefore PT^2 = \frac{[a^2\sin^2(\theta°) + b^2]^2}{a^2\cos^2(\theta°)} - \frac{b^2[b^2 + a^2\sin^2(\theta°)]}{a^2\cos^2(\theta°)}$$

$$\therefore PT^2 = \frac{[a^2\sin^2(\theta°) + b^2]^2 - b^2[b^2 + a^2\sin^2(\theta°)]}{a^2\cos^2(\theta°)}$$

$$\therefore PT^2 = \frac{a^4\sin^4(\theta°) \cancel{+ b^4} + 2a^2b^2\sin^2(\theta°) \cancel{- b^4} - a^2b^2\sin^2(\theta°)}{a^2\cos^2(\theta°)}$$

$$\therefore PT^2 = \frac{a^4\sin^4(\theta°) + a^2b^2\sin^2(\theta°)}{a^2\cos^2(\theta°)}$$

$$\therefore PT^2 = \frac{\cancel{a^2}\sin^2(\theta°)(a^2\sin^2(\theta°) + b^2)}{\cancel{a^2}\cos^2(\theta°)}$$

$$\therefore PT^2 = \frac{\sin^2(\theta°)(a^2\sin^2(\theta°) + b^2)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{PT} = \frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)} \qquad \ldots(A.15)$$

Referring fig. 2, in right-triangle F_1OS,

$$F_1S^2 = OF_1{}^2 + OS^2$$

Substituting eqns. (A.1) & (A.7) in above,

$$\therefore F_1S^2 = (a^2 + b^2) + \left(\frac{b\cos(\theta°)}{\sin(\theta°)}\right)^2$$

$$\therefore F_1S^2 = a^2 + b^2 + \frac{b^2\cos^2(\theta°)}{\sin^2(\theta°)}$$

$$\therefore F_1S^2 = \frac{a^2\sin^2(\theta°) + b^2\sin^2(\theta°) + b^2\cos^2(\theta°)}{\sin^2(\theta°)}$$

$$\therefore F_1S^2 = \frac{a^2\sin^2(\theta°) + b^2[\sin^2(\theta°) + \cos^2(\theta°)]}{\sin^2(\theta°)}$$

$$\therefore F_1S^2 = \frac{a^2\sin^2(\theta°) + b^2}{\sin^2(\theta°)}$$

$$\therefore F_1S = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)} \qquad \ldots(A.16)$$

Referring fig. 2, in right triangle F_1ON

$$\therefore F_1N^2 = OF_1{}^2 + ON^2$$

Substituting eqns. (A.1) & (A.11) in above eqn.,

$$\therefore F_1N^2 = a^2 + b^2 + \frac{(a^2 + b^2)^2\sin^2(\theta°)}{b^2\cos^2(\theta°)}$$

$$\therefore F_1N^2 = \frac{(a^2 + b^2)b^2\cos^2(\theta°) + (a^2 + b^2)^2\sin^2(\theta°)}{b^2\cos^2(\theta°)}$$

$$\therefore F_1N^2 = \frac{(a^2 + b^2)[b^2\cos^2(\theta°) + (a^2 + b^2)\sin^2(\theta°)]}{b^2\cos^2(\theta°)}$$

$$\therefore F_1N^2 = \frac{(a^2 + b^2)[b^2\cos^2(\theta°) + a^2\sin^2(\theta°) + b^2\sin^2(\theta°)]}{b^2\cos^2(\theta°)}$$

$$\therefore F_1N^2 = \frac{(a^2 + b^2)[a^2\sin^2(\theta°) + b^2\{\cos^2(\theta°) + \sin^2(\theta°)\}]}{b^2\cos^2(\theta°)}$$

$$\therefore F_1N^2 = \frac{(a^2 + b^2)(a^2\sin^2(\theta°) + b^2)}{b^2\cos^2(\theta°)}$$

$$\therefore F_1N = \frac{\sqrt{(a^2 + b^2)(a^2\sin^2(\theta°) + b^2)}}{b\cos(\theta°)} \qquad \ldots(A.17)$$

Referring fig. 2, $SN = OS + ON$

Substituting eqns. (A.7) & (A.11) in above eqn.,

$$\therefore \mathrm{SN} = \left(\frac{\mathrm{bcos}(\theta°)}{\sin(\theta°)}\right) + \left(\frac{(\mathrm{a}^2 + \mathrm{b}^2)\sin(\theta°)}{\mathrm{bcos}(\theta°)}\right)$$

$$\therefore \mathrm{SN} = \frac{\mathrm{b}^2\cos^2(\theta°) + (\mathrm{a}^2 + \mathrm{b}^2)\sin^2(\theta°)}{\mathrm{bsin}(\theta°)\cos(\theta°)}$$

$$\therefore \mathrm{SN} = \frac{\mathrm{b}^2\cos^2(\theta°) + \mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2\sin^2(\theta°)}{\mathrm{bsin}(\theta°)\cos(\theta°)}$$

$$\therefore \mathrm{SN} = \frac{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2[\cos^2(\theta°) + \sin^2(\theta°)]}{\mathrm{bsin}(\theta°)\cos(\theta°)}$$

$$\therefore \mathrm{SN} = \frac{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}{\mathrm{bsin}(\theta°)\cos(\theta°)} \qquad \ldots(\mathrm{A}.18)$$

PL, TL, RS, RN

The following table is abstract of derived equations for various elements of a Hyperbola

A.1	$\mathrm{OF_1} = \mathrm{OF_2} = \sqrt{\mathrm{a}^2 + \mathrm{b}^2}$	A.10	$\mathrm{OM} = \dfrac{\mathrm{a}^2 + \mathrm{b}^2}{\mathrm{acos}(\theta°)}$
A.2	$\mathrm{OQ} = \dfrac{\mathrm{a}}{\cos(\theta°)}$	A.11	$\mathrm{ON} = \dfrac{(\mathrm{a}^2 + \mathrm{b}^2)\sin(\theta°)}{\mathrm{bcos}(\theta°)}$
A.3	$\mathrm{PQ} = \mathrm{OR} = \dfrac{\mathrm{bsin}(\theta°)}{\cos(\theta°)}$	A.12	$\mathrm{TM} = \dfrac{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}{\mathrm{acos}(\theta°)}$
A.4	$\mathrm{F_1Q} = \dfrac{\mathrm{a} - \sqrt{\mathrm{a}^2 + \mathrm{b}^2}\cos(\theta°)}{\cos(\theta°)}$	A.13	$\mathrm{F_1T} = \sqrt{\mathrm{a}^2 + \mathrm{b}^2} - \mathrm{acos}(\theta°)$
A.5	$\mathrm{PF_1} = \dfrac{\sqrt{\mathrm{a}^2 + \mathrm{b}^2} - \mathrm{acos}(\theta°)}{\cos(\theta°)}$	A.14	$\mathrm{PM} = \dfrac{\mathrm{b}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{acos}(\theta°)}$
A.6	$\mathrm{OT} = \mathrm{acos}(\theta°)$	A.15	$\mathrm{PT} = \dfrac{\sin(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\cos(\theta°)}$
A.7	$\mathrm{OS} = \dfrac{\mathrm{bcos}(\theta°)}{\sin(\theta°)}$	A.16	$\mathrm{F_1S} = \dfrac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)}$
A.8	$\mathrm{TQ} = \dfrac{\mathrm{asin}^2(\theta°)}{\cos(\theta°)}$	A.17	$\mathrm{F_1N} = \dfrac{\sqrt{\mathrm{a}^2 + \mathrm{b}^2} \times \sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{bcos}(\theta°)}$
A.9	$\mathrm{QM} = \dfrac{\mathrm{b}^2}{\mathrm{acos}(\theta°)}$	A.18	$\mathrm{SN} = \dfrac{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}{\mathrm{bsin}(\theta°)\cos(\theta°)}$

DERIVATIONS OF THE PROPERTIES & THEOREMS

PROPERTY- 1:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1,}$$

$$\boldsymbol{PF_1=\frac{b^2\left(a-\sqrt{a^2+b^2}\cos\beta^\circ\right)}{b^2\cos^2\beta^\circ-a^2\sin^2\beta^\circ}}$$

$$\boldsymbol{PF_2=\frac{2a\left(b^2\cos^2\beta^\circ-a^2\sin^2\beta^\circ\right)+b^2\left(a-\sqrt{a^2+b^2}\cos\beta^\circ\right)}{b^2\cos^2\beta^\circ-a^2\sin^2\beta^\circ}}$$

(i) Determination of Focal distance (r) of hyperbola at an angle β° (measured from transverse axis)

Referring fig. 3, In this fig., P is one of the points of the hyperbola.

F_1 & F_2 are the foci. OB is the major axis. OC is the minor axis. O is centre of the hyperbola. Q is the projection of point P on transverse axis. Let, r is the focal distance at any instant β°. $OA = OC = a$, $OB = OD = b$. $\angle QF_1P = \beta^\circ$.

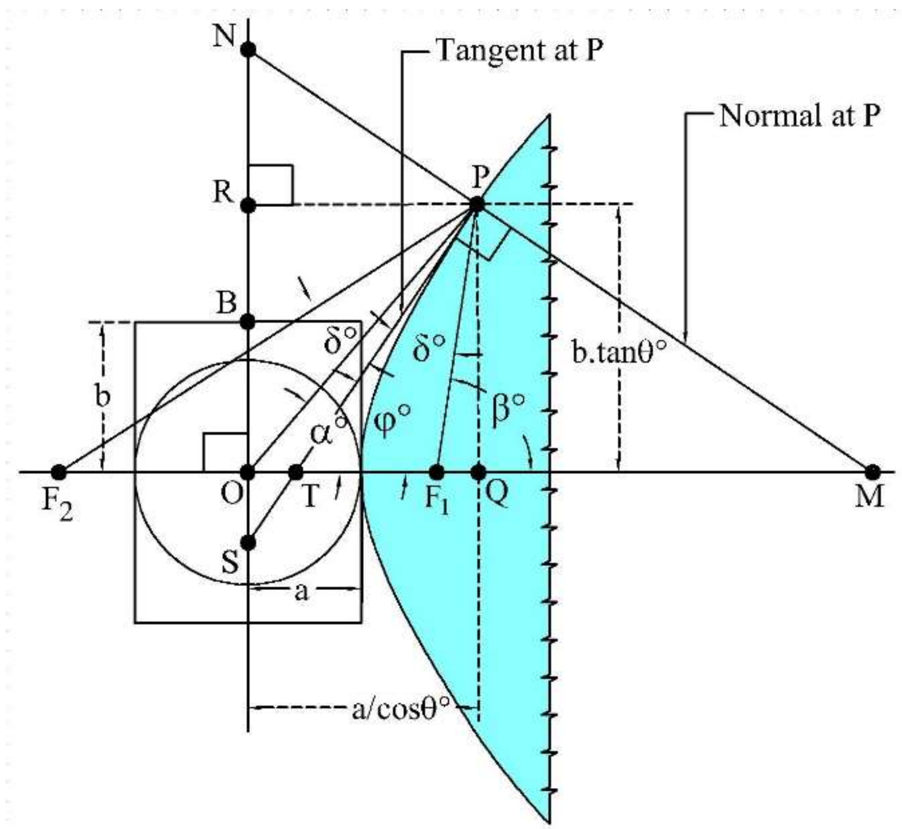

Fig. 3

In this figure. 3,

Point O	*is the centre of the hyperbola, otherwise it is intersection of transverse axis and conjugate axis.*
Point F_1 & F_2	*are foci of the hyperbola.*

PF_1 & PF_2	*are pair focal distances from point P.*
Point P	*is a point considered anywhere on hyperbola.*
Point Q	*is foot of perpendicular of point P upon transverse axis.*
Point R	*is foot of perpendicular of point P upon conjugate axis.*
Point T	*is intersection of the tangent and transverse axis, where PT is a tangent drawn at point P on hyperbola.*
Point S	*S is intersection of tangent and conjugate axis. Where, PS is a tangent drawn at point P on hyperbola.*
Point M	*is intersection of normal and transverse axis, where MN is a normal drawn at point P on hyperbola.*
Point N	*is intersection of normal & conjugate axis.*

$\angle \mathrm{POQ} = \propto °$
$\angle \mathrm{PF_1M} = \beta°$
$\angle \mathrm{PTQ} = \varphi°$
$\angle \mathrm{OPF_1} = \angle \mathrm{OPF_2} = \delta°$

We know already that, $\mathrm{OF_1} = \sqrt{a^2 + b^2}$

Let, (x, y) is the coordinate of point P with respect to O. Therefore, $\mathrm{OQ} = x$, $\mathrm{PQ} = y$,

Let, $\mathrm{PF_1} = r$, $\mathrm{PF_1}$ is the focal distance and $\angle \mathrm{QF_1P} = \beta°$, therefore

$$\mathrm{F_1Q} = r \times \cos\beta° \qquad \ldots(2.1)$$

$$y = \mathrm{PQ} = r\sin\beta° \qquad \ldots(2.2)$$

In fig. 5, $\mathrm{OQ} = x = \mathrm{OF_1} + \mathrm{F_1Q}$

Substituting eqn. (2.1) & $\mathrm{OF_1} = \sqrt{a^2 + b^2}$ in above,

$$\therefore x = \sqrt{a^2 + b^2} + r\cos\beta° \qquad \ldots(2.3)$$

We know that, the equation of hyperbola with respect to its centre O (0,0) is

$$\left(\frac{x^2}{a^2}\right) - \left(\frac{y^2}{b^2}\right) = 1$$

Substituting eqn. (2.3) & (2.2) in above eqn., we get

$$\left(\frac{\left(\sqrt{a^2 + b^2} + r\cos\beta°\right)^2}{a^2}\right) - \left(\frac{(r\sin\beta°)^2}{b^2}\right) = 1$$

$$\Rightarrow \frac{b^2\left(\sqrt{a^2 + b^2} + r\cos\beta°\right)^2 - a^2(r\sin\beta°)^2}{a^2b^2} = 1$$

$$\Rightarrow \frac{b^2\left(a^2 + b^2 + r^2\cos^2\beta° + 2r\sqrt{a^2 + b^2}\cos\beta°\right) - a^2(r^2\sin^2\beta°)}{a^2b^2} = 1$$

$$\Rightarrow \left(a^2b^2 + b^4 + r^2b^2\cos^2\beta° + 2b^2r\sqrt{a^2+b^2} \times \cos\beta°\right) - (r^2a^2\sin^2\beta°) = a^2b^2$$

$$\Rightarrow r^2b^2\cos^2\beta° + 2b^2r\sqrt{a^2+b^2} \times \cos\beta° - r^2a^2\sin^2\beta° + b^4 + a^2b^2 - a^2b^2 = 0$$

$$\Rightarrow r^2(b^2\cos^2\beta° - a^2\sin^2\beta°) + r\left(2b^2\sqrt{a^2+b^2} \times \cos\beta°\right) + b^4 = 0$$

Solving the above quadratic equation,

$$\Rightarrow r = \frac{-2b^2\sqrt{a^2+b^2}\cos\beta° \pm \sqrt{4b^4(a^2+b^2) \times \cos^2\beta° - 4b^4(b^2\cos^2\beta° - a^2\sin^2\beta°)}}{2(b^2\cos^2\beta° - a^2\sin^2\beta°)}$$

$$\Rightarrow r = \frac{-2b^2\sqrt{a^2+b^2}\cos\beta° \pm \sqrt{4b^4(a^2\cos^2\beta° + b^2\cos^2\beta°) - 4b^4(b^2\cos^2\beta° - a^2\sin^2\beta°)}}{2(b^2\cos^2\beta° - a^2\sin^2\beta°)}$$

$$\Rightarrow r = \frac{-2b^2\sqrt{a^2+b^2}\cos\beta° \pm \sqrt{4a^2b^4\cos^2\beta° \cancel{+ 4b^6\cos^2\beta° - 4b^6\cos^2\beta°} + 4a^2b^4\sin^2\beta°}}{2(b^2\cos^2\beta° - a^2\sin^2\beta°)}$$

$$\Rightarrow r = \frac{-2b^2\sqrt{a^2+b^2}\cos\beta° \pm \sqrt{4a^2b^4(\cos^2\beta° + \sin^2\beta°)}}{2(b^2\cos^2\beta° - a^2\sin^2\beta°)}$$

$$\Rightarrow r = \frac{-2b^2\sqrt{a^2+b^2}\cos\beta° \pm \sqrt{4a^2b^4}}{2(b^2\cos^2\beta° - a^2\sin^2\beta°)}$$

$$\Rightarrow r = \frac{-2b^2\sqrt{a^2+b^2}\cos\beta° \pm 2\sqrt{a^2b^4}}{2(b^2\cos^2\beta° - a^2\sin^2\beta°)}$$

$$\Rightarrow r = \frac{\cancel{2}\left[-b^2\sqrt{a^2+b^2}\cos\beta° \pm ab^2\right]}{\cancel{2}(b^2\cos^2\beta° - a^2\sin^2\beta°)}$$

$$\Rightarrow r = \frac{-b^2\sqrt{a^2+b^2}\cos\beta° \pm ab^2}{(b^2\cos^2\beta° - a^2\sin^2\beta°)}$$

$$\Rightarrow r_1 = \frac{-b^2\sqrt{a^2+b^2}\cos\beta° + ab^2}{(b^2\cos^2\beta° - a^2\sin^2\beta°)} \; or \; r_2 = \frac{-b^2\sqrt{a^2+b^2}\cos\beta° - ab^2}{(b^2\cos^2\beta° - a^2\sin^2\beta°)}$$

$$\Rightarrow r = r_2 = \frac{ab^2 + b^2\sqrt{a^2+b^2}\cos\beta°}{(b^2\cos^2\beta° - a^2\sin^2\beta°)} \quad (r_1 \; is \; invalid)$$

$$\Rightarrow r = \frac{b^2\left(a + \sqrt{a^2+b^2}\cos\beta°\right)}{(a^2\sin^2\beta° - b^2\cos^2\beta°)}$$

Resubstituting $r = PF_1$ in above,

$$\therefore PF_1 = \frac{b^2\left(a + \sqrt{a^2+b^2}\cos\beta°\right)}{a^2\sin^2\beta° - b^2\cos^2\beta°} \qquad \ldots(2.4)$$

We know that $PF_2 - PF_1 = 2a$
$\therefore PF_2 = 2a + PF_1$

$$\therefore \mathrm{PF_2} = 2\mathrm{a} + \left(\frac{\mathrm{b^2}\left(\mathrm{a} + \sqrt{\mathrm{a^2+b^2}}\cos\beta°\right)}{(\mathrm{a^2}\sin^2\beta° - \mathrm{b^2}\cos^2\beta°)}\right)$$

$$\therefore \mathrm{PF_2} = \frac{2\mathrm{a}(\mathrm{a^2}\sin^2\beta° - \mathrm{b^2}\cos^2\beta°) + \mathrm{b^2}\left(\mathrm{a} + \sqrt{\mathrm{a^2+b^2}}\cos\beta°\right)}{\mathrm{a^2}\sin^2\beta° - \mathrm{b^2}\cos^2\beta°} \quad \ldots(2.5)$$

Eqn. (2.5) is mathematical expression of the property.

PROPERTY- 2:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{P'F_1 = \frac{b^2\left(a - \sqrt{a^2+b^2}\cos(\beta°)\right)}{a^2\sin^2\beta° - b^2\cos^2\beta°}}$$

(ii) Determination of length of focal distance ($\mathrm{P'F_1}$) at an angle β° measured from transverse axis

$$\text{Eqn. (2.4)} \Longrightarrow \mathrm{PF_1} = \frac{\mathrm{b^2}\left(\mathrm{a} + \sqrt{\mathrm{a^2+b^2}}\cos\beta°\right)}{\mathrm{a^2}\sin^2\beta° - \mathrm{b^2}\cos^2\beta°} \quad \ldots(2.1)$$

Referring fig. 4,

If $\beta° = (180° + \beta°)$ and substitute in eqn. (102.1)

$$\mathrm{P'F_1} = \frac{\mathrm{b^2}\left(\mathrm{a} + \sqrt{\mathrm{a^2+b^2}}\cos(180° + \beta°)\right)}{\mathrm{a^2}\sin^2(180° + \beta°) - \mathrm{b^2}\cos^2(180° + \beta°)}$$

$$\therefore \mathrm{P'F_1} = \frac{\mathrm{b^2}\left(\mathrm{a} - \sqrt{\mathrm{a^2+b^2}}\cos(\beta°)\right)}{\mathrm{a^2}\sin^2\beta° - \mathrm{b^2}\cos^2\beta°} \quad \ldots(2.2)$$

Eqn. (2.2) is mathematical expression of the property.

PROPERTY- 3:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{PP' = \frac{2ab^2}{a^2\sin^2\beta° - b^2\cos^2\beta°}}$$

(iii) Determination of length of focal chord (PP') at an angle β° measured from transverse axis

Referring fig. 4,

$$\text{Eqn. (2.4)} \Longrightarrow \mathrm{PF_1} = \frac{\mathrm{b^2}\left(\mathrm{a} + \sqrt{\mathrm{a^2+b^2}}\cos\beta°\right)}{\mathrm{a^2}\sin^2\beta° - \mathrm{b^2}\cos^2\beta°} \quad \ldots(3.1)$$

$$\text{Eqn. (2.2)} \Longrightarrow \mathrm{P'F_1} = \frac{\mathrm{b^2}\left(\mathrm{a} - \sqrt{\mathrm{a^2+b^2}}\cos(\beta°)\right)}{\mathrm{a^2}\sin^2\beta° - \mathrm{b^2}\cos^2\beta°} \quad \ldots(3.2)$$

$$\mathrm{PP'} = \mathrm{PF_1} + \mathrm{P'F_1}$$

Adding eqns. (3.1) & (3.2),

$$PP' = PF_1 + P'F_1 = \left(\frac{b^2\left(a+\sqrt{a^2+b^2}\cos\beta°\right)}{a^2\sin^2\beta° - b^2\cos^2\beta°}\right) + \left(\frac{b^2\left(a-\sqrt{a^2+b^2}\cos(\beta°)\right)}{a^2\sin^2\beta° - b^2\cos^2\beta°}\right)$$

$$\therefore PP' = \frac{b^2\left(a+\sqrt{a^2+b^2}\cos\beta°\right)}{a^2\sin^2\beta° - b^2\cos^2\beta°} + \frac{b^2\left(a-\sqrt{a^2+b^2}\cos(\beta°)\right)}{a^2\sin^2\beta° - b^2\cos^2\beta°}$$

$$\therefore PP' = \frac{b^2\left(a+\sqrt{a^2+b^2}\cos\beta°\right) + b^2\left(a-\sqrt{a^2+b^2}\cos(\beta°)\right)}{a^2\sin^2\beta° - b^2\cos^2\beta°}$$

$$\therefore PP' = \frac{b^2\left(a\cancel{+\sqrt{a^2+b^2}\cos\beta°} + a\cancel{-\sqrt{a^2+b^2}\cos(\beta°)}\right)}{a^2\sin^2\beta° - b^2\cos^2\beta°}$$

$$\therefore PP' = \frac{2ab^2}{a^2\sin^2\beta° - b^2\cos^2\beta°} \qquad \ldots(3.3)$$

Eqn. (3.3) is mathematical expression of the property.

Determination of $\varphi°$ in terms of $\theta°$.

PROPERTY- 4:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$$\boldsymbol{\tan(\varphi°) = \frac{b}{a\sin(\theta°)}}$$

Determination of $\varphi°$ in terms of $\theta°$.

$$\text{Eqn. (A.3)} \Longrightarrow PQ = \frac{b\sin(\theta°)}{\cos(\theta°)} \qquad \ldots(4.1)$$

$$\text{Eqn. (A.8)} \Longrightarrow TQ = \frac{a\sin^2(\theta°)}{\cos(\theta°)} \qquad \ldots(4.2)$$

Referring fig. 4, in right-triangle PQT $\angle PTQ = \varphi°$

$$\tan(\varphi°) = \frac{PQ}{TQ}$$

Substituting eqns. (4.1) & (4.2),

$$\therefore \tan(\varphi°) = \left(\frac{b\sin(\theta°)}{\cos(\theta°)}\right) \div \left(\frac{a\sin^2(\theta°)}{\cos(\theta°)}\right)$$

$$\therefore \tan(\varphi°) = \frac{b\cancel{\sin(\theta°)}}{\cancel{\cos(\theta°)}} \times \frac{\cancel{\cos(\theta°)}}{a\sin^{\cancel{2}}(\theta°)}$$

$$\therefore \tan(\varphi°) = \frac{b}{a\sin(\theta°)} \qquad \ldots(4.3)$$

Eqn. (4.3) is mathematical expression of the property.

PROPERTY- 5:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{tan(\beta°) = \frac{bsin(\theta°)}{a - \sqrt{a^2 + b^2}cos(\theta°)}}$$

<u>Determination of β° in terms of θ°.</u>

$$\text{Eqn. (A.3)} \Longrightarrow \mathrm{PQ} = \frac{\mathrm{bsin}(\theta°)}{\cos(\theta°)} \qquad \ldots(5.1)$$

$$\text{Eqn. (A.4)} \Longrightarrow \mathrm{F_1Q} = \frac{\mathrm{a} - \sqrt{\mathrm{a}^2 + \mathrm{b}^2}\cos(\theta°)}{\cos(\theta°)} \qquad \ldots(5.2)$$

Referring fig. 4, in right-triangle PQT $\angle \mathrm{PF_1Q} = \beta°$

$$\tan(\beta°) = \frac{\mathrm{PQ}}{\mathrm{F_1Q}}$$

Substituting eqns. (5.1) & (5.2),

$$\therefore \tan(\beta°) = \left(\frac{\mathrm{bsin}(\theta°)}{\cos(\theta°)}\right) \div \left(\frac{\mathrm{a} - \sqrt{\mathrm{a}^2 + \mathrm{b}^2}\cos(\theta°)}{\cos(\theta°)}\right)$$

$$\therefore \tan(\beta°) = \frac{\mathrm{bsin}(\theta°)}{\cancel{\cos(\theta°)}} \times \frac{\cancel{\cos(\theta°)}}{\mathrm{a} - \sqrt{\mathrm{a}^2 + \mathrm{b}^2}\cos(\theta°)}$$

$$\therefore \tan(\beta°) = \frac{\mathrm{bsin}(\theta°)}{\mathrm{a} - \sqrt{\mathrm{a}^2 + \mathrm{b}^2}\cos(\theta°)} \qquad \ldots(5.3)$$

Eqn. (5.3) is mathematical expression of the property.

PROPERTY - 6:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{cos(\delta°) = \frac{sin(\theta°)\sqrt{a^2 + b^2}}{\sqrt{a^2 sin^2(\theta°) + b^2}}}$$

<u>Determination of δ° in terms of θ°.</u>

$$\text{Eqn. (A.5)} \Longrightarrow \mathrm{PF_1} = \frac{\sqrt{\mathrm{a}^2 + \mathrm{b}^2} - \mathrm{a}\cos(\theta°)}{\cos(\theta°)} \qquad \ldots(6.1)$$

$$\text{Eqn. (A.15)} \Longrightarrow \mathrm{PT} = \frac{\sin(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\cos(\theta°)} \qquad \ldots(6.2)$$

$$\text{Eqn. (A.13)} \Longrightarrow \mathrm{F_1T} = \sqrt{\mathrm{a}^2 + \mathrm{b}^2} - \mathrm{a}\cos(\theta°) \qquad \ldots(6.3)$$

Referring fig. 4, in triangle $\mathrm{TPF_1}$, $\angle \mathrm{TPF_1} = \angle \mathrm{TPF_2} = \delta°$

According to Cosine law, $\mathrm{c}^2 = \mathrm{a}^2 + \mathrm{b}^2 - 2\mathrm{ab}\cos(\mathrm{C}°)$

$$\therefore \cos(\mathrm{C}°) = \frac{\mathrm{a}^2 + \mathrm{b}^2 - \mathrm{c}^2}{2\mathrm{ab}}$$

In this case, $\mathrm{a} = \mathrm{PF}_1 . \mathrm{b} = \mathrm{PT}\ \&\ \mathrm{c} = \mathrm{TF}_1$

$$\therefore \cos(\delta°) = \frac{{\mathrm{PF}_1}^2 + \mathrm{PT}^2 - \mathrm{F}_1 T^2}{2 \times \mathrm{PF}_1 \times \mathrm{PT}}$$

Substituting eqns. (A.5), (A.15) & (A.13) in above eqn.,

$$\therefore \cos(\delta°) = \frac{\left(\dfrac{\sqrt{a^2+b^2} - a\cos(\theta°)}{\cos(\theta°)}\right)^2 + \left(\dfrac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)}\right)^2 - \left(\sqrt{a^2+b^2} - a\cos(\theta°)\right)^2}{2 \times \left(\dfrac{\sqrt{a^2+b^2} - a\cos(\theta°)}{\cos(\theta°)}\right) \times \left(\dfrac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)}\right)}$$

$$\therefore \cos(\delta°) = \left\{\frac{\begin{gathered}\dfrac{a^2 + b^2 + a^2\cos^2(\theta°) - 2a\sqrt{a^2+b^2}\cos(\theta°)}{\cos^2(\theta°)} + \dfrac{\sin^2(\theta°) \times (a^2\sin^2(\theta°) + b^2)}{\cos^2(\theta°)} \\ -\left[a^2 + b^2 + a^2\cos^2(\theta°) - 2a\sqrt{a^2+b^2}\cos(\theta°)\right]\end{gathered}}{2 \times \left(\dfrac{\sqrt{a^2+b^2}\sqrt{a^2\sin^2(\theta°) + b^2}\sin(\theta°) - a\sin(\theta°)\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos^2(\theta°)}\right)}\right\}$$

$$\therefore \cos(\delta°) = \left\{\frac{\dfrac{\begin{gathered}a^2 + b^2 + a^2\cos^2(\theta°) - 2a\sqrt{a^2+b^2}\cos(\theta°) + a^2\sin^4(\theta°) + b^2\sin^2(\theta°) \\ -a^2\cos^2(\theta°) - b^2\cos^2(\theta°) - a^2\cos^4(\theta°) + 2a\sqrt{a^2+b^2}\cos^3(\theta°)\end{gathered}}{\cos^2(\theta°)}}{\dfrac{2 \times \sqrt{a^2\sin^2(\theta°) + b^2} \times \sin(\theta°)\left(\sqrt{a^2+b^2} - a\cos(\theta°)\right)}{\cos^2(\theta°)}}\right\}$$

$$\therefore \cos(\delta°) = \left\{\frac{\dfrac{\begin{gathered}a^2 - a^2\cos^2(\theta°) + b^2 - b^2\cos^2(\theta°) + a^2\cos^2(\theta°) - 2a\sqrt{a^2+b^2}\cos(\theta°) \\ +a^2\sin^4(\theta°) + b^2\sin^2(\theta°) - a^2\cos^4(\theta°) + 2a\sqrt{a^2+b^2}\cos^3(\theta°)\end{gathered}}{\cancel{\cos^2(\theta°)}}}{\dfrac{2 \times \sqrt{a^2\sin^2(\theta°) + b^2} \times \sin(\theta°)\left(\sqrt{a^2+b^2} - a\cos(\theta°)\right)}{\cancel{\cos^2(\theta°)}}}\right\}$$

$$\therefore \cos(\delta°) = \frac{\left\{\begin{gathered}a^2[1 - \cos^2(\theta°)] + b^2[1 - \cos^2(\theta°)] + a^2\sin^4(\theta°) + b^2\sin^2(\theta°) \\ +a^2\cos^2(\theta°) - a^2\cos^4(\theta°) - 2a\sqrt{a^2+b^2}\cos(\theta°) + 2a\sqrt{a^2+b^2}\cos^3(\theta°)\end{gathered}\right\}}{2\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}\left(\sqrt{a^2+b^2} - a\cos(\theta°)\right)}$$

$$\therefore \cos(\delta°) = \frac{\left\{\begin{gathered}a^2\sin^2(\theta°) + b^2\sin^2(\theta°) + a^2\sin^4(\theta°) + b^2\sin^2(\theta°) \\ +a^2\cos^2(\theta°)[1 - \cos^2(\theta°)] - 2a\sqrt{a^2+b^2}\cos(\theta°)[1 - \cos^2(\theta°)]\end{gathered}\right\}}{2\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}\left(\sqrt{a^2+b^2} - a\cos(\theta°)\right)}$$

$$\therefore \cos(\delta^\circ) = \frac{\left\{\begin{matrix} a^2\sin^2(\theta^\circ) + b^2\sin^2(\theta^\circ) + a^2\sin^4(\theta^\circ) + a^2\cos^2(\theta^\circ)\sin^2(\theta^\circ) \\ +b^2\sin^2(\theta^\circ) - 2a\sqrt{a^2+b^2}\cos(\theta^\circ)\sin^2(\theta^\circ) \end{matrix}\right\}}{2\sin(\theta^\circ)\sqrt{a^2\sin^2(\theta^\circ)+b^2}\left(\sqrt{a^2+b^2} - a\cos(\theta^\circ)\right)}$$

$$\therefore \cos(\delta^\circ) = \frac{\left\{\begin{matrix} a^2\sin^2(\theta^\circ) + b^2\sin^2(\theta^\circ) + a^2\sin^2(\theta^\circ)[\sin^2(\theta^\circ) + \cos^2(\theta^\circ)] \\ +b^2\sin^2(\theta^\circ) - 2a\sqrt{a^2+b^2}\cos(\theta^\circ)\sin^2(\theta^\circ) \end{matrix}\right\}}{2\sin(\theta^\circ)\sqrt{a^2\sin^2(\theta^\circ)+b^2}\left(\sqrt{a^2+b^2} - a\cos(\theta^\circ)\right)}$$

$$\therefore \cos(\delta^\circ) = \frac{\left\{\begin{matrix} a^2\sin^2(\theta^\circ) + b^2\sin^2(\theta^\circ) + a^2\sin^2(\theta^\circ) + b^2\sin^2(\theta^\circ) \\ -2a\sqrt{a^2+b^2}\cos(\theta^\circ)\sin^2(\theta^\circ) \end{matrix}\right\}}{2\sin(\theta^\circ)\sqrt{a^2\sin^2(\theta^\circ)+b^2}\left(\sqrt{a^2+b^2} - a\cos(\theta^\circ)\right)}$$

$$\therefore \cos(\delta^\circ) = \frac{2a^2\sin^2(\theta^\circ) + 2b^2\sin^2(\theta^\circ) - 2a\sqrt{a^2+b^2}\cos(\theta^\circ)\sin^2(\theta^\circ)}{2\sin(\theta^\circ)\sqrt{a^2\sin^2(\theta^\circ)+b^2}\left(\sqrt{a^2+b^2} - a\cos(\theta^\circ)\right)}$$

$$\therefore \cos(\delta^\circ) = \frac{\cancel{2}\sin^{\cancel{2}}(\theta^\circ)\left[a^2+b^2 - a\cos(\theta^\circ)\sqrt{a^2+b^2}\right]}{\cancel{2\sin(\theta^\circ)}\sqrt{a^2\sin^2(\theta^\circ)+b^2}\left(\sqrt{a^2+b^2} - a\cos(\theta^\circ)\right)}$$

$$\therefore \cos(\delta^\circ) = \frac{\sin(\theta^\circ)\left[a^2+b^2 - a\cos(\theta^\circ)\sqrt{a^2+b^2}\right]}{\sqrt{a^2\sin^2(\theta^\circ)+b^2}\left(\sqrt{a^2+b^2} - a\cos(\theta^\circ)\right)}$$

$$\therefore \cos(\delta^\circ) = \frac{\sqrt{a^2+b^2}\sin(\theta^\circ)\left[\cancel{\sqrt{a^2+b^2} - a\cos(\theta^\circ)}\right]}{\sqrt{a^2\sin^2(\theta^\circ)+b^2}\left[\cancel{\sqrt{a^2+b^2} - a\cos(\theta^\circ)}\right]}$$

$$\therefore \cos(\delta^\circ) = \frac{\sin(\theta^\circ)\sqrt{a^2+b^2}}{\sqrt{a^2\sin^2(\theta^\circ)+b^2}} \qquad \ldots(6.4)$$

Eqn. (6.4) is mathematical expression of the property.

PROPERTY - 7:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$**,**

$$\boldsymbol{\tan(\alpha^\circ) = \frac{b\sin(\theta^\circ)}{a}}$$

<u>Determination of α° in terms of θ°.</u>

$$\text{Eqn. (A. 3)} \Longrightarrow PQ = \frac{b\sin(\theta^\circ)}{\cos(\theta^\circ)} \qquad \ldots(7.1)$$

$$\text{Eqn. (A. 2)} \Longrightarrow OQ = \frac{a}{\cos(\theta^\circ)} \qquad \ldots(7.2)$$

Referring fig. 4, in right-triangle POQ, $\angle POQ = \alpha^\circ$

$$\tan(\alpha^\circ) = \frac{PQ}{OQ}$$

Substituting eqns. (A.3) & (A.2) in above eqn.,

$$\tan(\alpha°) = \left(\frac{b\sin(\theta°)}{\cos(\theta°)}\right) \div \left(\frac{a}{\cos(\theta°)}\right)$$

$$\tan(\alpha°) = \frac{b\sin(\theta°)}{\cancel{\cos(\theta°)}} \times \frac{\cancel{\cos(\theta°)}}{a}$$

$$\tan(\alpha°) = \frac{b\sin(\theta°)}{a} \qquad \ldots(7.3)$$

Eqn. (7.3) is mathematical expression of the property.

THEOREM- 1:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

if point S is intersection of tangent & conjugate axis, point N is intersection of normal & conjugate axis and point $\boldsymbol{F_1}$ ***is one of the foci of a hyperbola, then*** $\boldsymbol{SF_1}$ ***is perpendicular to*** $\boldsymbol{NF_1}$***. In other words, the triangle*** $\boldsymbol{SF_1N}$ ***is right-angled triangle.***

Proof of the theorem

Referring fig. 4,

It is known that $\Delta\, NOF_1$ & $\Delta\, SOF_1$ are a right-triangle. Fig. 5 is detailed drawing of $\Delta\, SF_1N$.

Referring fig. 5,

Let $\angle NF_1O$ be $\gamma_1°$ and $\angle\ SF_1O$ be $\gamma_2°$.

$$\text{Eqn. (A.11)} \Longrightarrow ON = \frac{(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)} \qquad \ldots(8.1)$$

$$\text{Eqn. (A.7)} \Longrightarrow OS = \frac{(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)} \qquad \ldots(8.2)$$

$$\text{Eqn. (A.1)} \Longrightarrow OF_1 = \sqrt{a^2+b^2} \qquad \ldots(8.3)$$

In right triangle NOF_1,

$$\tan(\gamma_1°) = \frac{ON}{OF_1}$$

Substituting eqns. (8.1) & (8.3) in above,

$$\tan(\gamma_1°) = \frac{(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)} \times \frac{1}{\sqrt{a^2+b^2}}$$

$$\therefore \tan(\gamma_1°) = \frac{\sqrt{a^2+b^2} \times \sin(\theta°)}{b\cos(\theta°)} \qquad \ldots(8.3)$$

Referring fig. 5,

Let it be $\angle\ SF_1O = \gamma_2°$.

In right triangle SOF_1,

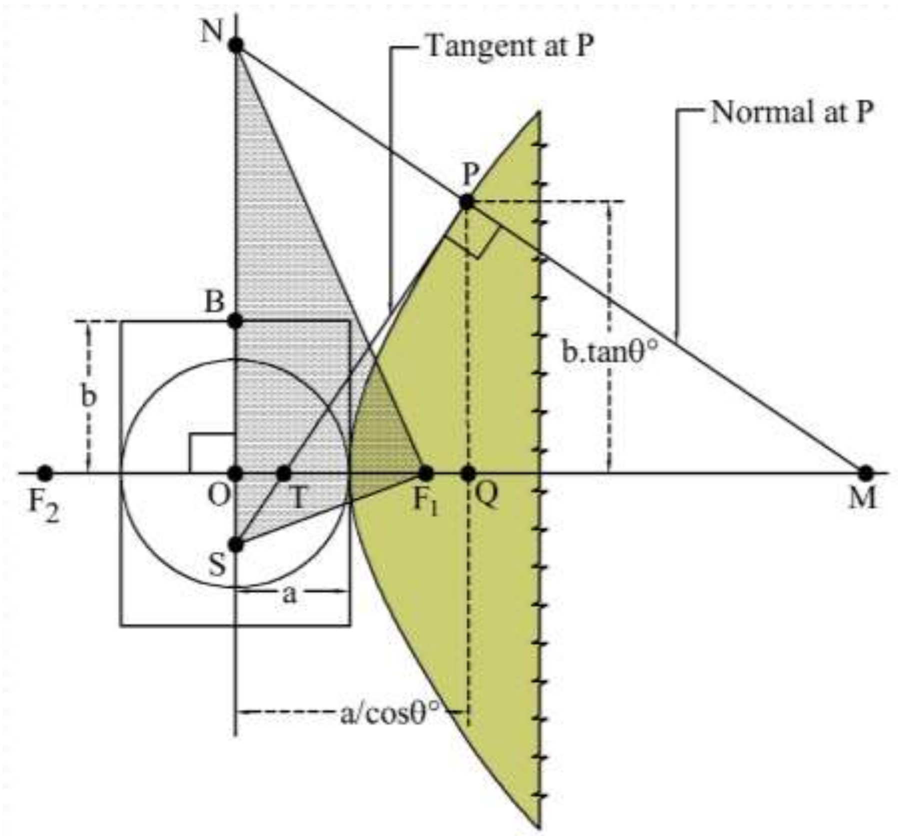

Fig. 4

$$\tan(\gamma_2°) = \frac{OS}{OF_1}$$

Substituting eqns. (A.7) & (A.1) in above,

$$\tan(\gamma_2°) = \frac{\left(\frac{b\cos(\theta°)}{\sin(\theta°)}\right)}{\sqrt{a^2 + b^2}}$$

$$\therefore \tan(\gamma_2°) = \frac{b\cos(\theta°)}{\sin(\theta°)} \times \frac{1}{\sqrt{a^2 + b^2}}$$

$$\therefore \tan(\gamma_2°) = \frac{b\cos(\theta°)}{\sqrt{a^2 + b^2} \times \sin(\theta°)} \qquad \dots(8.4)$$

Let $(\gamma_1° + \gamma_2°) = (\gamma°)$

$\therefore \tan(\gamma°) = \tan(\gamma_1° + \gamma_2°)$

We know already that

$$\tan(\gamma_1° + \gamma_2°) = \frac{\tan(\gamma_1°) + \tan(\gamma_2°)}{1 - [\tan(\gamma_1°) \times \tan(\gamma_2°)]}$$

To prove $\gamma_1° + \gamma_2° = 90°$, we need to prove $\tan(\gamma_1° + \gamma_2°) = \infty$. [since, $\tan(90°)$ is infinite]

To prove $\tan(90°) = \infty$, it is necessary to prove only denominator is equal to zero. Therefore, we need to prove $1 - \tan(\gamma_1°) \times \tan(\gamma_2°) = 0$ (since, anything is divided by zero is infinite).

Let L.H.S be $1 - \tan(\gamma_1°) \times \tan(\gamma_2°)$

Substituting eqns. (8.3) & (8.4) in above,

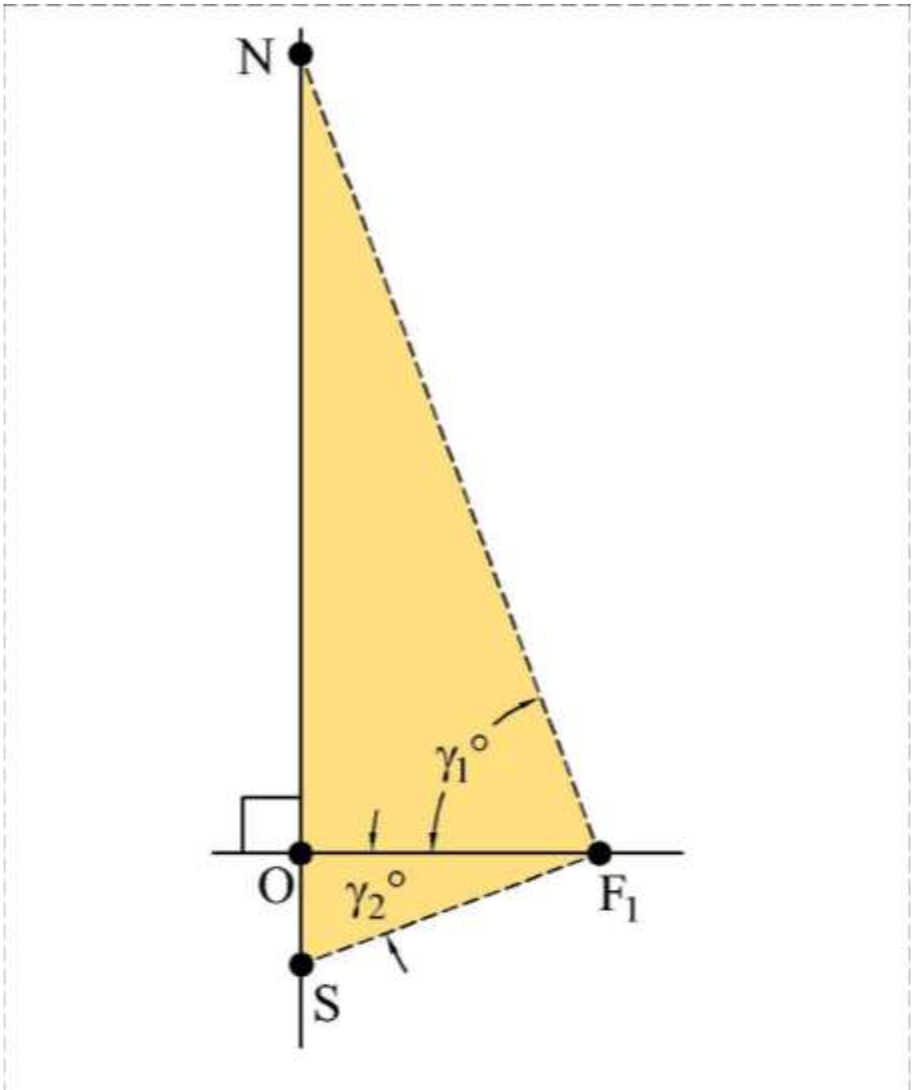

Fig. 5

$$\text{L.H.S} \Longrightarrow 1 - \left(\frac{\cancel{\sqrt{a^2+b^2} \times \sin(\theta°)}}{\cancel{b\cos(\theta°)}} \times \frac{\cancel{b\cos(\theta°)}}{\cancel{\sqrt{a^2+b^2} \times \sin(\theta°)}}\right)$$

$\therefore \text{L.H.S} \Longrightarrow 1 - 1$

$\therefore \text{L.H.S} \Longrightarrow 0$

Since $1 - \tan(\gamma_1°) \times \tan(\gamma_2°) = 0, \gamma°$ should be $90°$ (right angle).

Hence, the theorem is proven.

Alternate proof for the theorem

Referring fig. 5,

If this $\Delta\,\mathrm{SF_1N}$ is a right-triangle, according to Pythagoras theorem of right-angled triangle

$\mathrm{SF_1}^2 + \mathrm{F_1N}^2 = \mathrm{SN}^2$.

$$\text{Eqn. (A.16)} \Longrightarrow \mathrm{F_1S} = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)} \qquad \ldots(8.5)$$

$$\text{Eqn. (A.17)} \Longrightarrow \mathrm{F_1N} = \frac{\sqrt{a^2 + b^2} \times \sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)} \qquad \ldots(8.6)$$

Let LHS be $\mathrm{SF_1}^2 + \mathrm{F_1N}^2$ & RHS $= \mathrm{SN}^2$

LHS $\Longrightarrow \mathrm{F_1}S^2 + \mathrm{F_1N}^2$

Substituting eqns. (8.5) & (8.6) in above,

Substituting eqns.

$$\therefore \mathrm{F_1}S^2 + \mathrm{F_1N}^2 = \left(\frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)}\right)^2 + \left(\frac{\sqrt{a^2 + b^2} \times \sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)}\right)^2$$

$$\therefore \mathrm{F_1}S^2 + \mathrm{F_1N}^2 = \frac{a^2\sin^2(\theta°) + b^2}{\sin^2(\theta°)} + \frac{(a^2 + b^2) \times (a^2\sin^2(\theta°) + b^2)}{b^2\cos^2(\theta°)}$$

$$\therefore \mathrm{F_1}S^2 + \mathrm{F_1N}^2 = \frac{(a^2\sin^2(\theta°) + b^2) \times b^2\cos^2(\theta°) + (a^2 + b^2)\sin^2(\theta°) \times (a^2\sin^2(\theta°) + b^2)}{b^2\sin^2(\theta°)\cos^2(\theta°)}$$

$$\therefore \mathrm{F_1}S^2 + \mathrm{F_1N}^2 = \frac{[a^2\sin^2(\theta°) + b^2] \times [b^2\cos^2(\theta°) + (a^2 + b^2)\sin^2(\theta°)]}{b^2\sin^2(\theta°)\cos^2(\theta°)}$$

$$\therefore \mathrm{F_1}S^2 + \mathrm{F_1N}^2 = \frac{(a^2\sin^2(\theta°) + b^2) \times [b^2\cos^2(\theta°) + a^2\sin^2(\theta°) + b^2\sin^2(\theta°)]}{b^2\sin^2(\theta°)\cos^2(\theta°)}$$

$$\therefore \mathrm{F_1}S^2 + \mathrm{F_1N}^2 = \frac{(a^2\sin^2(\theta°) + b^2) \times [b^2\cos^2(\theta°) + b^2\sin^2(\theta°) + a^2\sin^2(\theta°)]}{b^2\sin^2(\theta°)\cos^2(\theta°)}$$

$$\therefore \mathrm{F_1}S^2 + \mathrm{F_1N}^2 = \frac{(a^2\sin^2(\theta°) + b^2) \times [b^2\{\cos^2(\theta°) + \sin^2(\theta°)\} + a^2\sin^2(\theta°)]}{b^2\sin^2(\theta°)\cos^2(\theta°)}$$

$$\therefore \mathrm{F_1}S^2 + \mathrm{F_1N}^2 = \frac{(a^2\sin^2(\theta°) + b^2) \times [a^2\sin^2(\theta°) + b^2]}{b^2\sin^2(\theta°)\cos^2(\theta°)}$$

$$\therefore \mathrm{F_1}S^2 + \mathrm{F_1N}^2 = \frac{(a^2\sin^2(\theta°) + b^2)^2}{b^2\sin^2(\theta°)\cos^2(\theta°)} \qquad \ldots(8.7)$$

$$\text{Eqn. (A.18)} \Longrightarrow \mathrm{SN} = \frac{a^2\sin^2(\theta°) + b^2}{b\sin(\theta°)\cos(\theta°)}$$

Squaring above eqn.,

$$SN^2 = \frac{(a^2\sin^2(\theta°) + b^2)^2}{b^2\sin^2(\theta°)\cos^2(\theta°)} \qquad \ldots(8.8)$$

Equating eqns. (8.7) & (8.8), ${SF_1}^2 + F_1N^2 = SN^2$, according to Pythagoras theorem.

Therefore, the triangle SF_1

$\therefore$ $\Delta\ SF_1N$ is a right-triangle. Hence the theorem is proved in two methods.

THEOREM- 2:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{\tan(\alpha°) \times \tan(\varphi°) = \frac{b^2}{a^2}}$$

Referring fig. 3,

Let $\angle$ POQ be $\alpha°$ & $\angle$ PTQ be $\varphi°$.

$$\text{Eqn. (7.3)} \Longrightarrow \tan(\alpha°) = \frac{b\sin(\theta°)}{a} \qquad \ldots(9.1)$$

$$\text{Eqn. (4.3)} \Longrightarrow \tan(\varphi°) = \frac{b}{a\sin(\theta°)} \qquad \ldots(9.2)$$

Multiplying eqns. (9.1) & (9.2),

$$\tan(\alpha°) \times \tan(\varphi°) = \left(\frac{b\cancel{\sin(\theta°)}}{a}\right) \times \left(\frac{b}{a\cancel{\sin(\theta°)}}\right)$$

$$\therefore \tan(\alpha°) \times \tan(\varphi°) = \frac{b^2}{a^2} \qquad \ldots(9.3)$$

where, a & b are arbitrary constant.

Eqn. (9.3) is mathematical expression of the property.

THEOREM- 3:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{OQ \times PQ = \frac{b^2}{\tan(\alpha°) - \tan(\varphi°)}}$$

Referring fig. 3,

Let $\angle$ POQ be $\alpha°$ & $\angle$ PTQ be $\varphi°$.

$$\text{Eqn. (7.3)} \Longrightarrow \tan(\alpha°) = \frac{b\sin(\theta°)}{a} \qquad \ldots(10.1)$$

$$\text{Eqn. (4.3)} \Longrightarrow \tan(\varphi°) = \frac{b}{a\sin(\theta°)} \qquad \ldots(10.2)$$

Subtracting eqn. (10.1) from (10.2),

$$\tan(\varphi^\circ) - \tan(\alpha^\circ) = \left(\frac{b}{a\sin(\theta^\circ)}\right) - \left(\frac{b\sin(\theta^\circ)}{a}\right)$$

$$\therefore \tan(\alpha^\circ) - \tan(\varphi^\circ) = \frac{b - b\sin^2(\theta^\circ)}{a\sin(\theta^\circ)}$$

$$\therefore \tan(\alpha^\circ) - \tan(\varphi^\circ) = \frac{b[1 - \sin^2(\theta^\circ)]}{a\sin(\theta^\circ)}$$

$$\therefore \tan(\alpha^\circ) - \tan(\varphi^\circ) = \frac{b[\cos^2(\theta^\circ)]}{a\sin(\theta^\circ)} \qquad \ldots(10.3)$$

$$\text{Eqn. (A. 2)} \Longrightarrow OQ = \frac{a}{\cos(\theta^\circ)}$$

$$\therefore \frac{1}{OQ} = \frac{\cos(\theta^\circ)}{a} \qquad \ldots(10.4)$$

$$\text{Eqn. (A. 3)} \Longrightarrow PQ = \frac{b\sin(\theta^\circ)}{\cos(\theta^\circ)}$$

$$\therefore \frac{1}{PQ} = \frac{\cos(\theta^\circ)}{b\sin(\theta^\circ)} \qquad \ldots(10.5)$$

Multiplying eqns. (10.4), (10.5) & b^2

$$\frac{1}{OQ} \times \frac{1}{PQ} \times b^2 = \left(\frac{\cos(\theta^\circ)}{a}\right) \times \left(\frac{\cos(\theta^\circ)}{\cancel{b}\sin(\theta^\circ)}\right) \times b^{\cancel{2}}$$

$$\therefore \frac{1}{OQ} \times \frac{1}{PQ} \times b^2 = \frac{b[\cos^2(\theta^\circ)]}{a\sin(\theta^\circ)}$$

$$\therefore \frac{b^2}{OQ \times PQ} = \frac{b[\cos^2(\theta^\circ)]}{a\sin(\theta^\circ)} \qquad \ldots(10.6)$$

Equating eqns. (10.3) & (10.6),

$$\tan(\alpha^\circ) - \tan(\varphi^\circ) = \frac{b^2}{OQ \times PQ}$$

The above equation may be rewritten as

$$OQ \times PQ = \frac{b^2}{\tan(\alpha^\circ) - \tan(\varphi^\circ)} \qquad \ldots(10.7)$$

where,
OQ is abscissa & PQ is ordinate of point P where the tangent is drawn on the hyperbola.

b is an arbitrary constant.

Eqn. (10.7) is mathematical expression of the property.

CONCLUSION

This research provides a comprehensive mathematical exploration of the hyperbola, with particular focus on the behavior of tangents, normals, and focal distances. By employing parametric techniques, seven new properties and three theorems have been systematically developed and rigorously established, accompanied by clear illustrative diagrams that enhance understanding. These contributions deepen and extend the classical theory of the hyperbola, offering refined insights into its geometric nature.

Beyond theoretical significance, the results hold practical relevance for several applied domains, including radar technology, antenna engineering, and optical systems such as torch reflectors, where hyperbolic geometry plays a functional role. The integration of parametric analysis with detailed visual representations strengthens both the precision and accessibility of the study.

Overall, this work offers a valuable resource for scholars and researchers in conic geometry, celestial mechanics, and related mathematical fields, representing a meaningful addition to both the theoretical framework of conic sections and their diverse interdisciplinary applications.

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