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Research Article

NEW THEORETICAL DEVELOPMENTS AND INNOVATIVE PROPERTIES OF HYPERBOLA

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ABSTRACT

A hyperbola is a set of points where the absolute difference of distances from any point on the hyperbola to two fixed points (foci) is constant. The hyperbola has key components such as the center, transverse axis, conjugate axis, vertex, focus, and asymptotes. The properties of hyperbola include relationship between the lengths of the axes and the distances between elements like the focal distances, tangent, normal, directrix and vertex.

This research article examines the geometric and mathematical significance of the hyperbola, a fundamental conic section. Moving beyond its classical definition, the study introduces a set of 104 newly derived theorems concerning the tangent, normal, and focal distances of the hyperbola. Each theorem is rigorously developed with detailed proofs, step-by-step derivations, and illustrative diagrams to aid comprehension. Collectively, these contributions expand the theoretical framework of hyperbola geometry and serve as a significant reference for scholars engaged in advanced studies of conic sections and related fields. These newly identified properties support the development of Artificial intelligence techniques for analyzing hyperbolic structures and its applications in mathematical modeling.

Keywords: *Hyperbola, Conic sections, Tangent, Normal & Focal distances*

INTRODUTION

A hyperbola *[Borowski E.J & Borwein J.M (1991)]* can be defined in two equivalent ways: as the locus of a point whose distance from a fixed point (focus) *[Christopher Clapham & James Nicolson (2009)]* and a fixed line (directrix) *[Borowski E.J & Borwein J.M (1991)]* are equal, or as the set of points for which the absolute difference of distances from two foci remains constant. Concepts such as foci, directrix, latus rectum *[Borowski E.J & Borwein J.M (1991)]*, and eccentricity *[Borowski E.J & Borwein J.M (1991)]* are central to its study. Common occurrences of hyperbolas include the path traced by the tip of a sundial's shadow and the scattering trajectories of subatomic particles. Hyperbolas are significant in various fields, including Astronomy, Physics, and Engineering, for their unique geometric properties and applications.

This research article extends the traditional study of the hyperbola by introducing newly derived theorems concerning its tangent *[Borowski E.J & Borwein J.M (1991)]*, normal *[Borowski E.J & Borwein J.M (1991)]*, and focal distances *[Borowski E.J & Borwein J.M (1991)]*. Each theorem is rigorously developed with comprehensive proofs, detailed step-by-step derivations, and carefully constructed diagrams to facilitate understanding. Beyond these results, the article also establishes refined mathematical relationships between the tangent, normal, focus, and other essential geometric elements of the hyperbola.

To support these derivations, a preamble is provided containing the fundamental formulas and parametric equations *[Borowski E.J & Borwein J.M (1991)]* of the hyperbola, which serve as the basis for the logical progression of proofs. Collectively, these contributions enrich the theoretical framework of its geometry, offering new perspectives for advanced studies in conic sections and related mathematical fields.

Orbit Paths: In astronomy, the paths of comets around the sun often form hyperbolic trajectories.

Navigation Systems: Hyperbolic navigation systems use the properties of hyperbolas for accurate positioning.

Acoustics: Hyperbolic shapes are used in the design of certain types of acoustic mirrors and reflectors to focus sound waves.

In the present scenario, 3I/ATLAS, which is an interstellar object (comet), travels through our solar system along a hyperbolic path, meaning its orbit is open and it is not gravitationally bound to the Sun. Its high eccentricity (e > 1) confirms that it originated from interstellar space and will leave the Solar System again. In such an orbit, the Sun lies at one focus, and the object makes its closest approach at perihelion before moving back outward. Interstellar objects like this are rare, so observing 3I/ATLAS helps scientists study material formed in other star systems, offering insights into how planets and comets develop across the galaxy.

ANALYSIS & DERIVATIONS OF THE EQUATIONS

Analysis Part - A

Referring fig.1,

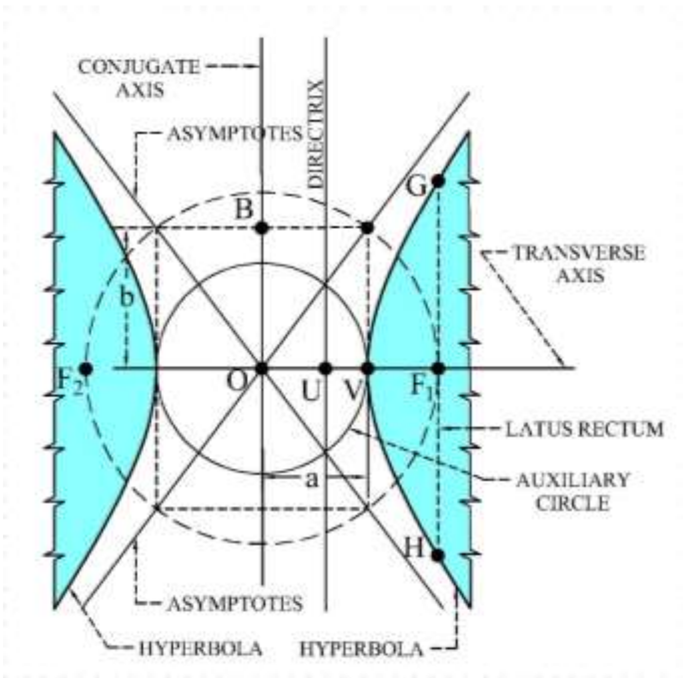

Fig. 1

In this figure,	
Point O	*is the centre of the hyperbola, otherwise it is intersection of transverse axis & conjugate axis. It is also intersection of asymptotes [Borowski E.J & Borwein J.M (1991)].*
Points F_1 & F_1	*are foci of the hyperbola.*
Point U	*is intersection of transverse axis & directrix.*
Point V	*is a vertex which is a point of intersection of transverse axis & hyperbola.*
GH	*is latus rectum of the hyperbola and its value is equal to $2b^2/a$*

Referring fig. 1,

$$\mathrm{OV} = \mathrm{a} \qquad \ldots(\mathrm{A.1})$$

$$\mathrm{OB} = \mathrm{b} \qquad \ldots(\mathrm{A.2})$$

$$\mathrm{OF_1} = \mathrm{OF_2} = \sqrt{\mathrm{a^2 + b^2}} \qquad \ldots(\mathrm{A.3})$$

$$\mathrm{F_1F_2} = 2\sqrt{\mathrm{a^2 + b^2}} \qquad \ldots(\mathrm{A.4})$$

Referring fig. 1, $\mathrm{VF_1} = \mathrm{OF_1} - \mathrm{OV}$

Substituting eqns. (A.3) & (A.1) in above,

$$\mathrm{VF_1} = \mathrm{OF_1} - \mathrm{OV}$$

$$\therefore \mathrm{VF_1} = \sqrt{\mathrm{a^2 + b^2}} - \mathrm{a} \qquad \ldots(\mathrm{A.5})$$

Referring fig. 1, $\mathrm{VF_2} = \mathrm{OF_1} - \mathrm{OV}$

Substituting eqns. (A.3) & (A.1) in above,

$$\mathrm{VF_2} = \mathrm{OF_2} + \mathrm{OV}$$

$$\therefore \mathrm{VF_2} = \sqrt{\mathrm{a^2 + b^2}} + \mathrm{a} \qquad \ldots(\mathrm{A.6})$$

Referring fig. 1, $\mathrm{OU} = \frac{\mathrm{a}}{\mathrm{e}}$

Substituting $\mathrm{e} = \frac{\sqrt{\mathrm{a^2 + b^2}}}{\mathrm{a}}$

$$\therefore \mathrm{OU} = \mathrm{a} \div \left(\frac{\sqrt{\mathrm{a^2 + b^2}}}{\mathrm{a}}\right)$$

$$\therefore \mathrm{OU} = \mathrm{a} \times \frac{\mathrm{a}}{\sqrt{\mathrm{a^2 + b^2}}}$$

$$\therefore \mathrm{OU} = \frac{\mathrm{a^2}}{\sqrt{\mathrm{a^2 + b^2}}} \qquad \ldots(\mathrm{A.7})$$

Referring fig. 1, $\mathrm{VU} = \mathrm{OV} - \mathrm{OU}$

Substituting eqns. (A.1) & (A.7) in above,

$$\therefore \mathrm{VU} = \mathrm{OV} - \mathrm{OU}$$

$$\therefore \mathrm{VU} = \mathrm{a} - \left(\frac{\mathrm{a^2}}{\sqrt{\mathrm{a^2 + b^2}}}\right)$$

$$\therefore \mathrm{VU} = \frac{\mathrm{a}\sqrt{\mathrm{a^2 + b^2}} - \mathrm{a^2}}{\sqrt{\mathrm{a^2 + b^2}}}$$

$$\therefore \mathrm{VU} = \frac{\mathrm{a}\left(\sqrt{\mathrm{a^2 + b^2}} - \mathrm{a}\right)}{\sqrt{\mathrm{a^2 + b^2}}} \qquad \ldots(\mathrm{A.8})$$

Referring fig. 1, $\mathrm{UF_1} = \mathrm{OF_1} - \mathrm{OU}$

Substituting eqns. (A.3) & (A.7) in above,

$$\therefore \mathrm{UF_1} = \sqrt{\mathrm{a^2 + b^2}} - \left(\frac{\mathrm{a^2}}{\sqrt{\mathrm{a^2 + b^2}}}\right)$$

$$\therefore \mathrm{UF_1} = \frac{\cancel{\mathrm{a^2}} + \mathrm{b^2} \cancel{- \mathrm{a^2}}}{\sqrt{\mathrm{a^2 + b^2}}}$$

$$\therefore \mathrm{UF_1} = \frac{\mathrm{b^2}}{\sqrt{\mathrm{a^2+b^2}}} \quad \ldots(\mathrm{A.9})$$

Referring fig. 1, $\mathrm{UF_2} = \mathrm{OF_2} + \mathrm{OU}$

Substituting eqns. (A.3) & (A.7) in above,

$$\mathrm{UF_2} = \sqrt{\mathrm{a^2+b^2}} + \left(\frac{\mathrm{a^2}}{\sqrt{\mathrm{a^2+b^2}}}\right)$$

$$\therefore \mathrm{UF_2} = \frac{\mathrm{a^2+b^2+a^2}}{\sqrt{\mathrm{a^2+b^2}}}$$

$$\therefore \mathrm{UF_2} = \frac{\mathrm{2a^2+b^2}}{\sqrt{\mathrm{a^2+b^2}}} \quad \ldots(\mathrm{A.10})$$

Referring fig. 1, $\mathrm{BU^2} = \mathrm{OU^2} + \mathrm{OB^2}$

Substituting eqns. (A.7) & (A.2) in above,

$$\therefore \mathrm{BU^2} = \left(\frac{\mathrm{a^2}}{\sqrt{\mathrm{a^2+b^2}}}\right)^2 + \mathrm{b^2}$$

$$\therefore \mathrm{BU^2} = \frac{\mathrm{a^4}}{\mathrm{a^2+b^2}} + \mathrm{b^2}$$

$$\therefore \mathrm{BU^2} = \frac{\mathrm{a^4+b^2(a^2+b^2)}}{\mathrm{a^2+b^2}}$$

$$\therefore \mathrm{BU^2} = \frac{\mathrm{a^4+a^2b^2+b^4}}{\mathrm{a^2+b^2}}$$

$$\therefore \mathrm{BU^2} = \frac{\mathrm{a^4+b^4+a^2b^2}}{\mathrm{a^2+b^2}}$$

$$\therefore \mathrm{BU^2} = \frac{\mathrm{(a^2+b^2)^2-a^2b^2}}{\mathrm{a^2+b^2}}$$

$$\therefore \mathrm{BU} = \frac{\sqrt{\mathrm{(a^2+b^2)^2-a^2b^2}}}{\sqrt{\mathrm{a^2+b^2}}} \quad \ldots(\mathrm{A.11})$$

Referring fig. 1, in right-triangle VOB,

$$\mathrm{BV^2} = \mathrm{OV^2} + \mathrm{OB^2}$$

Substituting eqns. (A.1) & (A.2),

$$\therefore \mathrm{BV^2} = \mathrm{a^2+b^2}$$

$$\therefore \mathrm{BV} = \sqrt{\mathrm{a^2+b^2}} \quad \ldots(\mathrm{A.12})$$

Analysis Part - B

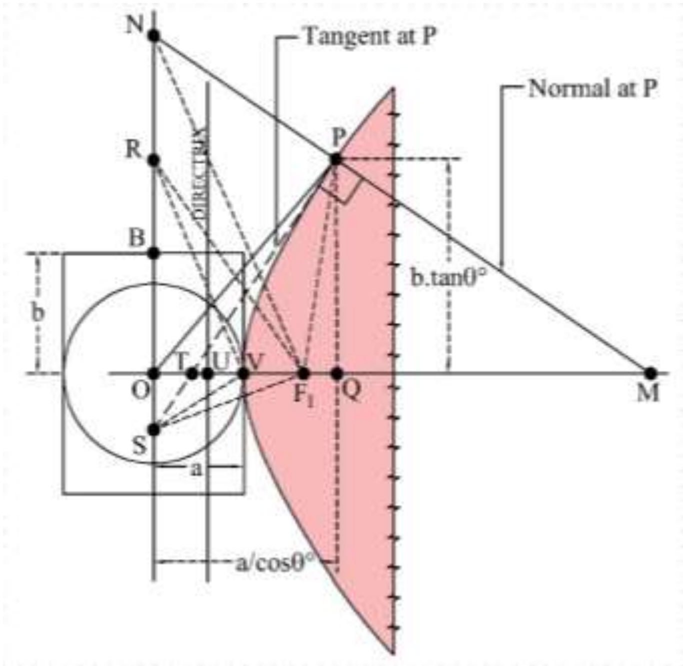

Fig.2

In this figure,	
Point O	*is centre of the hyperbola, otherwise it is intersection of transverse [Christopher Clapham & James Nicolson (2009)] & conjugate axis [Borowski E.J & Borwein J.M (1991)].*
Point V	*is vertex [Borowski E.J & Borwein J.M (1991)] of the hyperbola*
Points F_1	*is one of the foci of the hyperbola.*
Point P	*is a point considered anywhere on hyperbola.*
PF_1 & PF_1	*are focal distances of point P.*
PP'	*is focal chord of the hyperbola.*
Point Q	*is foot of perpendicular of point P upon transverse axis.*
OQ	*is abscissa [Borowski E.J & Borwein J.M (1991)] of point P.*
PQ or OR	*is ordinate [Borowski E.J & Borwein J.M (1991)] of point P.*
Point R	*is foot of perpendicular of point P upon conjugate axis.*
Point T	*is intersection of the tangent and transverse axis, where PT is a tangent drawn at point P on hyperbola.*
QT	*is sub-tangent [Borowski E.J & Borwein J.M (1991)] about transverse axis.*
Point S	*PS is a tangent drawn at point P on hyperbola and S is intersection of tangent and conjugate axis.*
RS	*is sub-tangent about conjugate axis.*
Point M	*is intersection of normal and transverse axis, where MN is a normal drawn at point P on hyperbola.*
QM	*is sub-normal [Mark D Licker (2003)] about transverse axis.*
Point N	*is intersection of normal & conjugate axis.*
RN	*is sub-normal about conjugate axis.*

Referring fig. 2,

$$OQ = \frac{a}{\cos(\theta°)} \quad \ldots (B.1)$$

$$PQ = OR = \frac{b\sin(\theta°)}{\cos(\theta°)} \quad \ldots (B.2)$$

Referring fig. 2, $UQ = OQ - OU$

Substituting eqns. (B.1) & (A.7),

$$\therefore UQ = \left(\frac{a}{\cos(\theta°)}\right) - \left(\frac{a^2}{\sqrt{a^2 + b^2}}\right)$$

$$\therefore UQ = \frac{a\sqrt{a^2 + b^2} - a^2\cos(\theta°)}{\sqrt{a^2 + b^2}\cos(\theta°)}$$

$$\therefore UQ = \frac{a\left[\sqrt{a^2 + b^2} - a\cos(\theta°)\right]}{\sqrt{a^2 + b^2} \times \cos(\theta°)} \quad \ldots (B.3)$$

Referring fig. 2, $VQ = OQ - OV$

Substituting eqns. (B.1) & (A.1),

$$\therefore VQ = \left(\frac{a}{\cos(\theta°)}\right) - a$$

$$\therefore VQ = \frac{a - a\cos(\theta°)}{\cos(\theta°)}$$

$$\therefore VQ = \frac{a[1 - \cos(\theta°)]}{\cos(\theta°)} \quad \ldots (B.4)$$

Referring fig. 2, in right-triangle OQP,

$$OP^2 = OQ^2 + PQ^2$$

Substituting eqns. (B.1) & (B.2),

$$OP^2 = \left(\frac{a}{\cos(\theta°)}\right)^2 + \left(\frac{b\sin(\theta°)}{\cos(\theta°)}\right)^2$$

$$\therefore OP^2 = \frac{a^2}{\cos^2(\theta°)} + \frac{b^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore OP^2 = \frac{a^2 + b^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore OP = \frac{\sqrt{a^2 + b^2\sin^2(\theta°)}}{\cos(\theta°)} \quad \ldots (B.5)$$

Referring fig. 2, $F_1Q = OQ - OF_1$

Substituting eqns. (B.1) & (A.3),

$$\therefore F_1Q = \left(\frac{a}{\cos(\theta°)}\right) - \sqrt{a^2 + b^2}$$

$$\therefore F_1Q = \frac{a - \sqrt{a^2 + b^2}\cos(\theta°)}{\cos(\theta°)} \quad \ldots (B.6)$$

$F_2Q = OQ + OF_2$

Substituting eqns. (B.1) & (A.3),

$$\therefore F_2Q = \left(\frac{a}{\cos(\theta°)}\right) + \sqrt{a^2+b^2}$$

$$\therefore F_2Q = \frac{a + \sqrt{a^2+b^2}\cos(\theta°)}{\cos(\theta°)} \qquad \ldots(B.7)$$

Referring fig. 2, $F_1M = OM - OF_1$

Substituting eqns. (C.7) & (A.3) in above eqn.,

$$\therefore F_1M = \left(\frac{a^2+b^2}{a\cos(\theta°)}\right) - \sqrt{a^2+b^2}$$

$$\therefore F_1M = \frac{a^2+b^2 - a\sqrt{a^2+b^2}\cos(\theta°)}{a\cos(\theta°)} \qquad \ldots(B.8)$$

Referring fig. 2, $F_2M = OM + OF_2$

Substituting eqns. (C.7) & (A.3) in above eqn.,

$$\therefore F_2M = \left(\frac{a^2+b^2}{a\cos(\theta°)}\right) + \sqrt{a^2+b^2}$$

$$\therefore F_2M = \frac{a^2+b^2 + a\sqrt{a^2+b^2}\cos(\theta°)}{a\cos(\theta°)} \qquad \ldots(B.9)$$

Referring fig. 2, in right triangle F_1ON

Substituting eqns. (A.3) & (C.8) in above eqn.,

$$\therefore F_1N^2 = {OF_1}^2 + ON^2$$

$$\therefore F_1N^2 = a^2+b^2 + \frac{(a^2+b^2)^2\sin^2(\theta°)}{b^2\cos^2(\theta°)}$$

$$\therefore F_1N^2 = \frac{(a^2+b^2)b^2\cos^2(\theta°) + (a^2+b^2)^2\sin^2(\theta°)}{b^2\cos^2(\theta°)}$$

$$\therefore F_1N^2 = \frac{(a^2+b^2)[b^2\cos^2(\theta°) + (a^2+b^2)\sin^2(\theta°)]}{b^2\cos^2(\theta°)}$$

$$\therefore F_1N^2 = \frac{(a^2+b^2)[b^2\cos^2(\theta°) + a^2\sin^2(\theta°) + b^2\sin^2(\theta°)]}{b^2\cos^2(\theta°)}$$

$$\therefore F_1N^2 = \frac{(a^2+b^2)[a^2\sin^2(\theta°) + b^2\{\cos^2(\theta°) + \sin^2(\theta°)\}]}{b^2\cos^2(\theta°)}$$

$$\therefore F_1N^2 = \frac{(a^2+b^2)(a^2\sin^2(\theta°) + b^2)}{b^2\cos^2(\theta°)}$$

$$\therefore F_1N = \frac{\sqrt{(a^2+b^2)(a^2\sin^2(\theta°) + b^2)}}{b\cos(\theta°)} \qquad \ldots(B.10)$$

$$\therefore F_2N = \frac{\sqrt{(a^2+b^2)(a^2\sin^2(\theta°) + b^2)}}{b\cos(\theta°)} \qquad \ldots(B.11)$$

Referring fig. 2, in right-triangle F_1QP,

$PF_1{}^2 = F_1Q^2 + PQ^2$

Substituting eqns. (B.6) & (B.2),

$$\therefore PF_1{}^2 = \left(\frac{a - \sqrt{a^2+b^2}\cos(\theta°)}{\cos(\theta°)}\right)^2 + \left(\frac{b\sin(\theta°)}{\cos(\theta°)}\right)^2$$

$$\therefore PF_1{}^2 = \frac{a^2 + (a^2+b^2)\cos^2(\theta°) - 2a\sqrt{a^2+b^2}\cos(\theta°)}{\cos^2(\theta°)} + \frac{b^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore PF_1{}^2 = \frac{a^2 + a^2\cos^2(\theta°) + b^2\cos^2(\theta°) - 2a\sqrt{a^2+b^2}\cos(\theta°) + b^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore PF_1{}^2 = \frac{a^2 + b^2[\cos^2(\theta°) + \sin^2(\theta°)] + a^2\cos^2(\theta°) - 2a\sqrt{a^2+b^2}\cos(\theta°)}{\cos^2(\theta°)}$$

$$\therefore PF_1{}^2 = \frac{a^2 + b^2 + a^2\cos^2(\theta°) - 2a\sqrt{a^2+b^2}\cos(\theta°)}{\cos^2(\theta°)}$$

$$\therefore PF_1{}^2 = \frac{\left(\sqrt{a^2+b^2}\right)^2 + [a\cos(\theta°)]^2 - 2\sqrt{a^2+b^2}\,a\cos(\theta°)}{\cos^2(\theta°)}$$

$$\therefore PF_1{}^2 = \left(\frac{\sqrt{a^2+b^2} - a\cos(\theta°)}{\cos(\theta°)}\right)^2$$

$$\therefore PF_1 = \frac{\sqrt{a^2+b^2} - a\cos(\theta°)}{\cos(\theta°)} \qquad \ldots(B.12)$$

Similarly,

Referring fig. 1 & fig. 2, in right-triangle F_2QP,

$PF_2{}^2 = F_2Q^2 + PQ^2$

Substituting eqns. (B.7) & (B.2),

$$\therefore PF_2{}^2 = \left(\frac{\sqrt{a^2+b^2} + a\cos(\theta°)}{\cos(\theta°)}\right)^2$$

$$\therefore PF_2 = \frac{\sqrt{a^2+b^2} + a\cos(\theta°)}{\cos(\theta°)} \qquad \ldots(B.13)$$

Referring fig. 2, in right-triangle V_1QP,

$VP^2 = VQ^2 + PQ^2$

Substituting eqns. (B.15) & (B.4),

$$VP^2 = \left(\frac{a[1-\cos(\theta°)]}{\cos(\theta°)}\right)^2 + \left(\frac{b\sin(\theta°)}{\cos(\theta°)}\right)^2$$

$$\therefore VP^2 = \frac{a^2[1-\cos(\theta°)]^2}{\cos^2(\theta°)} + \frac{b^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore VP^2 = \frac{a^2[1-\cos(\theta°)]^2 + b^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{VP} = \frac{\sqrt{a^2[1-\cos(\theta°)]^2 + b^2\sin^2(\theta°)}}{\cos(\theta°)} \qquad \dots(\mathrm{B}.14)$$

Referring fig. 2, in right-triangle VOS,

$\mathrm{VS}^2 = \mathrm{OV}^2 + \mathrm{OS}^2$

Substituting eqns. (A.1) & (C.2),

$$\therefore \mathrm{VS}^2 = a^2 + \left(\frac{b\cos(\theta°)}{\sin(\theta°)}\right)^2$$

$$\therefore \mathrm{VS}^2 = a^2 + \frac{b^2\cos^2(\theta°)}{\sin^2(\theta°)}$$

$$\therefore \mathrm{VS}^2 = \frac{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}{\sin^2(\theta°)}$$

$$\therefore \mathrm{VS} = \frac{\sqrt{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}}{\sin(\theta°)} \qquad \dots(\mathrm{B}.15)$$

Referring fig. 2, in right-triangle VOR,

$\mathrm{VR}^2 = \mathrm{OV}^2 + \mathrm{OR}^2$

Substituting eqns. (A.1) & (B.2) in above,

$$\therefore \mathrm{VR}^2 = a^2 + \left(\frac{b\sin(\theta°)}{\cos(\theta°)}\right)^2$$

$$\therefore \mathrm{VR}^2 = a^2 + \frac{b^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{VR}^2 = \frac{a^2\cos^2(\theta°) + b^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{VR} = \frac{\sqrt{a^2\cos^2(\theta°) + b^2\sin^2(\theta°)}}{\cos(\theta°)} \qquad \dots(\mathrm{B}.16)$$

Referring fig. 2, in right-triangle $\mathrm{F_1OR}$,

$\mathrm{F_1R}^2 = \mathrm{OF_1}^2 + \mathrm{OR}^2$

Substituting eqns. (A.3) & (B.2) in above,

$$\therefore \mathrm{F_1R}^2 = \left(\sqrt{a^2+b^2}\right)^2 + \left(\frac{b\sin(\theta°)}{\cos(\theta°)}\right)^2$$

$$\therefore \mathrm{F_1R}^2 = a^2 + b^2 + \frac{b^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{F_1R}^2 = \frac{a^2\cos^2(\theta°) + b^2\cos^2(\theta°) + b^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{F_1R}^2 = \frac{a^2\cos^2(\theta°) + b^2[\cos^2(\theta°) + \sin^2(\theta°)]}{\cos^2(\theta°)}$$

$$\therefore \mathrm{F_1R}^2 = \frac{a^2\cos^2(\theta°) + b^2}{\cos^2(\theta°)}$$

$$\therefore F_1R = \frac{\sqrt{a^2\cos^2(\theta°) + b^2}}{\cos(\theta°)} \quad \ldots(B.17)$$

Analysis Part - C

Referring fig. 2,

We know that $OT \times OQ = a^2$

$$\therefore OT = \frac{a^2}{OQ}$$

Substituting eqns. (B.1) in above eqn.,

$$\therefore OT = \frac{a^2}{\left(\frac{a}{\cos(\theta°)}\right)}$$

$$\therefore OT = a^{\cancel{2}} \times \frac{\cos(\theta°)}{\cancel{a}}$$

$$\therefore OT = a\cos(\theta°) \quad \ldots(C.1)$$

We know that $OS \times OR = b^2$

$$\therefore OS = \frac{b^2}{OR}$$

Substituting eqns. (B.2) in above eqn.,

$$\therefore OS = \frac{b^2}{\left(\frac{b\sin(\theta°)}{\cos(\theta°)}\right)}$$

$$\therefore OS = b^{\cancel{2}} \times \frac{\cos(\theta°)}{\cancel{b}\sin(\theta°)}$$

$$\therefore OS = \frac{b\cos(\theta°)}{\sin(\theta°)} \quad \ldots(C.2)$$

Referring fig. 2, in right-triangle TOB,

$BT^2 = OT^2 + OB^2$

Substituting eqns. (C.1) & (A.2),

$$\therefore BT^2 = a^2\cos^2(\theta°) + b^2$$

$$\therefore BT = \sqrt{a^2\cos^2(\theta°) + b^2} \quad \ldots(C.3)$$

Referring fig. 2, $TQ = OQ - OT$

Substituting eqns. (B.1) & (C.1) in above eqn.,

$$\therefore TQ = \left(\frac{a}{\cos(\theta°)}\right) - a\cos(\theta°)$$

$$\therefore TQ = \frac{a - a\cos^2(\theta°)}{\cos(\theta°)}$$

$$\therefore TQ = \frac{a[1 - \cos^2(\theta°)]}{\cos(\theta°)}$$

$$\therefore \mathrm{TQ} = \frac{\mathrm{a}\sin^2(\theta°)}{\cos(\theta°)} \qquad \ldots(\mathrm{C.4})$$

Referring fig. 2, Δ MQP & Δ PQT are similar.

$$\therefore \frac{\mathrm{PQ}}{\mathrm{QM}} = \frac{\mathrm{TQ}}{\mathrm{PQ}}$$

$$\therefore \mathrm{QM} = \frac{\mathrm{PQ} \times \mathrm{PQ}}{\mathrm{TQ}}$$

$$\therefore \mathrm{QM} = \frac{\mathrm{PQ}^2}{\mathrm{TQ}}$$

Substituting eqns. (B.2) & (C.4),

$$\therefore \mathrm{QM} = \left(\frac{\mathrm{b}\sin(\theta°)}{\cos(\theta°)}\right)^2 \div \left(\frac{\mathrm{a}\sin^2(\theta°)}{\cos(\theta°)}\right)$$

$$\therefore \mathrm{QM} = \left(\frac{\mathrm{b}\sin(\theta°)}{\cos(\theta°)}\right)^2 \times \left(\frac{\cos(\theta°)}{\mathrm{a}\sin^2(\theta°)}\right)$$

$$\therefore \mathrm{QM} = \frac{\mathrm{b}^2\cancel{\sin^2(\theta°)}}{\cos^{\cancel{2}}(\theta°)} \times \frac{\cancel{\cos(\theta°)}}{\mathrm{a}\cancel{\sin^2(\theta°)}}$$

$$\therefore \mathrm{QM} = \frac{\mathrm{b}^2}{\mathrm{a}\cos(\theta°)} \qquad \ldots(\mathrm{C.5})$$

Referring fig. 2, $\mathrm{OM} = \mathrm{OQ} + \mathrm{QM}$

Substituting eqns. (B.1) & (C.5) in above eqn.,

$$\therefore \mathrm{OM} = \left(\frac{\mathrm{a}}{\cos(\theta°)}\right) + \left(\frac{\mathrm{b}^2}{\mathrm{a}\cos(\theta°)}\right)$$

$$\therefore \mathrm{OM} = \frac{\mathrm{a}^2 + \mathrm{b}^2}{\mathrm{a}\cos(\theta°)} \qquad \ldots(\mathrm{C.6})$$

Referring fig. 2, Δ MON & Δ MQP are similar.

$$\therefore \frac{\mathrm{ON}}{\mathrm{OM}} = \frac{\mathrm{PQ}}{\mathrm{QM}}$$

$$\therefore \mathrm{ON} = \frac{\mathrm{PQ} \times \mathrm{OM}}{\mathrm{QM}}$$

Substituting eqns. (B.2), (C.6) & (C.5),

$$\therefore \mathrm{ON} = \frac{\left(\frac{\mathrm{b}\sin(\theta°)}{\cos(\theta°)}\right) \times \left(\frac{\mathrm{a}^2 + \mathrm{b}^2}{\mathrm{a}\cos(\theta°)}\right)}{\left(\frac{\mathrm{b}^2}{\mathrm{a}\cos(\theta°)}\right)}$$

$$\therefore \mathrm{ON} = \frac{\cancel{\mathrm{b}}\sin(\theta°)}{\cos(\theta°)} \times \frac{\mathrm{a}^2 + \mathrm{b}^2}{\cancel{\mathrm{a}\cos(\theta°)}} \times \frac{\cancel{\mathrm{a}\cos(\theta°)}}{\mathrm{b}^{\cancel{2}}}$$

$$\therefore \mathrm{ON} = \frac{(\mathrm{a}^2 + \mathrm{b}^2)\sin(\theta°)}{\mathrm{b}\cos(\theta°)} \qquad \ldots(\mathrm{C.7})$$

Referring fig. 2, $\mathrm{VT} = \mathrm{OV} - \mathrm{OT}$

Substituting eqns. (A.1) & (C.1) in above eqn.,

$$\therefore \mathrm{VT} = \mathrm{a} - \mathrm{acos}(\theta°)$$

$$\therefore \mathrm{VT} = \mathrm{a}[1 - \cos(\theta°)] \qquad \ldots (\mathrm{C}.8)$$

Referring fig. 2, $\mathrm{TM} = \mathrm{OM} - \mathrm{OT}$

Substituting eqns. (C.6) & (C.1) in above eqn.,

$$\therefore \mathrm{TM} = \left(\frac{\mathrm{a}^2 + \mathrm{b}^2}{\mathrm{acos}(\theta°)}\right) - \mathrm{acos}(\theta°)$$

$$\therefore \mathrm{TM} = \frac{\mathrm{a}^2 + \mathrm{b}^2 - \mathrm{a}^2\cos^2(\theta°)}{\mathrm{acos}(\theta°)}$$

$$\therefore \mathrm{TM} = \frac{\mathrm{a}^2[1 - \cos^2(\theta°)] + \mathrm{b}^2}{\mathrm{acos}(\theta°)}$$

$$\therefore \mathrm{TM} = \frac{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}{\mathrm{acos}(\theta°)} \qquad \ldots (\mathrm{C}.9)$$

Referring fig. 2, in right-triangle TOS,

$$\mathrm{TS}^2 = \mathrm{OT}^2 + \mathrm{OS}^2$$

Substituting eqns. (C.1) & (C.2) in above eqn.,

$$\therefore \mathrm{TS}^2 = [\mathrm{acos}(\theta°)]^2 + \left(\frac{\mathrm{bcos}(\theta°)}{\sin(\theta°)}\right)^2$$

$$\therefore \mathrm{TS}^2 = \mathrm{a}^2\cos^2(\theta°) + \frac{\mathrm{b}^2\cos^2(\theta°)}{\sin^2(\theta°)}$$

$$\therefore \mathrm{TS}^2 = \frac{\mathrm{a}^2\sin^2(\theta°)\cos^2(\theta°) + \mathrm{b}^2\cos^2(\theta°)}{\sin^2(\theta°)}$$

$$\therefore \mathrm{TS}^2 = \frac{(\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2)\cos^2(\theta°)}{\sin^2(\theta°)}$$

$$\mathrm{TS} = \frac{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)} \qquad \ldots (\mathrm{C}.10)$$

Referring fig. 2, $\mathrm{F_1T} = \mathrm{OF_1} - \mathrm{OT}$

Substituting eqns. (A.3) & (C.1),

$$\mathrm{F_1T} = \sqrt{\mathrm{a}^2 + \mathrm{b}^2} - \mathrm{acos}(\theta°) \qquad \ldots (\mathrm{C}.11)$$

Referring fig. 2, $\mathrm{F_1T} = \mathrm{OF_2} + \mathrm{OT}$

Substituting eqns. (A.3) & (C.1),

$$\mathrm{F_2T} = \sqrt{\mathrm{a}^2 + \mathrm{b}^2} + \mathrm{acos}(\theta°) \qquad \ldots (\mathrm{C}.12)$$

Referring fig. 2, in right-triangle $\mathrm{F_1OS}$,

$$\mathrm{F_1S}^2 = \mathrm{OF_1}^2 + \mathrm{OS}^2$$

Substituting eqns. (A.3) & (C.2) in above,

$$\therefore F_1S^2 = (a^2 + b^2) + \left(\frac{b\cos(\theta°)}{\sin(\theta°)}\right)^2$$

$$\therefore F_1S^2 = a^2 + b^2 + \frac{b^2\cos^2(\theta°)}{\sin^2(\theta°)}$$

$$\therefore F_1S^2 = \frac{a^2\sin^2(\theta°) + b^2\sin^2(\theta°) + b^2\cos^2(\theta°)}{\sin^2(\theta°)}$$

$$\therefore F_1S^2 = \frac{a^2\sin^2(\theta°) + b^2[\sin^2(\theta°) + \cos^2(\theta°)]}{\sin^2(\theta°)}$$

$$\therefore F_1S^2 = \frac{a^2\sin^2(\theta°) + b^2}{\sin^2(\theta°)}$$

$$\therefore F_1S = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)} \qquad \ldots(C.13)$$

Referring fig. 2, in right-triangle MQP,

$PM^2 = QM^2 + PQ^2$

Substituting eqns. (C.5) & (B.2) in above eqn.,

$$\therefore PM^2 = \left(\frac{b^2}{a\cos(\theta°)}\right)^2 + \left(\frac{b\sin(\theta°)}{\cos(\theta°)}\right)^2$$

$$\therefore PM^2 = \frac{b^4}{a^2\cos^2(\theta°)} + \frac{b^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore PM^2 = \frac{b^4 + a^2b^2\sin^2(\theta°)}{a^2\cos^2(\theta°)}$$

$$\therefore PM^2 = \frac{b^2[b^2 + a^2\sin^2(\theta°)]}{a^2\cos^2(\theta°)}$$

$$\therefore PM = \frac{b\sqrt{b^2 + a^2\sin^2(\theta°)}}{a\cos(\theta°)} \qquad \ldots(C.14)$$

Referring fig. 2, in right-triangle MPT,

$PT^2 + PM^2 = TM^2$

$\therefore PT^2 = TM^2 - PM^2$

Substituting eqns. (C.9) & (C.14),

$$\therefore PT^2 = \left(\frac{a^2\sin^2(\theta°) + b^2}{a\cos(\theta°)}\right)^2 - \left(\frac{b\sqrt{b^2 + a^2\sin^2(\theta°)}}{a\cos(\theta°)}\right)^2$$

$$\therefore PT^2 = \frac{[a^2\sin^2(\theta°) + b^2]^2}{a^2\cos^2(\theta°)} - \frac{b^2[b^2 + a^2\sin^2(\theta°)]}{a^2\cos^2(\theta°)}$$

$$\therefore PT^2 = \frac{[a^2\sin^2(\theta°) + b^2]^2 - b^2[b^2 + a^2\sin^2(\theta°)]}{a^2\cos^2(\theta°)}$$

$$\therefore PT^2 = \frac{a^4\sin^4(\theta°) \cancel{+ b^4} + 2a^2b^2\sin^2(\theta°) \cancel{- b^4} - a^2b^2\sin^2(\theta°)}{a^2\cos^2(\theta°)}$$

$$\therefore \mathrm{PT}^2 = \frac{a^4\sin^4(\theta°) + a^2b^2\sin^2(\theta°)}{a^2\cos^2(\theta°)}$$

$$\therefore \mathrm{PT}^2 = \frac{\cancel{a^2}\sin^2(\theta°)(a^2\sin^2(\theta°) + b^2)}{\cancel{a^2}\cos^2(\theta°)}$$

$$\therefore \mathrm{PT}^2 = \frac{\sin^2(\theta°)(a^2\sin^2(\theta°) + b^2)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{PT} = \frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)} \qquad \ldots (\mathrm{C}.15)$$

Referring fig. 2, PS = PT + TS

Substituting eqns. (C.15) & (C.10),

$$\therefore \mathrm{PS} = \left(\frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)}\right) + \left(\frac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)}\right)$$

$$\therefore \mathrm{PS} = \frac{\sin^2(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2} + \cos^2(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)\cos(\theta°)}$$

$$\therefore \mathrm{PS} = \frac{[\sin^2(\theta°) + \cos^2(\theta°)]\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)\cos(\theta°)}$$

$$\therefore \mathrm{PS} = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)\cos(\theta°)} \qquad \ldots (\mathrm{C}.16)$$

Referring fig. 2, RN = ON − OR

Substituting eqns. (C.7) & (B.2),

$$\therefore \mathrm{RN} = \left(\frac{(a^2 + b^2)\sin(\theta°)}{b\cos(\theta°)}\right) - \left(\frac{b\sin(\theta°)}{\cos(\theta°)}\right)$$

$$\therefore \mathrm{RN} = \frac{(a^2 + b^2)\sin(\theta°) - b^2\sin(\theta°)}{b\cos(\theta°)}$$

$$\therefore \mathrm{RN} = \frac{a^2\sin(\theta°) \cancel{+ b^2\sin(\theta°) - b^2\sin(\theta°)}}{b\cos(\theta°)}$$

$$\therefore \mathrm{RN} = \frac{a^2\sin(\theta°)}{b\cos(\theta°)} \qquad \ldots (\mathrm{C}.17)$$

Referring fig. 2, RS = OR + OS

Substituting eqns. (B.2) & (C.2),

$$\therefore \mathrm{RS} = \frac{b\sin(\theta°)}{\cos(\theta°)} + \frac{b\cos(\theta°)}{\sin(\theta°)}$$

$$\therefore \mathrm{RS} = \frac{b\sin^2(\theta°) + b\cos^2(\theta°)}{\sin(\theta°)\cos(\theta°)}$$

$$\therefore \mathrm{RS} = \frac{b[\sin^2(\theta°) + \cos^2(\theta°)]}{\sin(\theta°)\cos(\theta°)}$$

$$\therefore \mathrm{RS} = \frac{b}{\sin(\theta°)\cos(\theta°)} \qquad \ldots (\mathrm{C}.18)$$

Referring fig. 2, in right triangle PRN

$$PN^2 = PR^2 + RN^2$$

$$\therefore PN^2 = OQ^2 + RN^2$$

Substituting eqns. (B.1) & (C.17) in above,

$$\therefore PN^2 = \left(\frac{a}{\cos(\theta°)}\right)^2 + \left(\frac{a^2\sin(\theta°)}{b\cos(\theta°)}\right)^2$$

$$\therefore PN^2 = \frac{a^2}{\cos^2(\theta°)} + \frac{a^4\sin^2(\theta°)}{b^2\cos^2(\theta°)}$$

$$\therefore PN^2 = \frac{a^2b^2 + a^4\sin^2(\theta°)}{b^2\cos^2(\theta°)}$$

$$\therefore PN^2 = \frac{a^2[b^2 + a^2\sin^2(\theta°)]}{b^2\cos^2(\theta°)}$$

$$\therefore PN = \frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)} \qquad \ldots(C.19)$$

Referring fig. 2, $SN = OS + ON$

Substituting eqns. (C.2) & (C.7) in above eqn.,

$$\therefore SN = \left(\frac{b\cos(\theta°)}{\sin(\theta°)}\right) + \left(\frac{(a^2 + b^2)\sin(\theta°)}{b\cos(\theta°)}\right)$$

$$\therefore SN = \frac{b^2\cos^2(\theta°) + (a^2 + b^2)\sin^2(\theta°)}{b\sin(\theta°)\cos(\theta°)}$$

$$\therefore SN = \frac{b^2\cos^2(\theta°) + a^2\sin^2(\theta°) + b^2\sin^2(\theta°)}{b\sin(\theta°)\cos(\theta°)}$$

$$\therefore SN = \frac{a^2\sin^2(\theta°) + b^2[\cos^2(\theta°) + \sin^2(\theta°)]}{b\sin(\theta°)\cos(\theta°)}$$

$$\therefore SN = \frac{a^2\sin^2(\theta°) + b^2}{b\sin(\theta°)\cos(\theta°)} \qquad \ldots(C.20)$$

Referring fig. 2, $MN = PM + PN$

Substituting eqns. (C.14) & (C.19) in above,

$$\therefore MN = \left(\frac{b\sqrt{b^2 + a^2\sin^2(\theta°)}}{a\cos(\theta°)}\right) + \left(\frac{a\sqrt{b^2 + a^2\sin^2(\theta°)}}{b\cos(\theta°)}\right)$$

$$\therefore MN = \frac{(a^2 + b^2)\sqrt{b^2 + a^2\sin^2(\theta°)}}{ab\cos(\theta°)} \qquad \ldots(C.21)$$

Referring fig. 2, in right-triangle MOS,

$$SM^2 = OM^2 + OS^2$$

Substituting eqns. (C.6) & (C.2),

$$\therefore SM^2 = \left(\frac{a^2 + b^2}{a\cos(\theta°)}\right)^2 + \left(\frac{b\cos(\theta°)}{\sin(\theta°)}\right)^2$$

$$\therefore \mathrm{SM}^2 = \frac{(a^2+b^2)^2}{a^2\cos^2(\theta°)} + \frac{b^2\cos^2(\theta°)}{\sin^2(\theta°)}$$

$$\therefore \mathrm{SM}^2 = \frac{(a^2+b^2)^2\sin^2(\theta°) + a^2b^2\cos^4(\theta°)}{a^2\sin^2(\theta°)\cos^2(\theta°)}$$

$$\therefore \mathrm{SM} = \frac{\sqrt{(a^2+b^2)^2\sin^2(\theta°) + a^2b^2\cos^4(\theta°)}}{a\sin(\theta°)\cos(\theta°)} \quad \ldots(C.22)$$

Referring fig. 2, in right-triangle TON,

$$\mathrm{TN}^2 = \mathrm{OT}^2 + \mathrm{ON}^2$$

Substituting eqns. (C.1) & (C.7),

$$\therefore \mathrm{TN}^2 = \left(a\cos(\theta°)\right)^2 + \left(\frac{(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)}\right)^2$$

$$\therefore \mathrm{TN}^2 = a^2\cos^2(\theta°) + \frac{(a^2+b^2)^2\sin^2(\theta°)}{b^2\cos^2(\theta°)}$$

$$\therefore \mathrm{TN}^2 = \frac{a^2b^2\cos^4(\theta°) + (a^2+b^2)^2\sin^2(\theta°)}{b^2\cos^2(\theta°)}$$

$$\therefore \mathrm{TN} = \frac{\sqrt{(a^2+b^2)^2\sin^2(\theta°) + a^2b^2\cos^4(\theta°)}}{b\cos(\theta°)} \quad \ldots(C.23)$$

Analysis Part - D

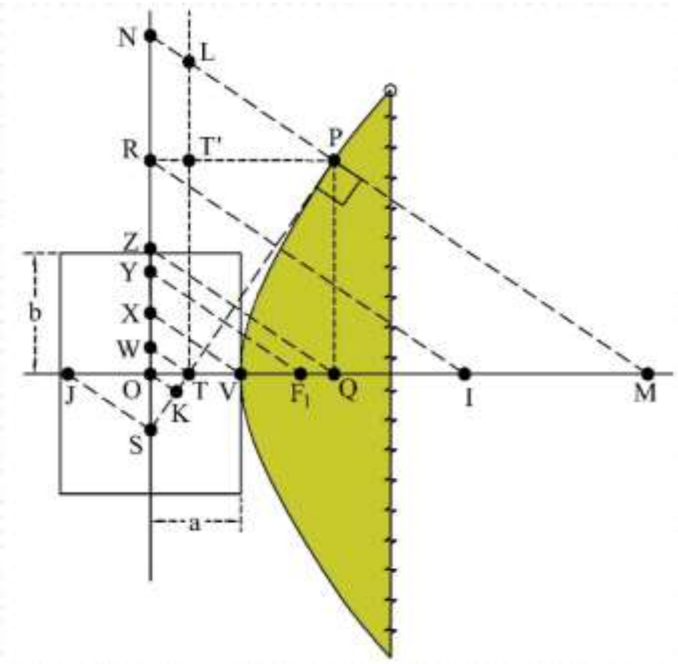

Fig. 3

In this figure,
Point P   is a point considered anywhere on hyperbola. In this figure,
Point Q   is foot of perpendicular of point P upon transverse axis.

Point R	*is foot of perpendicular of point P upon conjugate axis.*
Point L	*is intersection of normal & vertical line drawn at point T*
Point I	*is intersection of transverse axis & parallel line drawn to normal at point R*
Point J	*is intersection of transverse axis &parallel line drawn to normal at point S*
Point K	*is intersection of transverse axis &parallel line drawn to normal at point O*
Point W	*is intersection of conjugate axis & parallel line drawn to normal at point T*
Point X	*is intersection of conjugate axis & parallel line drawn to normal at point V*
Point Y	*is intersection of conjugate axis & parallel line drawn to normal at point F_1*
Point Z	*is intersection of conjugate axis & parallel line drawn to normal at point Q*

Referring fig. 3, Δ MON & Δ MTL are similar.

$$\therefore \frac{\mathrm{ON}}{\mathrm{OM}} = \frac{\mathrm{TL}}{\mathrm{TM}}$$

$$\therefore \mathrm{TL} = \frac{\mathrm{ON} \times \mathrm{TM}}{\mathrm{OM}}$$

Substituting eqns. (C.7), (C.9) & (C.6) in above eqn.,

$$\therefore \mathrm{TL} = \frac{\left(\frac{(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)}\right) \times \left(\frac{a^2\sin^2(\theta°)+b^2}{a\cos(\theta°)}\right)}{\left(\frac{a^2+b^2}{a\cos(\theta°)}\right)}$$

$$\therefore \mathrm{TL} = \frac{\cancel{(a^2+b^2)}\sin(\theta°)}{b\cos(\theta°)} \times \frac{a^2\sin^2(\theta°)+b^2}{\cancel{a\cos(\theta°)}} \times \frac{\cancel{a\cos(\theta°)}}{\cancel{a^2+b^2}}$$

$$\therefore \mathrm{TL} = \frac{\sin(\theta°)(a^2\sin^2(\theta°)+b^2)}{b\cos(\theta°)} \qquad \ldots (D.1)$$

Referring fig. 3, Δ MQP & Δ PT′L are similar.

$$\therefore \frac{\mathrm{PL}}{\mathrm{T'P}} = \frac{\mathrm{PM}}{\mathrm{QM}}$$

$$\therefore \mathrm{PL} = \frac{\mathrm{PM} \times \mathrm{T'P}}{\mathrm{QM}}$$

$$\therefore \mathrm{PL} = \frac{\mathrm{PM} \times \mathrm{TQ}}{\mathrm{QM}} \qquad (\because \mathrm{T'P} = \mathrm{TQ})$$

$$\therefore \mathrm{PL} = \frac{\mathrm{PM} \times \mathrm{TQ}}{\mathrm{QM}}$$

Substituting eqns. (C.14), (C.4) & (C.5) in above eqn.,

$$\therefore \mathrm{PL} = \frac{\left(\frac{b\sqrt{b^2+a^2\sin^2(\theta°)}}{a\cos(\theta°)}\right) \times \left(\frac{a\sin^2(\theta°)}{\cos(\theta°)}\right)}{\left(\frac{b^2}{a\cos(\theta°)}\right)}$$

$$\therefore \mathrm{PL} = \frac{\cancel{b}\sqrt{b^2+a^2\sin^2(\theta°)}}{\cancel{a\cos(\theta°)}} \times \frac{a\sin^2(\theta°)}{\cos(\theta°)} \times \frac{\cancel{a\cos(\theta°)}}{b^{\cancel{2}}}$$

$$\therefore \mathrm{PL} = \frac{\mathrm{a}\sin^2(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \qquad \ldots(\mathrm{D.2})$$

Referring fig. 3, Δ MON & Δ TOW are similar.

$$\therefore \frac{\mathrm{MN}}{\mathrm{OM}} = \frac{\mathrm{TW}}{\mathrm{OT}}$$

$$\therefore \mathrm{TW} = \frac{\mathrm{MN}\times\mathrm{OT}}{\mathrm{OM}}$$

Substituting eqns. (C.21), (C.1) & (C.6) in above eqn.,

$$\therefore \mathrm{TW} = \frac{\left(\frac{(\mathrm{a}^2+\mathrm{b}^2)\sqrt{\mathrm{b}^2+\mathrm{a}^2\sin^2(\theta°)}}{\mathrm{a}\mathrm{b}\cos(\theta°)}\right)\times \mathrm{a}\cos(\theta°)}{\left(\frac{\mathrm{a}^2+\mathrm{b}^2}{\mathrm{a}\cos(\theta°)}\right)}$$

$$\therefore \mathrm{TW} = \frac{\cancel{(\mathrm{a}^2+\mathrm{b}^2)}\sqrt{\mathrm{b}^2+\mathrm{a}^2\sin^2(\theta°)}}{\mathrm{ab}\cancel{\cos(\theta°)}}\times \cancel{\mathrm{a}\cos(\theta°)}\times\frac{\mathrm{a}\cos(\theta°)}{\cancel{\mathrm{a}^2+\mathrm{b}^2}}$$

$$\therefore \mathrm{TW} = \frac{\mathrm{a}\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{b}} \qquad \ldots(\mathrm{D.3})$$

Referring fig. 3, Δ MON & Δ VOX are similar.

$$\therefore \frac{\mathrm{MN}}{\mathrm{OM}} = \frac{\mathrm{VX}}{\mathrm{OV}}$$

$$\therefore \mathrm{VX} = \frac{\mathrm{MN}\times\mathrm{OV}}{\mathrm{OM}}$$

Substituting eqns. (C.21), (A.1) & (C.6) in above eqn.,

$$\therefore \mathrm{VX} = \frac{\left(\frac{(\mathrm{a}^2+\mathrm{b}^2)\sqrt{\mathrm{b}^2+\mathrm{a}^2\sin^2(\theta°)}}{\mathrm{a}\mathrm{b}\cos(\theta°)}\right)\times \mathrm{a}}{\left(\frac{\mathrm{a}^2+\mathrm{b}^2}{\mathrm{a}\cos(\theta°)}\right)}$$

$$\therefore \mathrm{VX} = \frac{\cancel{(\mathrm{a}^2+\mathrm{b}^2)}\sqrt{\mathrm{b}^2+\mathrm{a}^2\sin^2(\theta°)}}{\mathrm{ab}\cancel{\cos(\theta°)}}\times \mathrm{a}\times\frac{\cancel{\mathrm{a}\cos(\theta°)}}{\cancel{\mathrm{a}^2+\mathrm{b}^2}}$$

$$\therefore \mathrm{VX} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{b}} \qquad \ldots(\mathrm{D.4})$$

Referring fig. 3, Δ MON & Δ FOY are similar.

$$\therefore \frac{\mathrm{MN}}{\mathrm{OM}} = \frac{\mathrm{F_1Y}}{\mathrm{OF_1}}$$

$$\therefore \mathrm{F_1Y} = \frac{\mathrm{MN}\times\mathrm{OF_1}}{\mathrm{OM}}$$

Substituting eqns. (C.21), (A.3) & (C.6) in above eqn.,

$$\therefore \mathrm{F_1Y} = \frac{\left(\frac{(\mathrm{a}^2+\mathrm{b}^2)\sqrt{\mathrm{b}^2+\mathrm{a}^2\sin^2(\theta°)}}{\mathrm{a}\mathrm{b}\cos(\theta°)}\right)\times \sqrt{\mathrm{a}^2+\mathrm{b}^2}}{\left(\frac{\mathrm{a}^2+\mathrm{b}^2}{\mathrm{a}\cos(\theta°)}\right)}$$

$$\therefore F_1Y = \frac{\cancel{(a^2+b^2)}\sqrt{b^2+a^2\sin^2(\theta°)}}{ab\cancel{\cos(\theta°)}} \times \sqrt{a^2+b^2} \times \frac{\cancel{a\cos(\theta°)}}{\cancel{a^2+b^2}}$$

$$\therefore F_1Y = \frac{\sqrt{(a^2+b^2)(a^2\sin^2(\theta°)+b^2)}}{b} \qquad \ldots(D.5)$$

Referring fig. 3, Δ MON & Δ FOY are similar.

$$\therefore \frac{MN}{OM} = \frac{QZ}{OQ}$$

$$\therefore QZ = \frac{MN \times OQ}{OM}$$

Substituting eqns. (C.21), (B.1) & (C.6) in above eqn.,

$$\therefore QZ = \frac{\left(\frac{(a^2+b^2)\sqrt{b^2+a^2\sin^2(\theta°)}}{ab\cos(\theta°)}\right) \times \left(\frac{a}{\cos(\theta°)}\right)}{\left(\frac{a^2+b^2}{a\cos(\theta°)}\right)}$$

$$\therefore QZ = \frac{\cancel{(a^2+b^2)}\sqrt{b^2+a^2\sin^2(\theta°)}}{\cancel{ab\cos(\theta°)}} \times \frac{a}{\cos(\theta°)} \times \frac{\cancel{a\cos(\theta°)}}{\cancel{a^2+b^2}}$$

$$\therefore QZ = \frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)} \qquad \ldots(D.6)$$

Referring fig. 3, Δ MON & Δ IOR are similar.

$$\therefore \frac{OI}{OR} = \frac{OM}{ON}$$

$$\therefore OI = \frac{OM \times OR}{ON}$$

$$\therefore OI = \frac{OM \times PQ}{ON}$$

Substituting eqns. (C.2), (B.4) & (C.3),

$$\therefore OI = \frac{\left(\frac{a^2+b^2}{a\cos(\theta°)}\right) \times \left(\frac{b\sin(\theta°)}{\cos(\theta°)}\right)}{\left(\frac{(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)}\right)}$$

$$\therefore OI = \frac{\cancel{a^2+b^2}}{a\cos(\theta°)} \times \frac{b\cancel{\sin(\theta°)}}{\cancel{\cos(\theta°)}} \times \frac{b\cancel{\cos(\theta°)}}{\cancel{(a^2+b^2)\sin(\theta°)}}$$

$$\therefore OI = \frac{b^2}{a\cos(\theta°)} \qquad \ldots(D.7)$$

Referring fig. 3, Δ MON & Δ IOR are similar.

$$\therefore \frac{MN}{OM} = \frac{IR}{OI}$$

$$\therefore IR = \frac{MN \times OI}{OM}$$

Substituting eqns. (C.21), (D.7) & (C.6) in above eqn.,

$$\therefore \mathrm{IR} = \frac{\left(\frac{(a^2+b^2)\sqrt{b^2+a^2\sin^2(\theta°)}}{ab\cos(\theta°)}\right)\times\left(\frac{b^2}{a\cos(\theta°)}\right)}{\left(\frac{a^2+b^2}{a\cos(\theta°)}\right)}$$

$$\therefore \mathrm{IR} = \frac{\cancel{(a^2+b^2)}\sqrt{b^2+a^2\sin^2(\theta°)}}{a\cancel{b}\cos(\theta°)}\times\frac{b^{\cancel{2}}}{\cancel{a\cos(\theta°)}}\times\frac{\cancel{a\cos(\theta°)}}{\cancel{a^2+b^2}}$$

$$\therefore \mathrm{IR} = \frac{b\sqrt{a^2\sin^2(\theta°)+b^2}}{a\cos(\theta°)} \qquad \ldots(D.8)$$

Referring fig. 3, Δ MON & Δ TOW are similar.

$$\therefore \frac{\mathrm{ON}}{\mathrm{OM}} = \frac{\mathrm{OW}}{\mathrm{OT}}$$

$$\therefore \mathrm{OW} = \frac{\mathrm{ON}\times\mathrm{OT}}{\mathrm{OM}}$$

Substituting eqns. (C.7), (C.1) & (C.6) in above eqn.,

$$\therefore \mathrm{OW} = \frac{\left(\frac{(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)}\right)\times\left(a\cos(\theta°)\right)}{\left(\frac{a^2+b^2}{a\cos(\theta°)}\right)}$$

$$\therefore \mathrm{OW} = \frac{\cancel{(a^2+b^2)}\sin(\theta°)}{b\cancel{\cos(\theta°)}}\times a\cancel{\cos(\theta°)}\times\frac{a\cos(\theta°)}{\cancel{a^2+b^2}}$$

$$\therefore \mathrm{OW} = \frac{\sin(\theta°)}{b}\times a\times\frac{a\cos(\theta°)}{}$$

$$\therefore \mathrm{OW} = \frac{a^2\sin(\theta°)\cos(\theta°)}{b} \qquad \ldots(D.9)$$

Referring fig. 3, Δ MON & Δ V_1OX are similar.

$$\therefore \frac{\mathrm{OX}}{\mathrm{OV_1}} = \frac{\mathrm{ON}}{\mathrm{OM}}$$

$$\therefore \mathrm{OX} = \frac{\mathrm{ON}\times\mathrm{O}V}{\mathrm{OM}}$$

Substituting eqns. (C.7), (A.1) & (C.6) in above eqn.,

$$\therefore \mathrm{OX} = \frac{\left(\frac{(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)}\right)\times(a)}{\left(\frac{a^2+b^2}{a\cos(\theta°)}\right)}$$

$$\therefore \mathrm{OX} = \frac{\cancel{(a^2+b^2)}\sin(\theta°)}{b\cancel{\cos(\theta°)}}\times a\times\frac{a\cancel{\cos(\theta°)}}{\cancel{a^2+b^2}}$$

$$\therefore \mathrm{OX} = \frac{a^2\sin(\theta°)}{b} \qquad \ldots(D.10)$$

Referring fig. 3, Δ MON & Δ F_1OY are similar.

$$\therefore \frac{\mathrm{OY}}{\mathrm{OF_1}} = \frac{\mathrm{ON}}{\mathrm{OM}}$$

$$\therefore \mathrm{OY} = \frac{\mathrm{ON} \times \mathrm{OF_1}}{\mathrm{OM}}$$

Substituting eqns. (C.7), (A.3) & (C.6) in above eqn.,

$$\therefore \mathrm{OY} = \frac{\left(\frac{(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)}\right) \times \sqrt{a^2+b^2}}{\left(\frac{a^2+b^2}{a\cos(\theta°)}\right)}$$

$$\therefore \mathrm{OY} = \frac{\cancel{(a^2+b^2)}\sin(\theta°)}{b\cancel{\cos(\theta°)}} \times \sqrt{a^2+b^2} \times \frac{a\cancel{\cos(\theta°)}}{\cancel{a^2+b^2}}$$

$$\therefore \mathrm{OY} = \frac{a\sin(\theta°)\sqrt{a^2+b^2}}{b} \qquad \ldots(\mathrm{D.11})$$

Referring fig. 3, Δ MON & Δ QOZ are similar.

$$\therefore \frac{\mathrm{OZ}}{\mathrm{OQ}} = \frac{\mathrm{ON}}{\mathrm{OM}}$$

$$\therefore \mathrm{OZ} = \frac{\mathrm{ON} \times \mathrm{OQ}}{\mathrm{OM}}$$

Substituting eqns. (C.7), (B.1) & (C.6) in above eqn.,

$$\therefore \mathrm{OZ} = \frac{\left(\frac{(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)}\right) \times \left(\frac{a}{\cos(\theta°)}\right)}{\left(\frac{a^2+b^2}{a\cos(\theta°)}\right)}$$

$$\therefore \mathrm{OZ} = \frac{\cancel{(a^2+b^2)}\sin(\theta°)}{b\cancel{\cos(\theta°)}} \times \frac{a}{\cos(\theta°)} \times \frac{a\cancel{\cos(\theta°)}}{\cancel{a^2+b^2}}$$

$$\therefore \mathrm{OZ} = \frac{a^2\sin(\theta°)}{b\cos(\theta°)} \qquad \ldots(\mathrm{D.12})$$

Referring fig. 3, Δ JOS & Δ MON are similar.

$$\therefore \frac{\mathrm{OJ}}{\mathrm{OM}} = \frac{\mathrm{OS}}{\mathrm{ON}}$$

$$\therefore \mathrm{OJ} = \frac{\mathrm{OS} \times \mathrm{OM}}{\mathrm{ON}}$$

Substituting eqns. (C.2), (C.6) & (C.7) in above,

$$\therefore \mathrm{OJ} = \frac{\left(\frac{b\cancel{\cos(\theta°)}}{\sin(\theta°)}\right) \times \left(\frac{a^2+b^2}{a\cancel{\cos(\theta°)}}\right)}{\left(\frac{(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)}\right)}$$

$$\therefore \mathrm{OJ} = \frac{b\cancel{(a^2+b^2)}}{a\sin(\theta°)} \times \frac{b\cos(\theta°)}{\cancel{(a^2+b^2)}\sin(\theta°)}$$

$$\therefore \mathrm{OJ} = \frac{b^2\cos(\theta°)}{a\sin^2(\theta°)} \qquad \ldots(\mathrm{D.13})$$

Referring fig. 3, Δ JOS & Δ MON are similar.

$$\therefore \frac{SJ}{OS} = \frac{MN}{ON}$$

$$\therefore SJ = \frac{OS \times MN}{ON}$$

Substituting eqns. (C.2), (C.21) & (C.7) in above,

$$\therefore SJ = \frac{\left(\frac{\cancel{b\cos(\theta°)}}{\sin(\theta°)}\right) \times \left(\frac{(a^2+b^2)\sqrt{a^2\sin^2(\theta°)+b^2}}{a\cancel{b\cos(\theta°)}}\right)}{\left(\frac{(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)}\right)}$$

$$\therefore SJ = \frac{\cancel{(a^2+b^2)}\sqrt{a^2\sin^2(\theta°)+b^2}}{a\sin(\theta°)} \times \frac{b\cos(\theta°)}{\cancel{(a^2+b^2)}\sin(\theta°)}$$

$$\therefore SJ = \frac{b\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{a\sin^2(\theta°)} \qquad \ldots (D.14)$$

Referring fig. 3, Δ OKT & Δ MON are similar.

$$\therefore \frac{OK}{OT} = \frac{OM}{MN}$$

$$\therefore OK = \frac{OT \times OM}{MN}$$

Substituting eqns. (C.1), (C.6) & (C.21) in above,

$$\therefore OK = \frac{a\cos(\theta°) \times \left(\frac{a^2+b^2}{a\cos(\theta°)}\right)}{\left(\frac{(a^2+b^2)\sqrt{b^2+a^2\sin^2(\theta°)}}{ab\cos(\theta°)}\right)}$$

$$\therefore OK = \cancel{a\cos(\theta°)} \times \left(\frac{\cancel{a^2+b^2}}{\cancel{a\cos(\theta°)}}\right) \times \frac{ab\cos(\theta°)}{\cancel{(a^2+b^2)}\sqrt{b^2+a^2\sin^2(\theta°)}}$$

$$\therefore OK = \frac{ab\cos(\theta°)}{\sqrt{a^2\sin^2(\theta°)+b^2}} \qquad \ldots (D.15)$$

Referring fig. 3, Δ OKT & Δ JST are similar.

$$\therefore \frac{TK}{OK} = \frac{TS}{SJ}$$

$$\therefore TK = \frac{TS \times OK}{SJ}$$

Substituting eqns. (C.10), (D.15) & (D.14) in above,

$$\therefore TK = \frac{\left(\frac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)}\right) \times \left(\frac{ab\cos(\theta°)}{\sqrt{a^2\sin^2(\theta°)+b^2}}\right)}{\left(\frac{b\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{a\sin^2(\theta°)}\right)}$$

$$\therefore TK = \frac{\cos(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}{\cancel{\sin(\theta°)}} \times \frac{a\cancel{b\cos(\theta°)}}{\sqrt{a^2\sin^2(\theta°)+b^2}} \times \frac{a\sin^{\cancel{2}}(\theta°)}{\cancel{b\cos(\theta°)}\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}$$

$$\therefore \mathrm{TK} = \frac{a^2 \sin(\theta°)\cos(\theta°)}{\sqrt{a^2\sin^2(\theta°) + b^2}} \quad \ldots (\mathrm{D.16})$$

Analysis Part - E

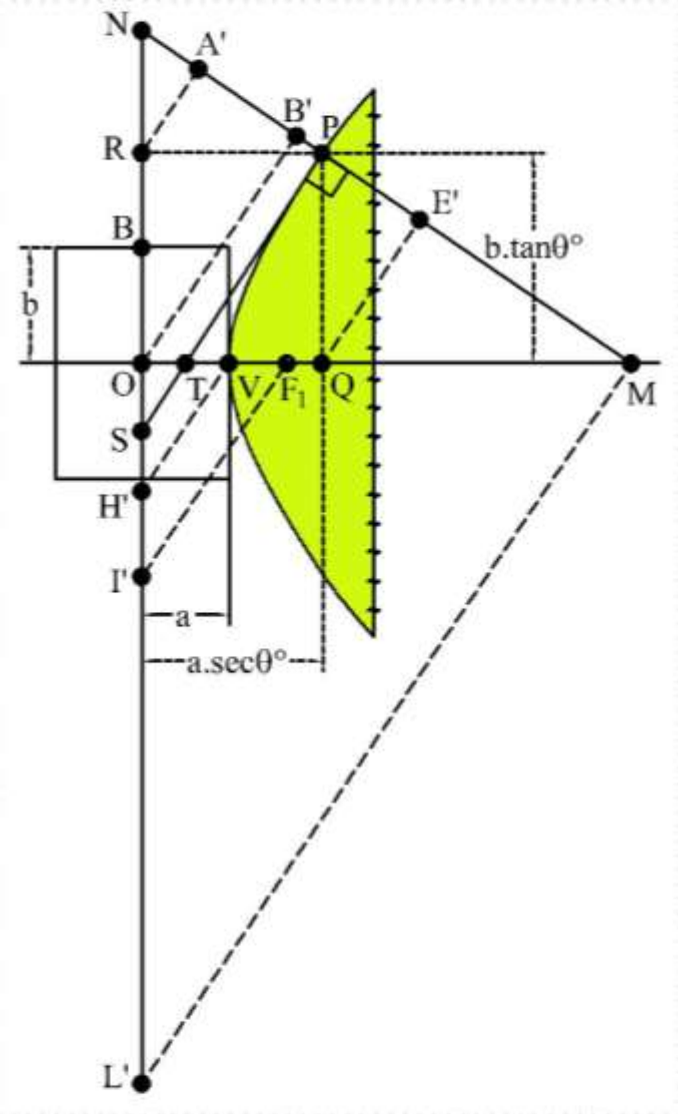

Fig. 4

In this figure,

Point V	*is vertex of the hyperbola.*
Points F_1	*is one of the foci of the hyperbola.*
Point P	*is a point considered anywhere on hyperbola.*
PF_1 & PF_1	*are focal distances of a point P*
Point Q	*is foot of perpendicular of point P upon transverse axis.*
Point R	*is foot of perpendicular of point P upon conjugate axis.*
PS	*is tangent drawn to the hyperbola at point P.*
MN	*is normal drawn to the hyperbola at point P.*
Point A'	*is intersection of normal & parallel line drawn to tangent at point R*
Point B'	*is intersection of normal & parallel line drawn to tangent at point O*

Point E'	*is intersection of normal & parallel line drawn to tangent at point Q*
Point H'	*is intersection of conjugate axis & parallel line drawn to tangent at point V*
Point I'	*is intersection of conjugate axis & parallel line drawn to tangent at point F_1*
Point L'	*is intersection of conjugate axis &parallel line drawn to tangent at point M*

Referring fig. 4, Δ OB′M & Δ TPM are similar.

$$\therefore \frac{\mathrm{OB'}}{\mathrm{OM}} = \frac{\mathrm{PT}}{\mathrm{TM}}$$

$$\therefore \mathrm{OB'} = \frac{\mathrm{PT} \times \mathrm{OM}}{\mathrm{TM}}$$

Substituting eqns. (C.15), (C.6) & (C.9) in above eqn.,

$$\therefore \mathrm{OB'} = \frac{\left(\frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)}\right) \times \left(\frac{(a^2 + b^2)\sin(\theta°)}{\cancel{b}\cos(\theta°)}\right)}{\left(\frac{a^2\sin^2(\theta°) + b^2}{a\cos(\theta°)}\right)}$$

$$\therefore \mathrm{OB'} = \frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)} \times \frac{(a^2 + b^2)\sin(\theta°)}{b\cancel{\cos(\theta°)}} \times \frac{a\cancel{\cos(\theta°)}}{a^2\sin^2(\theta°) + b^2}$$

$$\therefore \mathrm{OB'} = \frac{a(a^2 + b^2)\sin^2(\theta°)}{b\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}} \qquad \ldots(\mathrm{E.1})$$

Referring fig. 4, Δ OB′M & Δ QE′M are similar.

$$\frac{\mathrm{QE'}}{\mathrm{QM}} = \frac{\mathrm{OB'}}{\mathrm{OM}}$$

$$\therefore \mathrm{QE'} = \frac{\mathrm{OB'} \times \mathrm{QM}}{\mathrm{OM}}$$

Substituting eqns. (E.1), (C.5) & (C.6) in above eqn.,

$$\therefore \mathrm{QE'} = \frac{\left(\frac{a(a^2 + b^2)\sin^2(\theta°)}{b\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}\right) \times \left(\frac{b^{\cancel{2}}}{\cancel{a}\cos(\theta°)}\right)}{\left(\frac{(a^2 + b^2)}{a\cos(\theta°)}\right)}$$

$$\therefore \mathrm{QE'} = \frac{a\cancel{(a^2 + b^2)}\sin^2(\theta°)}{\cancel{b}\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}} \times \frac{b^{\cancel{2}}}{\cancel{a\cos(\theta°)}} \times \frac{\cancel{a\cos(\theta°)}}{\cancel{(a^2 + b^2)}}$$

$$\therefore \mathrm{QE'} = \frac{ab\sin^2(\theta°)}{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}} \qquad \ldots(\mathrm{E.2})$$

Referring fig. 4, Δ SPN & Δ RA′N are similar.

$$\therefore \frac{\mathrm{PS}}{\mathrm{SN}} = \frac{\mathrm{RA'}}{\mathrm{RN}}$$

$$\therefore \mathrm{RA'} = \frac{\mathrm{PS} \times \mathrm{RN}}{\mathrm{SN}}$$

Substituting eqns. (C.16), (C.17) & (C.20) in above eqn.,

$$\therefore \mathrm{RA}' = \frac{\left(\dfrac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)\cos(\theta°)}\right)}{\left(\dfrac{a^2\sin^2(\theta°) + b^2}{b\sin(\theta°)\cos(\theta°)}\right)} \times \left(\frac{a^2\sin(\theta°)}{\cancel{b}\cos(\theta°)}\right)$$

$$\therefore \mathrm{RA}' = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\cancel{\sin(\theta°)\cos(\theta°)}} \times \frac{\cancel{b\sin(\theta°)\cos(\theta°)}}{a^2\sin^2(\theta°) + b^2} \times \frac{a^2\sin(\theta°)}{\cancel{b}\cos(\theta°)}$$

$$\therefore \mathrm{RA}' = \frac{a^2\sin(\theta°)}{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}} \qquad \ldots(\mathrm{E.3})$$

Referring fig. 4, Δ TOS & Δ VOH$'$ are similar.

$$\therefore \frac{\mathrm{TS}}{\mathrm{OT}} = \frac{\mathrm{VH}'}{\mathrm{OV}}$$

$$\therefore \mathrm{VH}' = \frac{\mathrm{TS} \times \mathrm{OV}}{\mathrm{OT}}$$

Substituting eqns. (C.10). (A.1) & (C.1) in above eqn.,

$$\therefore \mathrm{VH}' = \frac{\left(\dfrac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)}\right) \times a}{a\cos(\theta°)}$$

$$\therefore \mathrm{VH}' = \frac{\cancel{\cos(\theta°)}\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)} \times \cancel{a} \times \frac{1}{\cancel{a\cos(\theta°)}}$$

$$\therefore \mathrm{VH}' = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)} \qquad \ldots(\mathrm{E.4})$$

Referring fig. 4, Δ TOS & Δ F_1OI' are similar.

$$\therefore \frac{\mathrm{TS}}{\mathrm{OT}} = \frac{F_1I'}{OF_1}$$

$$\therefore F_1I' = \frac{\mathrm{TS} \times OF_1}{\mathrm{OT}}$$

Substituting eqns. (C.10), (A.3) & (C.1) in above eqn.,

$$\therefore F_1I' = \frac{\left(\dfrac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)}\right) \times \sqrt{a^2 + b^2}}{a\cos(\theta°)}$$

$$\therefore F_1I' = \frac{\cancel{\cos(\theta°)}\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)} \times \sqrt{a^2 + b^2} \times \frac{1}{\cancel{a\cos(\theta°)}}$$

$$\therefore F_1I' = \frac{\sqrt{(a^2 + b^2)(a^2\sin^2(\theta°) + b^2)}}{\cancel{a}\sin(\theta°)} \qquad \ldots(\mathrm{E.5})$$

Referring fig. 4, Δ TOS & Δ QOJ$'$ are similar.

$$\therefore \frac{\mathrm{TS}}{\mathrm{OT}} = \frac{\mathrm{QJ}'}{\mathrm{OQ}}$$

$$\therefore \mathrm{QJ}' = \frac{\mathrm{TS} \times \mathrm{OQ}}{\mathrm{OT}}$$

Substituting eqns. (C.10), (B.1) & (C.1) in above eqn.,

$$\therefore \mathrm{QJ}' = \frac{\left(\frac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)}\right)\times\left(\frac{\cancel{a}}{\cos(\theta°)}\right)}{a\cos(\theta°)}$$

$$\therefore \mathrm{QJ}' = \frac{\cancel{\cos(\theta°)}\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)}\times\frac{\cancel{a}}{\cancel{\cos(\theta°)}}\times\frac{1}{\cancel{a}\cos(\theta°)}$$

$$\therefore \mathrm{QJ}' = \frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)\cos(\theta°)} \quad \ldots(\mathrm{E.6})$$

Referring fig. 4, Δ TOS & Δ MOL′ are similar.

$$\therefore \frac{\mathrm{TS}}{\mathrm{OT}} = \frac{\mathrm{ML}'}{\mathrm{OM}}$$

$$\therefore \mathrm{ML}' = \frac{\mathrm{TS}\times\mathrm{OM}}{\mathrm{OT}}$$

Substituting eqns. (C.10), (C.6) & (C.1) in above eqn.,

$$\therefore \mathrm{ML}' = \frac{\left(\frac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)}\right)\times\left(\frac{a^2+b^2}{a\cos(\theta°)}\right)}{a\cos(\theta°)}$$

$$\therefore \mathrm{ML}' = \frac{\cancel{\cos(\theta°)}\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)}\times\frac{a^2+b^2}{a\cos(\theta°)}\times\frac{1}{a\cancel{\cos(\theta°)}}$$

$$\therefore \mathrm{ML}' = \frac{(a^2+b^2)\sqrt{a^2\sin^2(\theta°)+b^2}}{a^2\sin(\theta°)\cos(\theta°)} \quad \ldots(\mathrm{E.7})$$

Referring fig. 4, Δ TOS & Δ MOL′ are similar.

$$\therefore \frac{\mathrm{OT}}{\mathrm{OS}} = \frac{\mathrm{OM}}{\mathrm{OL}'}$$

$$\therefore \mathrm{OL}' = \frac{\mathrm{OM}\times\mathrm{OS}}{\mathrm{OT}}$$

Substituting eqns. (C.6), (C.2) & (C.1) in above eqn.,

$$\mathrm{OL}' = \frac{\left(\frac{a^2+b^2}{a\cos(\theta°)}\right)\times\left(\frac{b\cos(\theta°)}{\sin(\theta°)}\right)}{a\cos(\theta°)}$$

$$\therefore \mathrm{OL}' = \frac{a^2+b^2}{a\cancel{\cos(\theta°)}}\times\frac{b\cancel{\cos(\theta°)}}{\sin(\theta°)}\times\frac{1}{a\cos(\theta°)}$$

$$\therefore \mathrm{OL}' = \frac{b(a^2+b^2)}{a^2\sin(\theta°)\cos(\theta°)} \quad \ldots(\mathrm{E.8})$$

Referring fig. 1,

Substituting eqn. (A.3) & (A.1) in above,

$$\text{Eqn.}\,(\mathrm{B.12}) \Longrightarrow \mathrm{PF}_1 = \frac{\sqrt{a^2+b^2}-a\cos(\theta°)}{\cos(\theta°)}$$

$$\text{Eqn. (B.13)} \Longrightarrow \mathrm{PF_2} = \frac{\sqrt{a^2+b^2}+a\cos(\theta°)}{\cos(\theta°)}$$

Multiplying eqns. (B.12) & (B.13),

$$\mathrm{PF_1} \times \mathrm{PF_2} = \left(\frac{\sqrt{a^2+b^2}-a\cos(\theta°)}{\cos(\theta°)}\right) \times \left(\frac{\sqrt{a^2+b^2}+a\cos(\theta°)}{\cos(\theta°)}\right)$$

$$\therefore \mathrm{PF_1} \times \mathrm{PF_2} = \frac{a^2+b^2-a^2\cos^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{PF_1} \times \mathrm{PF_2} = \frac{a^2[1-\cos^2(\theta°)]+b^2}{\cos^2(\theta°)}$$

$$\therefore \mathrm{PF_1} \times \mathrm{PF_2} = \frac{a^2\sin^2(\theta°)+b^2}{\cos^2(\theta°)} \qquad \ldots(\mathrm{E}.9)$$

$$\text{Eqn. (C.13)} \Longrightarrow \mathrm{F_1T} = \sqrt{a^2+b^2}-a\cos(\theta°)$$

$$\text{Eqn. (C.14)} \Longrightarrow \mathrm{F_2T} = \sqrt{a^2+b^2}+a\cos(\theta°)$$

Multiplying eqns. (C.13) & (C.14),

$$\mathrm{F_1T} \times \mathrm{F_2T} = \left[\sqrt{a^2+b^2}-a\cos(\theta°)\right] \times \left[\sqrt{a^2+b^2}+a\cos(\theta°)\right]$$

$$\therefore \mathrm{F_1T} \times \mathrm{F_2T} = a^2+b^2-a^2\cos^2(\theta°)$$

$$\therefore \mathrm{F_1T} \times \mathrm{F_2T} = a^2\left(1-\cos^2(\theta°)\right)+b^2$$

$$\therefore \mathrm{F_1T} \times \mathrm{F_2T} = a^2\sin^2(\theta°)+b^2 \qquad \ldots(\mathrm{E}.10)$$

$$\text{Eqn. (B.8)} \Longrightarrow \mathrm{F_1M} = \frac{\sqrt{a^2+b^2}\left(\sqrt{a^2+b^2}-a\cos(\theta°)\right)}{a\cos(\theta°)}$$

$$\text{Eqn. (B.9)} \Longrightarrow \mathrm{F_2M} = \frac{\sqrt{a^2+b^2}\left(\sqrt{a^2+b^2}+a\cos(\theta°)\right)}{a\cos(\theta°)}$$

Multiplying eqns. (B.8) & (B.9),

$$\mathrm{F_1M} \times \mathrm{F_2M} = \left(\frac{\sqrt{a^2+b^2}\left(\sqrt{a^2+b^2}-a\cos^2(\theta°)\right)}{a\cos(\theta°)}\right) \times \left(\frac{\sqrt{a^2+b^2}\left(\sqrt{a^2+b^2}+a\cos(\theta°)\right)}{a\cos(\theta°)}\right)$$

$$\therefore \mathrm{F_1M} \times \mathrm{F_2M} = \frac{(a^2+b^2)\left(a^2+b^2-a^2\cos^2(\theta°)\right)}{a^2\cos^2(\theta°)}$$

$$\therefore \mathrm{F_1M} \times \mathrm{F_2M} = \frac{(a^2+b^2)(a^2-a^2\cos^2(\theta°)+b^2)}{a^2\cos^2(\theta°)}$$

$$\therefore \mathrm{F_1M} \times \mathrm{F_2M} = \frac{(a^2+b^2)(a^2[1-\cos^2(\theta°)]+b^2)}{a^2\cos^2(\theta°)}$$

$$\therefore \mathrm{F_1M} \times \mathrm{F_2M} = \frac{(a^2+b^2)(a^2\sin^2(\theta°)+b^2)}{a^2\cos^2(\theta°)} \qquad \ldots(\mathrm{E}.11)$$

$$\text{Eqn. (B.6)} \Longrightarrow \mathrm{F_1Q} = \frac{a-\sqrt{a^2+b^2}\cos(\theta°)}{\cos(\theta°)}$$

$$\text{Eqn. (B.7)} \Longrightarrow F_2Q = \frac{a + \sqrt{a^2+b^2}\cos(\theta°)}{\cos(\theta°)}$$

Multiplying eqns. (B.6) & (B.7),

$$F_1Q \times F_2Q = \left(\frac{a - \sqrt{a^2+b^2}\cos(\theta°)}{\cos(\theta°)}\right) \times \left(\frac{a + \sqrt{a^2+b^2}\cos(\theta°)}{\cos(\theta°)}\right)$$

$$\therefore F_1Q \times F_2Q = \frac{a^2 - (a^2+b^2)\cos^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore F_1Q \times F_2Q = \frac{a^2 - \left(a^2\cos^2(\theta°) + b^2\cos^2(\theta°)\right)}{\cos^2(\theta°)}$$

$$\therefore F_1Q \times F_2Q = \frac{a^2[1-\cos^2(\theta°)] - b^2\cos^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore F_1Q \times F_2Q = \frac{a^2\sin^2(\theta°) - b^2\cos^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots (E.12)$$

Abstract of the analysis Part – A, B, C, D & E

The following table is abstract of derived equations in Analysis A, B, C, D & E for various elements of a Hyperbola

A.1	$OV = a$	C.12	$F_2T = \sqrt{a^2+b^2} + a\cos(\theta°)$
A.2	$OB = b$	C.13	$F_1S = \frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)}$
A.3	$OF_1 = OF_2 = \sqrt{a^2+b^2}$	C.14	$PM = \frac{b\sqrt{a^2\sin^2(\theta°)+b^2}}{a\cos(\theta°)}$
A.4	$F_1F_2 = 2\sqrt{a^2+b^2}$	C.15	$PT = \frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{\cos(\theta°)}$
A.5	$VF_1 = \sqrt{a^2+b^2} - a$	C.16	$PS = \frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)\cos(\theta°)}$
A.6	$VF_2 = \sqrt{a^2+b^2} + a$	C.17	$RN = \frac{a^2\sin(\theta°)}{b\cos(\theta°)}$
A.7	$OU = \frac{a^2}{\sqrt{a^2+b^2}}$	C.18	$RS = \frac{b}{\sin(\theta°)\cos(\theta°)}$
A.8	$VU = \frac{a\left(\sqrt{a^2+b^2} - a\right)}{\sqrt{a^2+b^2}}$	C.19	$PN = \frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}$
A.9	$UF_1 = \frac{b^2}{\sqrt{a^2+b^2}}$	C.20	$SN = \frac{a^2\sin^2(\theta°)+b^2}{b\sin(\theta°)\cos(\theta°)}$
A.10	$UF_2 = \frac{2a^2+b^2}{\sqrt{a^2+b^2}}$	C.21	$MN = \frac{(a^2+b^2)\sqrt{b^2+a^2\sin^2(\theta°)}}{ab\cos(\theta°)}$
A.11	$BU = \frac{\sqrt{(a^2+b^2)^2 - a^2b^2}}{\sqrt{a^2+b^2}}$	C.22	$SM = \frac{\sqrt{(a^2+b^2)^2\sin^2(\theta°) + a^2b^2\cos^4(\theta°)}}{a\sin(\theta°)\cos(\theta°)}$

A.12	$\mathrm{BV} = \sqrt{a^2+b^2}$	C.23	$\mathrm{TN} = \dfrac{\sqrt{(a^2+b^2)^2\sin^2(\theta°)+a^2b^2\cos^4(\theta°)}}{b\cos(\theta°)}$
B.1	$\mathrm{OQ} = \dfrac{a}{\cos(\theta°)}$	D.1	$\mathrm{TL} = \dfrac{\sin(\theta°)[a^2\sin^2(\theta°)+b^2]}{b\cos(\theta°)}$
B.2	$\mathrm{PQ} = \mathrm{OR} = \dfrac{b\sin(\theta°)}{\cos(\theta°)}$	D.2	$\mathrm{PL} = \dfrac{a\sin^2(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}$
B.3	$\mathrm{UQ} = \dfrac{a\left[\sqrt{a^2+b^2}-a\cos(\theta°)\right]}{\sqrt{a^2+b^2}\times\cos(\theta°)}$	D.3	$\mathrm{TW} = \dfrac{a\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{b}$
B.4	$\mathrm{VQ} = \dfrac{a[1-\cos(\theta°)]}{\cos(\theta°)}$	D.4	$\mathrm{VX} = \dfrac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b}$
B.5	$\mathrm{OP} = \dfrac{\sqrt{a^2+b^2\sin^2(\theta°)}}{\cos(\theta°)}$	D.5	$\mathrm{F_1Y} = \dfrac{\sqrt{a^2+b^2}\times\sqrt{a^2\sin^2(\theta°)+b^2}}{b}$
B.6	$\mathrm{F_1Q} = \dfrac{a-\sqrt{a^2+b^2}\cos(\theta°)}{\cos(\theta°)}$	D.6	$\mathrm{QZ} = \dfrac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}$
B.7	$\mathrm{F_2Q} = \dfrac{a+\sqrt{a^2+b^2}\cos(\theta°)}{\cos(\theta°)}$	D.7	$\mathrm{OI} = \dfrac{b^2}{a\cos(\theta°)}$
B.8	$\mathrm{F_1M} = \dfrac{\sqrt{a^2+b^2}\left(\sqrt{a^2+b^2}-a\cos(\theta°)\right)}{a\cos(\theta°)}$	D.8	$\mathrm{IR} = \dfrac{b\sqrt{a^2\sin^2(\theta°)+b^2}}{a\cos(\theta°)}$
B.9	$\mathrm{F_2M} = \dfrac{\sqrt{a^2+b^2}\left(\sqrt{a^2+b^2}+a\cos(\theta°)\right)}{a\cos(\theta°)}$	D.9	$\mathrm{OW} = \dfrac{a^2\sin(\theta°)\cos(\theta°)}{b}$
B.10	$\mathrm{F_1N} = \dfrac{\sqrt{a^2+b^2}\times\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}$	D.10	$\mathrm{OX} = \dfrac{a^2\sin(\theta°)}{b}$
B.11	$\mathrm{F_2N} = \dfrac{\sqrt{a^2+b^2}\times\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}$	D.11	$\mathrm{OY} = \dfrac{a\sin(\theta°)\sqrt{a^2+b^2}}{b}$
B.12	$\mathrm{PF_1} = \dfrac{\sqrt{a^2+b^2}-a\cos(\theta°)}{\cos(\theta°)}$	D.12	$\mathrm{OZ} = \dfrac{a^2\sin(\theta°)}{b\cos(\theta°)}$
B.13	$\mathrm{PF_2} = \dfrac{\sqrt{a^2+b^2}+a\cos(\theta°)}{\cos(\theta°)}$	D.13	$\mathrm{OJ} = \dfrac{b^2\cos(\theta°)}{a\sin^2(\theta°)}$
B.14	$\mathrm{VP} = \dfrac{\sqrt{a^2[1-\cos(\theta°)]^2+b^2\sin^2(\theta°)}}{\cos(\theta°)}$	D.14	$\mathrm{SJ} = \dfrac{b\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{a\sin^2(\theta°)}$
B.15	$\mathrm{VS} = \dfrac{\sqrt{a^2\sin^2(\theta°)+b^2\cos^2(\theta°)}}{\sin(\theta°)}$	D.15	$\mathrm{OK} = \dfrac{ab\cos(\theta°)}{\sqrt{a^2\sin^2(\theta°)+b^2}}$
B.16	$\mathrm{VR} = \dfrac{\sqrt{a^2\cos^2(\theta°)+b^2\sin^2(\theta°)}}{\cos(\theta°)}$	D.16	$\mathrm{TK} = \dfrac{a^2\sin(\theta°)\cos(\theta°)}{\sqrt{a^2\sin^2(\theta°)+b^2}}$
B.17	$\mathrm{F_1R} = \dfrac{\sqrt{a^2\cos^2(\theta°)+b^2}}{\cos(\theta°)}$	E.1	$\mathrm{OB'} = \dfrac{a(a^2+b^2)\sin^2(\theta°)}{b\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}$
C.1	$\mathrm{OT} = a\cos(\theta°)$	E.2	$\mathrm{QE'} = \dfrac{ab\sin^2(\theta°)}{\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}$

C.2	$OS = \frac{b\cos(\theta°)}{\sin(\theta°)}$	E.3	$RA' = \frac{a^2\sin(\theta°)}{\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}$
C.3	$BT = \sqrt{a^2\cos^2(\theta°)+b^2}$	E.4	$VH' = \frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)}$
C.4	$TQ = \frac{a\sin^2(\theta°)}{\cos(\theta°)}$	E.5	$F_1I' = \frac{\sqrt{a^2+b^2}\times\sqrt{a^2\sin^2(\theta°)+b^2}}{a\sin(\theta°)}$
C.5	$QM = \frac{b^2}{a\cos(\theta°)}$	E.6	$QJ' = \frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)\cos(\theta°)}$
C.6	$OM = \frac{a^2+b^2}{a\cos(\theta°)}$	E.7	$ML' = \frac{(a^2+b^2)\sqrt{a^2\sin^2(\theta°)+b^2}}{a^2\sin(\theta°)\cos(\theta°)}$
C.7	$ON = \frac{(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)}$	E.8	$OL' = \frac{b(a^2+b^2)}{a^2\sin(\theta°)\cos(\theta°)}$
C.8	$VT = a[1-\cos(\theta°)]$	E.9	$PF_1\times PF_2 = \frac{a^2\sin^2(\theta°)+b^2}{\cos^2(\theta°)}$
C.9	$TM = \frac{a^2\sin^2(\theta°)+b^2}{a\cos(\theta°)}$	E.10	$F_1T\times F_2T = a^2\sin^2(\theta°)+b^2$
C.10	$TS = \frac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)}$	E.11	$F_1M\times F_2M = \frac{(a^2+b^2)(a^2\sin^2(\theta°)+b^2)}{a^2\cos^2(\theta°)}$
C.11	$F_1T = \sqrt{a^2+b^2}-a\cos(\theta°)$	E.12	$F_1Q\times F_2Q = \frac{a^2\sin^2(\theta°)-b^2\cos^2(\theta°)}{\cos^2(\theta°)}$

DERIVATIONS OF THE THEOREMS

Previously, equations have been derived and tabulated above. By using the equations from the table, now theorems have been developed as below. Totally 104 theorems have been developed and proven by using these above equations.

THEOREM- 1:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1}$,

$\boldsymbol{PN = QZ}$

Proof of the theorem

$$\text{Eqn. (C.19)} \Longrightarrow PN = \frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)} \qquad \ldots(1.1)$$

$$\text{Eqn. (D.6)} \Longrightarrow QZ = \frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)} \qquad \ldots(1.2)$$

Equating Eqns. (1.1) & (1.2),

$$PN = QZ \qquad \ldots(1.3)$$

Eqn. (1.3) is mathematical expression of the theorem.

THEOREM- 2:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{PM = IR}$$

Proof of the theorem

$$\text{Eqn. (C.14)} \Longrightarrow \mathrm{PM} = \frac{\mathrm{b}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\cos(\theta°)} \qquad \ldots(2.1)$$

$$\text{Eqn. (D.8)} \Longrightarrow \mathrm{IR} = \frac{\mathrm{b}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\cos(\theta°)} \qquad \ldots(2.2)$$

Equating Eqns. (2.1) & (2.2),

$$\mathrm{PM} = \mathrm{IR} \qquad \ldots(2.3)$$

Eqn. (2.3) is mathematical expression of the theorem.

THEOREM- 3:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{OZ = RN}$$

Proof of the theorem

$$\text{Eqn. (D.12)} \Longrightarrow \mathrm{OZ} = \frac{\mathrm{a}^2\sin(\theta°)}{\mathrm{b}\cos(\theta°)} \qquad \ldots(3.1)$$

$$\text{Eqn. (C.17)} \Longrightarrow \mathrm{RN} = \frac{\mathrm{a}^2\sin(\theta°)}{\mathrm{b}\cos(\theta°)} \qquad \ldots(3.2)$$

Equating Eqns. (3.1) & (3.2),

$$\mathrm{OZ} = \mathrm{RN} \qquad \ldots(3.3)$$

Eqn. (3.3) is mathematical expression of the theorem.

THEOREM- 4:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{OI = QM}$$

Proof of the theorem

$$\text{Eqn. (D.7)} \Longrightarrow \mathrm{OI} = \frac{\mathrm{b}^2}{\mathrm{a}\cos(\theta°)} \qquad \ldots(4.1)$$

$$\text{Eqn. (C.5)} \Longrightarrow \mathrm{QM} = \frac{\mathrm{b}^2}{\mathrm{a}\cos(\theta°)} \qquad \ldots(4.2)$$

Equating Eqns. (4.1) & (4.2),

$$\mathrm{OI} = \mathrm{QM} \qquad \ldots(4.3)$$

Eqn. (4.3) is mathematical expression of the theorem.

THEOREM- 5:

$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$

$\boldsymbol{VH' = F_1S}$

Proof of the theorem

$$\text{Eqn. (E. 4)} \Longrightarrow \mathrm{VH'} = \frac{\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{sin(\theta°)}} \quad \ldots(5.1)$$

$$\text{Eqn. (C. 13)} \Longrightarrow \mathrm{F_1S} = \frac{\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{sin(\theta°)}} \quad \ldots(5.2)$$

Equating Eqns. (5.1) & (5.2),

$$\mathrm{VH' = F_1S} \quad \ldots(5.3)$$

Eqn. (5.3) is mathematical expression of the theorem.

THEOREM- 6:

$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$

$\boldsymbol{PS = QJ'}$

Proof of the theorem

$$\text{Eqn. (C. 16)} \Longrightarrow \mathrm{PS} = \frac{\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{sin(\theta°)cos(\theta°)}} \quad \ldots(6.1)$$

$$\text{Eqn. (E. 6)} \Longrightarrow \mathrm{QJ'} = \frac{\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{sin(\theta°)cos(\theta°)}} \quad \ldots(6.2)$$

Equating Eqns. (6.1) & (6.2),

$$\mathrm{PS = QJ'} \quad \ldots(6.3)$$

Eqn. (6.3) is mathematical expression of the theorem.

THEOREM- 7:

$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$

$\boldsymbol{BV = OF_1}$

Proof of the theorem

$$\text{Eqn. (A. 12)} \Longrightarrow \mathrm{BV} = \sqrt{\mathrm{a^2 + b^2}} \quad \ldots(7.1)$$

$$\text{Eqn. (A. 3)} \Longrightarrow \mathrm{OF_1} = \sqrt{\mathrm{a^2 + b^2}} \quad \ldots(7.2)$$

Equating eqns. (7.1) & (7.2),

$$\mathrm{BV = OF_1} \quad \ldots(7.3)$$

Eqn. (7.3) is mathematical expression of the theorem.

THEOREM- 8:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$

$\boldsymbol{MN \times PN = F_1N^2}$

Proof of the theorem

$$\text{Eqn. (C.21)} \Longrightarrow \mathrm{MN} = \frac{(a^2+b^2)\sqrt{a^2\sin^2(\theta°)+b^2}}{ab\cos(\theta°)} \qquad \ldots(8.1)$$

$$\text{Eqn. (C.19)} \Longrightarrow \mathrm{PN} = \frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)} \qquad \ldots(8.2)$$

Multiplying eqns. (8.1) & (8.2),

$$\mathrm{MN} \times \mathrm{PN} = \left(\frac{(a^2+b^2)\sqrt{a^2\sin^2(\theta°)+b^2}}{\cancel{a}b\cos(\theta°)}\right) \times \left(\frac{\cancel{a}\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}\right)$$

$$\therefore \mathrm{MN} \times \mathrm{PN} = \frac{(a^2+b^2) \times (a^2\sin^2(\theta°)+b^2)}{b^2\cos^2(\theta°)} \qquad \ldots(8.3)$$

$$\text{Eqn. (C.20)} \Longrightarrow \mathrm{SN} = \frac{a^2\sin^2(\theta°)+b^2}{b\sin(\theta°)\cos(\theta°)} \qquad \ldots(8.4)$$

$$\text{Eqn. (C.7)} \Longrightarrow \mathrm{ON} = \frac{(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)} \qquad \ldots(8.5)$$

Multiplying eqns. (76.4) & (76.5),

$$\mathrm{SN} \times \mathrm{ON} = \left(\frac{a^2\sin^2(\theta°)+b^2}{b\cancel{\sin(\theta°)}\cos(\theta°)}\right) \times \left(\frac{(a^2+b^2)\cancel{\sin(\theta°)}}{b\cos(\theta°)}\right)$$

$$\therefore \mathrm{SN} \times \mathrm{ON} = \frac{(a^2+b^2) \times (a^2\sin^2(\theta°)+b^2)}{b^2\cos^2(\theta°)} \qquad \ldots(8.6)$$

$$\text{Eqn. (B.10)} \Longrightarrow \mathrm{F_1N} = \frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}$$

Squaring above eqn.,

$$\mathrm{F_1N}^2 = \frac{(a^2+b^2) \times (a^2\sin^2(\theta°)+b^2)}{b^2\cos^2(\theta°)} \qquad \ldots(8.7)$$

Equating Eqns. (8.3), (8.6) & (8.7),

$$\mathrm{MN} \times \mathrm{PN} = \mathrm{SN} \times \mathrm{ON} = \mathrm{F_1N}^2 \qquad \ldots(8.8)$$

Eqn. (8.8) is mathematical expression of the theorem.

THEOREM- 9:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$

$\boldsymbol{PM \times QE' = TQ \times QM = PQ^2}$

Proof of the theorem

$$\text{Eqn. (C.14)} \Longrightarrow \mathrm{PM} = \frac{\mathrm{b}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\cos(\theta°)} \qquad \ldots(9.1)$$

$$\text{Eqn. (E.2)} \Longrightarrow \mathrm{QE}' = \frac{\mathrm{ab}\sin^2(\theta°)}{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}} \qquad \ldots(9.2)$$

Multiplying eqns. (9.1) & (9.2),

$$\mathrm{PM} \times \mathrm{QE}' = \left(\frac{\mathrm{b}\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}{\cancel{\mathrm{a}}\cos(\theta°)}\right) \times \left(\frac{\cancel{\mathrm{a}}\mathrm{b}\sin^2(\theta°)}{\cos(\theta°)\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}\right)$$

$$\therefore \mathrm{PM} \times \mathrm{QE}' = \frac{\mathrm{b}^2\sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(9.3)$$

$$\text{Eqn. (C.4)} \Longrightarrow \mathrm{TQ} = \frac{\mathrm{a}\sin^2(\theta°)}{\cos(\theta°)} \qquad \ldots(9.4)$$

$$\text{Eqn. (C.5)} \Longrightarrow \mathrm{QM} = \frac{\mathrm{b}^2}{\mathrm{a}\cos(\theta°)} \qquad \ldots(9.5)$$

Multiplying eqns. (9.4) & (9.5),

$$\mathrm{TQ} \times \mathrm{QM} = \frac{\cancel{\mathrm{a}}\sin^2(\theta°)}{\cos(\theta°)} \times \frac{\mathrm{b}^2}{\cancel{\mathrm{a}}\cos(\theta°)}$$

$$\therefore \mathrm{TQ} \times \mathrm{QM} = \frac{\mathrm{b}^2\sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(9.6)$$

$$\text{Eqn. (B.2)} \Longrightarrow \mathrm{PQ} = \frac{\mathrm{b}\sin(\theta°)}{\cos(\theta°)}$$

Squaring above eqn.,

$$\mathrm{PQ}^2 = \frac{\mathrm{b}^2\sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(9.7)$$

Equating Eqns. (9.3), (9.6) & (9.7),

$$\mathrm{PM} \times \mathrm{QE}' = \mathrm{TQ} \times \mathrm{QM} = \mathrm{PQ}^2 \qquad \ldots(9.8)$$

Eqn. (9.8) is mathematical expression of the theorem.

THEOREM- 10:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{OJ \times OT = OS^2}$$

Proof of the theorem

$$\text{Eqn. (D.13)} \Longrightarrow \mathrm{OJ} = \frac{\mathrm{b}^2\cos(\theta°)}{\mathrm{a}\sin^2(\theta°)} \qquad \ldots(10.1)$$

$$\text{Eqn. (C.1)} \Longrightarrow \mathrm{OT} = \mathrm{a}\cos(\theta°) \qquad \ldots(10.2)$$

Multiplying eqns. (10.1) & (10.2),

$$\mathrm{OJ} \times \mathrm{OT} = \left(\frac{\mathrm{b}^2\cos(\theta°)}{\cancel{\mathrm{a}}\sin^2(\theta°)}\right) \times \cancel{\mathrm{a}}\cos(\theta°)$$

$$\therefore \mathrm{OJ} \times \mathrm{OT} = \frac{\mathrm{b}^2\cos^2(\theta°)}{\sin^2(\theta°)} \qquad \ldots(10.3)$$

$$\text{Eqn. (C. 2)} \Longrightarrow \mathrm{OS} = \frac{\mathrm{b}\cos(\theta°)}{\sin(\theta°)}$$

Squaring above eqn.,

$$\therefore \mathrm{OS}^2 = \frac{\mathrm{b}^2\cos^2(\theta°)}{\sin^2(\theta°)} \qquad \ldots(10.4)$$

Equating Eqns. (10.3) & (10.4),

$$\mathrm{OJ} \times \mathrm{OT} = \mathrm{OS}^2 \qquad \ldots(10.5)$$

Eqn. (10.5) is mathematical expression of the theorem.

THEOREM- 11:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{TW \times PN = VX^2}$$

Proof of the theorem

$$\text{Eqn. (D. 3)} \Longrightarrow \mathrm{TW} = \frac{\mathrm{a}\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}} \qquad \ldots(11.2)$$

$$\text{Eqn. (C. 19)} \Longrightarrow \mathrm{PN} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \qquad \ldots(11.2)$$

Multiplying eqns. (11.1) & (11.2),

$$\mathrm{TW} \times \mathrm{PN} = \left(\frac{\mathrm{a}\cancel{\cos(\theta°)}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}}\right) \times \left(\frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cancel{\cos(\theta°)}}\right)$$

$$\therefore \mathrm{TW} \times \mathrm{PN} = \frac{\mathrm{a}^2(\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2)}{\mathrm{b}^2} \qquad \ldots(11.3)$$

$$\text{Eqn. (D. 4)} \Longrightarrow \mathrm{VX} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}}$$

Squaring above eqn.,

$$\mathrm{VX}^2 = \frac{\mathrm{a}^2(\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2)}{\mathrm{b}^2} \qquad \ldots(11.4)$$

Equating Eqns. (11.3) & (11.4),

$$\mathrm{TW} \times \mathrm{PN} = \mathrm{VX}^2 \qquad \ldots(11.5)$$

Eqn. (11.5) is mathematical expression of the theorem.

THEOREM- 12:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{PS \times TS = PM \times VX = F_1S^2}$$

Proof of the theorem

$$\text{Eqn. (C. 16)} \Longrightarrow \text{PS} = \frac{\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\sin(\theta°)\cos(\theta°)} \quad \ldots(12.1)$$

$$\text{Eqn. (C. 10)} \Longrightarrow \text{TS} = \frac{\cos(\theta°)\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\sin(\theta°)} \quad \ldots(12.2)$$

Multiplying eqns. (12.1) & (12.2),

$$\text{PS} \times \text{TS} = \left(\frac{\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\sin(\theta°)\cancel{\cos(\theta°)}}\right) \times \left(\frac{\cancel{\cos(\theta°)}\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\sin(\theta°)}\right)$$

$$\therefore \text{PS} \times \text{TS} = \frac{\text{a}^2\sin^2(\theta°) + \text{b}^2}{\sin^2(\theta°)} \quad \ldots(12.3)$$

$$\text{Eqn. (C. 14)} \Longrightarrow \text{PM} = \frac{\text{b}\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\text{a}\cos(\theta°)} \quad \ldots(12.4)$$

$$\text{Eqn. (D. 4)} \Longrightarrow \text{VX} = \frac{\text{a}\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\text{b}} \quad \ldots(12.5)$$

Multiplying eqns. (12.4) & (12.5),

$$\text{PM} \times \text{VX} = \left(\frac{\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\sin(\theta°)}\right) \times \left(\frac{\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\sin(\theta°)}\right)$$

$$\therefore \text{PM} \times \text{VX} = \frac{\text{a}^2\sin^2(\theta°) + \text{b}^2}{\sin^2(\theta°)} \quad \ldots(12.6)$$

$$\text{Eqn. (C. 13)} \Longrightarrow \text{F}_1\text{S} = \frac{\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\sin(\theta°)}$$

Squaring the above eqn.,

$$\text{F}_1\text{S}^2 = \left(\frac{\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\sin(\theta°)}\right)^2$$

$$\therefore \text{F}_1\text{S}^2 = \frac{\text{a}^2\sin^2(\theta°) + \text{b}^2}{\sin^2(\theta°)} \quad \ldots(12.7)$$

Equating eqns. (12.3), (12.6) & (12.7),

$$\text{PS} \times \text{TS} = \text{PM} \times \text{VX} = \text{F}_1\text{S}^2 \quad \ldots(12.8)$$

Eqn. (12.8) is mathematical expression of the theorem.

THEOREM- 13:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{OL' \times ON = OM^2}$

Proof of the theorem

$$\text{Eqn. (E. 8)} \Longrightarrow \text{OL}' = \frac{\text{b}(\text{a}^2 + \text{b}^2)}{\text{a}^2\sin(\theta°)\cos(\theta°)} \quad \ldots(13.1)$$

$$\text{Eqn. (C. 7)} \Longrightarrow \mathrm{ON} = \frac{(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)} \qquad \ldots(13.2)$$

Multiplying eqns. (13.1) & (13.2),

$$\mathrm{OL}' \times \mathrm{ON} = \left(\frac{\cancel{b}(a^2+b^2)}{a^2\cancel{\sin(\theta°)}\cos(\theta°)}\right) \times \left(\frac{(a^2+b^2)\cancel{\sin(\theta°)}}{\cancel{b}\cos(\theta°)}\right)$$

$$\therefore \mathrm{OL}' \times \mathrm{OM} = \frac{(a^2+b^2)^2}{a^2\cos^2(\theta°)} \qquad \ldots(13.3)$$

$$\text{Eqn. (C. 6)} \Longrightarrow \mathrm{OM} = \frac{a^2+b^2}{a\cos(\theta°)}$$

$$\therefore \mathrm{OM}^2 = \frac{(a^2+b^2)^2}{a^2\cos^2(\theta°)} \qquad \ldots(13.4)$$

Equating eqns. (13.3) & (13.4),

$$\mathrm{OL}' \times \mathrm{ON} = \mathrm{OM}^2 \qquad \ldots(13.5)$$

Eqn. (13.5) is mathematical expression of the theorem.

THEOREM- 14:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{RS \times SN = PS^2}$

Proof of the theorem

$$\text{Eqn. (C. 18)} \Longrightarrow \mathrm{RS} = \frac{b}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(14.1)$$

$$\text{Eqn. (C. 20)} \Longrightarrow \mathrm{SN} = \frac{a^2\sin^2(\theta°)+b^2}{b\sin(\theta°)\cos(\theta°)} \qquad \ldots(14.2)$$

Multiplying eqns. (14.1) & (14.2),

$$\mathrm{RS} \times \mathrm{SN} = \left(\frac{\cancel{b}}{\sin(\theta°)\cos(\theta°)}\right) \times \left(\frac{a^2\sin^2(\theta°)+b^2}{\cancel{b}\sin(\theta°)\cos(\theta°)}\right)$$

$$\therefore \mathrm{RS} \times \mathrm{SN} = \frac{a^2\sin^2(\theta°)+b^2}{\sin^2(\theta°)\cos^2(\theta°)} \qquad \ldots(14.3)$$

$$\text{Eqn. (C. 16)} \Longrightarrow \mathrm{PS} = \frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)\cos(\theta°)}$$

Squaring above eqn.,

$$\mathrm{PS}^2 = \frac{a^2\sin^2(\theta°)+b^2}{\sin^2(\theta°)\cos^2(\theta°)} \qquad \ldots(14.4)$$

Equating eqns. (14.3) & (14.4),

$$\mathrm{RS} \times \mathrm{SN} = \mathrm{PS}^2 \qquad \ldots(14.5)$$

Eqn. (14.5) is mathematical expression of the theorem.

THEOREM- 15:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{RS \times RN = OQ^2}$$

Proof of the theorem

$$\text{Eqn. (C. 18)} \Longrightarrow \mathrm{RS} = \frac{\mathrm{b}}{\sin(\theta°)\cos(\theta°)} \quad \ldots(15.1)$$

$$\text{Eqn. (C. 17)} \Longrightarrow \mathrm{RN} = \frac{\mathrm{a}^2\sin(\theta°)}{\mathrm{b}\cos(\theta°)} \quad \ldots(15.2)$$

Multiplying eqns. (15.1) & (15.2),

$$\mathrm{RS} \times \mathrm{RN} = \left(\frac{\cancel{\mathrm{b}}}{\cancel{\sin(\theta°)}\cos(\theta°)}\right) \times \left(\frac{\mathrm{a}^2\cancel{\sin(\theta°)}}{\cancel{\mathrm{b}}\cos(\theta°)}\right)$$

$$\therefore \mathrm{RS} \times \mathrm{RN} = \frac{\mathrm{a}^2}{\cos^2(\theta°)} \quad \ldots(15.3)$$

$$\text{Eqn. (C. 16)} \Longrightarrow \mathrm{OQ} = \frac{\mathrm{a}}{\cos(\theta°)}$$

Squaring above eqn.,

$$\mathrm{OQ}^2 = \frac{\mathrm{a}^2}{\cos^2(\theta°)} \quad \ldots(15.4)$$

Equating eqns. (15.3) & (15.4),

$$\mathrm{RS} \times \mathrm{RN} = \mathrm{OQ}^2 \quad \ldots(15.5)$$

Eqn. (15.5) is mathematical expression of the theorem.

THEOREM- 16:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{OU \times OF_1 = OS \times RN = TS \times RA' = OK \times PN = TK \times PS = RS \times OW = a^2}$$

Proof of the theorem

$$\text{Eqn. (A. 7)} \Longrightarrow \mathrm{OU} = \frac{\mathrm{a}^2}{\sqrt{\mathrm{a}^2 + \mathrm{b}^2}} \quad \ldots(16.1)$$

$$\text{Eqn. (A. 3)} \Longrightarrow \mathrm{OF}_1 = \sqrt{\mathrm{a}^2 + \mathrm{b}^2} \quad \ldots(16.2)$$

Multiplying eqns. (16.1) & (16.2),

$$\mathrm{OU} \times \mathrm{OF}_1 = \left(\frac{\mathrm{a}^2}{\cancel{\sqrt{\mathrm{a}^2 + \mathrm{b}^2}}}\right) \times \cancel{\sqrt{\mathrm{a}^2 + \mathrm{b}^2}}$$

$$\therefore \mathrm{OU} \times \mathrm{OF}_1 = \mathrm{a}^2 \quad \ldots(16.3)$$

$$\text{Eqn. (C. 2)} \Longrightarrow \mathrm{OS} = \frac{\mathrm{b}\cos(\theta°)}{\sin(\theta°)} \quad \ldots(16.4)$$

$$\text{Eqn. (C. 17)} \Longrightarrow \mathrm{RN} = \frac{\mathrm{a}^2\sin(\theta°)}{\mathrm{b}\cos(\theta°)} \quad \ldots(16.5)$$

Multiplying eqns. (16.4) & (16.5),

$$\mathrm{OS} \times \mathrm{RN} = \left(\frac{\cancel{\mathrm{bcos}(\theta°)}}{\cancel{\sin(\theta°)}}\right) \times \left(\frac{\mathrm{a}^2\cancel{\sin(\theta°)}}{\cancel{\mathrm{bcos}(\theta°)}}\right)$$

$$\therefore \mathrm{OS} \times \mathrm{RN} = \mathrm{a}^2 \qquad \ldots(16.6)$$

$$\text{Eqn. (C. 10)} \Longrightarrow \mathrm{TS} = \frac{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)} \qquad \ldots(16.7)$$

$$\text{Eqn. (E. 3)} \Longrightarrow \mathrm{RA'} = \frac{\mathrm{a}^2\sin(\theta°)}{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}} \qquad \ldots(16.8)$$

Multiplying eqns. (16.7) & (16.8),

$$\mathrm{TS} \times \mathrm{RA'} = \left(\frac{\cancel{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}{\cancel{\sin(\theta°)}}\right) \times \left(\frac{\mathrm{a}^2\cancel{\sin(\theta°)}}{\cancel{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}\right)$$

$$\therefore \mathrm{TS} \times \mathrm{RA'} = \mathrm{a}^2 \qquad \ldots(16.9)$$

$$\text{Eqn. (D. 15)} \Longrightarrow \mathrm{OK} = \frac{\mathrm{abcos}(\theta°)}{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}} \qquad \ldots(16.10)$$

$$\text{Eqn. (C. 19)} \Longrightarrow \mathrm{PN} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{bcos}(\theta°)} \qquad \ldots(16.11)$$

Multiplying eqns. (16.10) & (16.11),

$$\mathrm{OK} \times \mathrm{PN} = \left(\frac{\cancel{\mathrm{abcos}(\theta°)}}{\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}\right) \times \left(\frac{\mathrm{a}\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}{\cancel{\mathrm{bcos}(\theta°)}}\right)$$

$$\therefore \mathrm{OK} \times \mathrm{PN} = \mathrm{a}^2 \qquad \ldots(16.12)$$

$$\text{Eqn. (D. 16)} \Longrightarrow \mathrm{TK} = \frac{\mathrm{a}^2\sin(\theta°)cos(\theta°)}{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}} \qquad \ldots(16.13)$$

$$\text{Eqn. (C. 16)} \Longrightarrow \mathrm{PS} = \frac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(16.14)$$

Multiplying eqns. (16.13) & (16.14),

$$\mathrm{TK} \times \mathrm{PS} = \left(\frac{\mathrm{a}^2\cancel{\sin(\theta°)cos(\theta°)}}{\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}\right) \times \left(\frac{\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}{\cancel{\sin(\theta°)\cos(\theta°)}}\right)$$

$$\therefore \mathrm{TK} \times \mathrm{PS} = \mathrm{a}^2 \qquad \ldots(16.15)$$

$$\text{Eqn. (C. 18)} \Longrightarrow \mathrm{RS} = \frac{\mathrm{b}}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(16.16)$$

$$\text{Eqn. (D. 9)} \Longrightarrow \mathrm{OW} = \frac{\mathrm{a}^2\sin(\theta°)\cos(\theta°)}{\mathrm{b}} \qquad \ldots(16.17)$$

Multiplying eqns. (16.16) & (16.17),

$$\mathrm{RS} \times \mathrm{OW} = \left(\frac{\cancel{\mathrm{b}}}{\cancel{\sin(\theta°)\cos(\theta°)}}\right) \times \left(\frac{\mathrm{a}^2\cancel{\sin(\theta°)\cos(\theta°)}}{\cancel{\mathrm{b}}}\right)$$

$$\therefore \mathrm{RS} \times \mathrm{OW} = \mathrm{a}^2 \qquad \ldots(16.18)$$

Equating eqns. (16.3), (16.6), (16.9), (16.12), (16.15) & (16.18),

$$OU \times OF_1 = OS \times RN = TS \times RA' = OK \times PN = TK \times PS = RS \times OW = a^2 \qquad \ldots(16.19)$$

Eqn. (16.19) is mathematical expression of the theorem.

THEOREM- 17:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{OS \times OR = VF_1 \times VF_2 = UF_1 \times OF_1 = OT \times OI = OJ \times TQ = SJ \times QE' = OK \times PM = b^2}$

Proof of the theorem

$$\text{Eqn. (C. 2)} \Longrightarrow OS = \frac{b\cos(\theta°)}{\sin(\theta°)} \qquad \ldots(17.1)$$

$$\text{Eqn. (B. 2)} \Longrightarrow OR = \frac{b\sin(\theta°)}{\cos(\theta°)} \qquad \ldots(17.2)$$

Multiplying eqns. (17.1) & (17.2),

$$OS \times OR = \left(\frac{b\cancel{\cos(\theta°)}}{\cancel{\sin(\theta°)}}\right) \times \left(\frac{b\cancel{\sin(\theta°)}}{\cancel{\cos(\theta°)}}\right)$$

$$\therefore OS \times OR = b^2 \qquad \ldots(17.3)$$

$$\text{Eqn. (A. 5)} \Longrightarrow VF_1 = \sqrt{a^2 + b^2} - a \qquad \ldots(17.4)$$

$$\text{Eqn. (A. 6)} \Longrightarrow VF_2 = \sqrt{a^2 + b^2} + a \qquad \ldots(17.5)$$

Multiplying eqns. (17.3) & (17.4),

$$VF_1 \times VF_2 = \left(\sqrt{a^2 + b^2} - a\right) \times \left(\sqrt{a^2 + b^2} + a\right)$$

$$\therefore VF_1 \times VF_2 = \cancel{a^2} + b^2 \cancel{- a^2}$$

$$\therefore VF_1 \times VF_2 = b^2 \qquad \ldots(17.6)$$

$$\text{Eqn. (A. 9)} \Longrightarrow UF_1 = \frac{b^2}{\sqrt{a^2 + b^2}} \qquad \ldots(17.7)$$

$$\text{Eqn. (A. 3)} \Longrightarrow OF_1 = \sqrt{a^2 + b^2} \qquad \ldots(17.8)$$

Multiplying the eqns. (17.7) & (17.8),

$$UF_1 \times OF_1 = \left(\frac{b^2}{\cancel{\sqrt{a^2 + b^2}}}\right) \times \cancel{\sqrt{a^2 + b^2}}$$

$$\therefore UF_1 \times OF_1 = b^2 \qquad \ldots(17.9)$$

$$\text{Eqn. (C. 1)} \Longrightarrow OT = a\cos(\theta°) \qquad \ldots(17.10)$$

$$\text{Eqn. (D. 7)} \Longrightarrow OI = \frac{b^2}{a\cos(\theta°)} \qquad \ldots(17.11)$$

Multiplying the eqns. (17.10) & (17.11),

$$OT \times OI = \cancel{a\cos(\theta°)} \times \frac{b^2}{\cancel{a\cos(\theta°)}}$$

$$\therefore OT \times OI = b^2 \qquad \ldots(17.12)$$

$$\text{Eqn. (D.13)} \Longrightarrow \mathrm{OJ} = \frac{\mathrm{b}^2\cos(\theta°)}{\mathrm{a}\sin^2(\theta°)} \quad \ldots(17.13)$$

$$\text{Eqn. (C.4)} \Longrightarrow \mathrm{TQ} = \frac{\mathrm{a}\sin^2(\theta°)}{\cos(\theta°)} \quad \ldots(17.14)$$

Multiplying the eqns. (17.13) & (17.14),

$$\mathrm{OJ} \times \mathrm{TQ} = \left(\frac{\mathrm{b}^2\cancel{\cos(\theta°)}}{\cancel{\mathrm{a}\sin^2(\theta°)}}\right) \times \left(\frac{\cancel{\mathrm{a}\sin^2(\theta°)}}{\cancel{\cos(\theta°)}}\right)$$

$$\therefore \mathrm{OJ} \times \mathrm{TQ} = \mathrm{b}^2 \quad \ldots(17.15)$$

$$\text{Eqn. (D.14)} \Longrightarrow \mathrm{SJ} = \frac{\mathrm{b}\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\sin^2(\theta°)} \quad \ldots(17.16)$$

$$\text{Eqn. (E.2)} \Longrightarrow \mathrm{QE}' = \frac{\mathrm{ab}\sin^2(\theta°)}{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}} \quad \ldots(17.17)$$

Multiplying the eqns. (17.16) & (17.17),

$$\mathrm{SJ} \times \mathrm{QE}' = \left(\frac{\cancel{\mathrm{b}\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}{\cancel{\mathrm{a}\sin^2(\theta°)}}\right) \times \left(\frac{\cancel{\mathrm{ab}\sin^2(\theta°)}}{\cancel{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}\right)$$

$$\therefore \mathrm{SJ} \times \mathrm{QE}' = \mathrm{b}^2 \quad \ldots(17.18)$$

$$\text{Eqn. (D.15)} \Longrightarrow \mathrm{OK} = \frac{\mathrm{ab}\cos(\theta°)}{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}} \quad \ldots(17.19)$$

$$\text{Eqn. (C.14)} \Longrightarrow \mathrm{PM} = \frac{\mathrm{b}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\cos(\theta°)} \quad \ldots(17.20)$$

Multiplying the eqns. (17.19) & (17.20),

$$\mathrm{OK} \times \mathrm{PM} = \left(\frac{\mathrm{ab}\cancel{\cos(\theta°)}}{\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}\right) \times \left(\frac{\mathrm{b}\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}{\cancel{\mathrm{a}\cos(\theta°)}}\right)$$

$$\therefore \mathrm{OK} \times \mathrm{PM} = \mathrm{b}^2 \quad \ldots(17.21)$$

Equating eqns. (17.3), (17.6), (17.9), (17.12), (17.15), (17.18) & (17.21),

$$\mathrm{OS} \times \mathrm{OR} = \mathrm{VF}_1 \times \mathrm{VF}_2 = \mathrm{UF}_1 \times \mathrm{OF}_1 = \mathrm{OT} \times \mathrm{OI} = \mathrm{OJ} \times \mathrm{TQ} = \mathrm{SJ} \times \mathrm{QE}' = \mathrm{OK} \times \mathrm{PM} = \mathrm{b}^2 \quad ..(17.22)$$

Eqn. (17.22) is mathematical expression of the theorem.

THEOREM- 18:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{(UF_2 - UF_1) \times OF_1 = 2a^2}$

Proof of the theorem

$$\text{Eqn. (A.10)} \Longrightarrow \mathrm{UF}_2 = \frac{2\mathrm{a}^2 + \mathrm{b}^2}{\sqrt{\mathrm{a}^2 + \mathrm{b}^2}} \quad \ldots(18.1)$$

$$\text{Eqn. (A.9)} \Longrightarrow \mathrm{UF}_1 = \frac{\mathrm{b}^2}{\sqrt{\mathrm{a}^2 + \mathrm{b}^2}} \quad \ldots(18.2)$$

$$\text{Eqn. (A. 3)} \Longrightarrow \mathrm{OF_1} = \sqrt{a^2 + b^2} \quad \ldots(18.3)$$

Subtracting eqn. (18.2) from (18.1),

$$\mathrm{UF_2} - \mathrm{UF_1} = \left(\frac{2a^2 + b^2}{\sqrt{a^2 + b^2}}\right) - \left(\frac{b^2}{\sqrt{a^2 + b^2}}\right)$$

$$\therefore \mathrm{UF_2} - \mathrm{UF_1} = \frac{2a^2 \cancel{+ b^2 - b^2}}{\sqrt{a^2 + b^2}}$$

$$\therefore \mathrm{UF_2} - \mathrm{UF_1} = \frac{2a^2}{\sqrt{a^2 + b^2}} \quad \ldots(18.4)$$

Multiplying above eqn. & eqn. (18.3)

$$\therefore (\mathrm{UF_2} - \mathrm{UF_1}) \times \mathrm{OF_1} = \left(\frac{2a^2}{\cancel{\sqrt{a^2 + b^2}}}\right) \times \cancel{\sqrt{a^2 + b^2}}$$

$$\therefore (\mathrm{UF_2} - \mathrm{UF_1}) \times \mathrm{OF_1} = 2a^2 \quad \ldots(18.5)$$

Eqn. (18.5) is mathematical expression of the theorem.

THEOREM- 19:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$$\boldsymbol{OT \times OM = OS \times ON = SJ \times OB' = OK \times MN = TK \times ML' = a^2 + b^2}$$

Proof of the theorem

$$\text{Eqn. (C. 1)} \Longrightarrow \mathrm{OT} = \mathrm{a}\cos(\theta°) \quad \ldots(19.1)$$

$$\text{Eqn. (C. 6)} \Longrightarrow \mathrm{OM} = \frac{a^2 + b^2}{\mathrm{a}\cos(\theta°)} \quad \ldots(19.2)$$

Multiplying eqns. (19.1) & (19.2),

$$\mathrm{OT} \times \mathrm{OM} = \cancel{\mathrm{a}\cos(\theta°)} \times \left(\frac{a^2 + b^2}{\cancel{\mathrm{a}\cos(\theta°)}}\right)$$

$$\therefore \mathrm{OM} \times \mathrm{OT} = a^2 + b^2 \quad \ldots(19.3)$$

$$\text{Eqn. (C. 2)} \Longrightarrow \mathrm{OS} = \frac{\mathrm{b}\cos(\theta°)}{\sin(\theta°)} \quad \ldots(19.4)$$

$$\text{Eqn. (C. 7)} \Longrightarrow \mathrm{ON} = \frac{(a^2 + b^2)\sin(\theta°)}{\mathrm{b}\cos(\theta°)} \quad \ldots(19.5)$$

Multiplying eqns. (19.4) & (19.5),

$$\mathrm{OS} \times \mathrm{ON} = \left(\frac{\cancel{\mathrm{b}\cos(\theta°)}}{\cancel{\sin(\theta°)}}\right) \times \left(\frac{(a^2 + b^2)\cancel{\sin(\theta°)}}{\cancel{\mathrm{b}\cos(\theta°)}}\right)$$

$$\therefore \mathrm{OS} \times \mathrm{ON} = a^2 + b^2 \quad \ldots(19.6)$$

$$\text{Eqn. (D. 14)} \Longrightarrow \mathrm{SJ} = \frac{\mathrm{b}\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\mathrm{a}\sin^2(\theta°)} \quad \ldots(19.7)$$

$$\text{Eqn. (E. 1)} \Longrightarrow \mathrm{OB'} = \frac{a(a^2 + b^2)\sin^2(\theta°)}{\mathrm{b}\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}} \quad \ldots(19.8)$$

Multiplying eqns. (19.7) & (19.8),

$$SJ \times OB' = \left(\frac{\cancel{b\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}}{\cancel{a\sin^2(\theta°)}}\right) \times \left(\frac{\cancel{a}(a^2+b^2)\cancel{\sin^2(\theta°)}}{\cancel{b\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}}\right)$$

$$\therefore SJ \times OB' = a^2 + b^2 \qquad \ldots(19.9)$$

$$\text{Eqn. (D.15)} \Longrightarrow OK = \frac{ab\cos(\theta°)}{\sqrt{a^2\sin^2(\theta°)+b^2}} \qquad \ldots(19.10)$$

$$\text{Eqn. (C.21)} \Longrightarrow MN = \frac{(a^2+b^2)\sqrt{a^2\sin^2(\theta°)+b^2}}{ab\cos(\theta°)} \qquad \ldots(19.11)$$

Multiplying eqns. (19.10) & (19.11),

$$OK \times MN = \left(\frac{\cancel{ab\cos(\theta°)}}{\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}\right) \times \left(\frac{(a^2+b^2)\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}{\cancel{ab\cos(\theta°)}}\right)$$

$$\therefore OK \times MN = a^2 + b^2 \qquad \ldots(19.12)$$

$$\text{Eqn. (D.16)} \Longrightarrow TK = \frac{a^2\sin(\theta°)\cos(\theta°)}{\sqrt{a^2\sin^2(\theta°)+b^2}} \qquad \ldots(19.13)$$

$$\text{Eqn. (E.7)} \Longrightarrow ML' = \frac{(a^2+b^2)\sqrt{a^2\sin^2(\theta°)+b^2}}{a^2\sin(\theta°)\cos(\theta°)} \qquad \ldots(19.14)$$

Multiplying eqns. (19.13) & (19.14),

$$TK \times ML' = \left(\frac{\cancel{a^2\sin(\theta°)\cos(\theta°)}}{\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}\right) \times \left(\frac{(a^2+b^2)\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}{\cancel{a^2\sin(\theta°)\cos(\theta°)}}\right)$$

$$\therefore TK \times ML' = a^2 + b^2 \qquad \ldots(19.15)$$

Equating Eqns. (19.3), (19.6), (19.9), (19.12) & (19.15),

$$OT \times OM = OS \times ON = SJ \times OB' = OK \times MN = TK \times ML' = a^2 + b^2 \qquad \ldots(19.16)$$

Eqn. (19.16) is mathematical expression of the theorem.

THEOREM- 20:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$

$\boldsymbol{OP^2 - (PF_1 \times PF_2) = a^2 - b^2}$

Proof of the theorem

$$\text{Eqn. (B.5)} \Longrightarrow OP = \frac{\sqrt{a^2 + b^2\sin^2(\theta°)}}{\cos(\theta°)}$$

Squaring above eqn.,

$$\therefore OP^2 = \frac{a^2 + b^2\sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(20.1)$$

$$\text{Eqn. (E.10)} \Longrightarrow F_1T \times F_2T = a^2\sin^2(\theta°) + b^2 \qquad \ldots(20.2)$$

Subtracting eqn. (20.2) from (20.1),

$$\mathrm{OP^2 - PF_1 \times PF_2 = \left(\frac{a^2 + b^2\sin^2(\theta°)}{\cos^2(\theta°)}\right) - \left(\frac{a^2\sin^2(\theta°) + b^2}{\cos^2(\theta°)}\right)}$$

$$\therefore \mathrm{OP^2 - PF_1 \times PF_2 = \frac{a^2 + b^2\sin^2(\theta°) - a^2\sin^2(\theta°) - b^2}{\cos^2(\theta°)}}$$

$$\therefore \mathrm{OP^2 - PF_1 \times PF_2 = \frac{a^2 - a^2\sin^2(\theta°) + b^2\sin^2(\theta°) - b^2}{\cos^2(\theta°)}}$$

$$\therefore \mathrm{OP^2 - PF_1 \times PF_2 = \frac{a^2[1 - \sin^2(\theta°)] + b^2[\sin^2(\theta°) - 1]}{\cos^2(\theta°)}}$$

$$\therefore \mathrm{OP^2 - PF_1 \times PF_2 = \frac{a^2\cos^2(\theta°) - b^2\cos^2(\theta°)}{\cos^2(\theta°)}}$$

$$\therefore \mathrm{OP^2 - PF_1 \times PF_2 = \frac{(a^2 - b^2) \times \cancel{\cos^2(\theta°)}}{\cancel{\cos^2(\theta°)}}}$$

$$\therefore \mathrm{OP^2 - PF_1 \times PF_2 = a^2 - b^2} \qquad \ldots(20.3)$$

Eqn. (20.3) is mathematical expression of the theorem.

THEOREM- 21:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{F_1R^2 - VR^2 = b^2}$

Proof of the theorem

$$\mathrm{Eqn.\,(B.17) \Longrightarrow F_1R = \frac{\sqrt{a^2\cos^2(\theta°) + b^2}}{\cos(\theta°)}}$$

Squaring above eqn.,

$$\mathrm{F_1R^2 = \frac{a^2\cos^2(\theta°) + b^2}{\cos^2(\theta°)}} \qquad \ldots(21.1)$$

$$\mathrm{Eqn.\,(B.16) \Longrightarrow VR = \frac{\sqrt{a^2\cos^2(\theta°) + b^2\sin^2(\theta°)}}{\cos(\theta°)}}$$

Squaring above eqn.,

$$\mathrm{VR^2 = \frac{a^2\cos^2(\theta°) + b^2\sin^2(\theta°)}{\cos^2(\theta°)}} \qquad \ldots(21.2)$$

Subtracting eqns. (21.2) from (21.1),

$$\mathrm{F_1R^2 - VR^2 = \left(\frac{a^2\cos^2(\theta°) + b^2}{\cos^2(\theta°)}\right) - \left(\frac{a^2\cos^2(\theta°) + b^2\sin^2(\theta°)}{\cos^2(\theta°)}\right)}$$

$$\therefore \mathrm{F_1R^2 - VR^2 = \frac{a^2\cos^2(\theta°) + b^2 - [a^2\cos^2(\theta°) + b^2\sin^2(\theta°)]}{\cos^2(\theta°)}}$$

$$\therefore \mathrm{F_1R^2 - VR^2 = \frac{\cancel{a^2\cos^2(\theta°)} + b^2 \cancel{- a^2\cos^2(\theta°)} - b^2\sin^2(\theta°)}{\cos^2(\theta°)}}$$

$$\therefore \mathrm{F_1R^2 - VR^2 = \frac{b^2\left(1 - \sin^2(\theta°)\right)}{\cos^2(\theta°)}}$$

$$\therefore \mathrm{F_1R^2 - VR^2} = \frac{\mathrm{b^2}\cancel{\cos^2(\theta°)}}{\cancel{\cos^2(\theta°)}}$$

$$\therefore \mathrm{F_1R^2 - VR^2 = b^2} \qquad \ldots(21.3)$$

Eqn. (21.3) is mathematical expression of the theorem.

THEOREM- 22:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{PL + TW = a \times PN}$

Proof of the theorem

$$\text{Eqn. (D. 2)} \Longrightarrow \mathrm{PL} = \frac{\mathrm{a}\sin^2(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \qquad \ldots(22.1)$$

$$\text{Eqn. (D. 3)} \Longrightarrow \mathrm{TW} = \frac{\mathrm{a}\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}} \qquad \ldots(22.2)$$

$$\text{Eqn. (C. 19)} \Longrightarrow \mathrm{PN} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \qquad \ldots(22.3)$$

Adding eqns. (22.1) & (22.2),

$$\mathrm{PL + TW} = \left(\frac{\mathrm{a}\sin^2(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)}\right) + \left(\frac{\mathrm{a}\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}}\right)$$

$$\therefore \mathrm{PL + TW} = \frac{[\mathrm{a}\sin^2(\theta°) + \mathrm{a}\cos^2(\theta°)]\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)}$$

$$\therefore \mathrm{PL + TW} = \frac{[\mathrm{a}\sin^2(\theta°) + \mathrm{a}\cos^2(\theta°)]\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)}$$

$$\therefore \mathrm{PL + TW} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \qquad \ldots(22.4)$$

Equating Eqns. (22.3) & (22.4),

$$\mathrm{PL + TW = PN} \qquad \ldots(22.5)$$

Eqn. (22.5) is mathematical expression of the theorem.

THEOREM- 23:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{F_1T \times F_2T = PT \times TS = TW \times PM = SJ \times PL}$

Proof of the theorem

$$\text{Eqn. (E. 10)} \Longrightarrow \mathrm{F_1T \times F_2T} = \mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2 \qquad \ldots(23.1)$$

$$\text{Eqn. (C. 15)} \Longrightarrow \mathrm{PT} = \frac{\sin(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\cos(\theta°)} \qquad \ldots(23.2)$$

$$\text{Eqn. (C. 10)} \Longrightarrow \mathrm{TS} = \frac{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)} \qquad \ldots(23.3)$$

Multiplying eqns. (23.2) & (23.3),

$$\mathrm{PT} \times \mathrm{TS} = \left(\frac{\cancel{\sin(\theta°)}\sqrt{a^2\sin^2(\theta°) + b^2}}{\cancel{\cos(\theta°)}}\right) \times \left(\frac{\cancel{\cos(\theta°)}\sqrt{a^2\sin^2(\theta°) + b^2}}{\cancel{\sin(\theta°)}}\right)$$

$$\therefore \mathrm{PT} \times \mathrm{TS} = a^2\sin^2(\theta°) + b^2 \qquad \ldots(23.4)$$

$$\text{Eqn. (D. 3)} \Longrightarrow \mathrm{TW} = \frac{a\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{b} \qquad \ldots(23.5)$$

$$\text{Eqn. (C. 14)} \Longrightarrow \mathrm{PM} = \frac{b\sqrt{a^2\sin^2(\theta°) + b^2}}{a\cos(\theta°)} \qquad \ldots(23.6)$$

Multiplying eqns. (23.5) & (23.6),

$$\mathrm{TW} \times \mathrm{PM} = \left(\frac{\cancel{a\cos(\theta°)}\sqrt{a^2\sin^2(\theta°) + b^2}}{\cancel{b}}\right) \times \left(\frac{\cancel{b}\sqrt{a^2\sin^2(\theta°) + b^2}}{\cancel{a\cos(\theta°)}}\right)$$

$$\therefore \mathrm{TW} \times \mathrm{PM} = a^2\sin^2(\theta°) + b^2 \qquad \ldots(23.7)$$

$$\text{Eqn. (D. 14)} \Longrightarrow \mathrm{SJ} = \frac{b\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{a\sin^2(\theta°)} \qquad \ldots(23.8)$$

$$\text{Eqn. (D. 2)} \Longrightarrow \mathrm{PL} = \frac{a\sin^2(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)} \qquad \ldots(23.9)$$

Multiplying eqns. (23.8) & (23.9),

$$\mathrm{SJ} \times \mathrm{PL} = \left(\frac{\cancel{b\cos(\theta°)}\sqrt{a^2\sin^2(\theta°) + b^2}}{\cancel{a\sin^2(\theta°)}}\right) \times \left(\frac{\cancel{a\sin^2(\theta°)}\sqrt{a^2\sin^2(\theta°) + b^2}}{\cancel{b\cos(\theta°)}}\right)$$

$$\therefore \mathrm{SJ} \times \mathrm{PL} = a^2\sin^2(\theta°) + b^2 \qquad \ldots(23.10)$$

Equating Eqns. (23.1), (23.4), (23.7) & (23.10),

$$F_1T \times F_2T = \mathrm{PT} \times \mathrm{TS} = \mathrm{TW} \times \mathrm{PM} = \mathrm{SJ} \times \mathrm{PL} \qquad \ldots(23.11)$$

Eqn. (23.11) is mathematical expression of the theorem.

THEOREM- 24:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1},$

$$\boldsymbol{\left(\frac{1}{F_1T \times F_2T}\right) - \left(\frac{1}{PF_1 \times PF_2}\right) = \frac{1}{F_1S^2}}$$

Proof of the theorem

$$\text{Eqn. (E. 10)} \Longrightarrow F_1T \times F_2T = a^2\sin^2(\theta°) + b^2$$

Reciprocating above eqn.,

$$\therefore \frac{1}{F_1T \times F_2T} = \frac{1}{a^2\sin^2(\theta°) + b^2} \qquad \ldots(24.1)$$

$$\text{Eqn. (E. 9)} \Longrightarrow PF_1 \times PF_2 = \frac{a^2\sin^2(\theta°) + b^2}{\cos^2(\theta°)}$$

Reciprocating above eqn.,

$$\frac{1}{\mathrm{PF_1 \times PF_2}} = \frac{\cos^2(\theta°)}{a^2\sin^2(\theta°) + b^2} \qquad \ldots(24.2)$$

Subtracting eqn. (24.1) from (24.2),

$$\left(\frac{1}{\mathrm{F_1T \times F_2T}}\right) - \left(\frac{1}{\mathrm{PF_1 \times PF_2}}\right) = \left(\frac{1}{a^2\sin^2(\theta°) + b^2}\right) - \left(\frac{\cos^2(\theta°)}{a^2\sin^2(\theta°) + b^2}\right)$$

$$\therefore \left(\frac{1}{\mathrm{F_1T \times F_2T}}\right) - \left(\frac{1}{\mathrm{PF_1 \times PF_2}}\right) = \left(\frac{1 - \cos^2(\theta°)}{a^2\sin^2(\theta°) + b^2}\right)$$

$$\therefore \left(\frac{1}{\mathrm{F_1T \times F_2T}}\right) - \left(\frac{1}{\mathrm{PF_1 \times PF_2}}\right) = \left(\frac{\sin^2(\theta°)}{a^2\sin^2(\theta°) + b^2}\right) \qquad \ldots(24.3)$$

$$\text{Eqn. (C. 13)} \Longrightarrow \mathrm{F_1S} = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)}$$

Reciprocal of square of above eqn.,

$$\frac{1}{\mathrm{F_1S^2}} = \frac{\sin^2(\theta°)}{a^2\sin^2(\theta°) + b^2} \qquad \ldots(24.4)$$

Equating eqns. (24.3) & (24.4)

$$\therefore \left(\frac{1}{\mathrm{F_1T \times F_2T}}\right) - \left(\frac{1}{\mathrm{PF_1 \times PF_2}}\right) = \frac{1}{\mathrm{F_1S^2}} \qquad \ldots(24.5)$$

Eqn. (24.5) is mathematical expression of the theorem.

THEOREM- 25:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{TM \times OT = F_1T \times F_2T}$

Proof of the theorem

$$\text{Eqn. (C. 9)} \Longrightarrow \mathrm{TM} = \frac{a^2\sin^2(\theta°) + b^2}{a\cos(\theta°)} \qquad \ldots(25.1)$$

$$\text{Eqn. (C. 1)} \Longrightarrow \mathrm{OT} = a\cos(\theta°) \qquad \ldots(25.2)$$

Multiplying eqn. (25.1) by (25.2),

$$\mathrm{TM \times OT} = \left(\frac{a^2\sin^2(\theta°) + b^2}{\cancel{a\cos(\theta°)}}\right) \times \cancel{a\cos(\theta°)}$$

$$\therefore \mathrm{TM \times OT} = a^2\sin^2(\theta°) + b^2 \qquad \ldots(25.3)$$

$$\text{Eqn. (E. 10)} \Longrightarrow \mathrm{F_1T \times F_2T} = a^2\sin^2(\theta°) + b^2 \qquad \ldots(25.4)$$

Equating eqns. (25.3) & (25.4)

$$\mathrm{TM \times OT = F_1T \times F_2T} \qquad \ldots(25.5)$$

Eqn. (25.5) is mathematical expression of the theorem.

THEOREM- 26:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{(OP^2 - VR^2) \times b^2 = PQ^2 \times a^2}$

Proof of the theorem

$$\text{Eqn. (B.5)} \Longrightarrow \mathrm{OP} = \frac{\sqrt{a^2 + b^2\sin^2(\theta°)}}{\cos(\theta°)}$$

$$\therefore \mathrm{OP}^2 = \frac{a^2 + b^2\sin^2(\theta°)}{\cos^2(\theta°)} \quad \ldots(26.1)$$

$$\text{Eqn. (B.16)} \Longrightarrow \mathrm{VR} = \frac{\sqrt{a^2\cos^2(\theta°) + b^2\sin^2(\theta°)}}{\cos(\theta°)}$$

Squaring above eqn.,

$$\therefore \mathrm{VR}^2 = \frac{a^2\cos^2(\theta°) + b^2\sin^2(\theta°)}{\cos^2(\theta°)} \quad \ldots(26.2)$$

Subtracting (26.2) from (26.1)

$$\mathrm{OP}^2 - \mathrm{VR}^2 = \left(\frac{a^2 + b^2\sin^2(\theta°)}{\cos^2(\theta°)}\right) - \left(\frac{a^2\cos^2(\theta°) + b^2\sin^2(\theta°)}{\cos^2(\theta°)}\right)$$

$$\therefore \mathrm{OP}^2 - \mathrm{VR}^2 = \frac{a^2 - a^2\cos^2(\theta°) \cancel{+ b^2\sin^2(\theta°) - b^2\sin^2(\theta°)}}{\cos^2(\theta°)}$$

$$\therefore \mathrm{OP}^2 - \mathrm{VR}^2 = \frac{a^2\left(1 - \cos^2(\theta°)\right)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{OP}^2 - \mathrm{VR}^2 = \frac{a^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

Multiplying the above eqn. by b^2

$$\therefore (\mathrm{OP}^2 - \mathrm{VR}^2) \times b^2 = \frac{a^2b^2\sin^2(\theta°)}{\cos^2(\theta°)} \quad \ldots(26.3)$$

$$\text{Eqn. (B.5)} \Longrightarrow \mathrm{PQ} = \frac{b\sin(\theta°)}{\cos(\theta°)}$$

$$\therefore \mathrm{PQ}^2 = \frac{b^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

Multiplying the above eqn. by a^2

$$\therefore \mathrm{PQ}^2 \times a^2 = \frac{a^2b^2\sin^2(\theta°)}{\cos^2(\theta°)} \quad \ldots(26.4)$$

Equating eqns. (26.3) & (26.4),

$$(\mathrm{OP}^2 - \mathrm{VR}^2) \times b^2 = \mathrm{PQ}^2 \times a^2 \quad \ldots(26.5)$$

Eqn. (26.5) is mathematical expression of the theorem.

THEOREM- 27:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{F_1R \times OT = BT \times a}$$

Proof of the theorem

$$\text{Eqn. (B.17)} \Longrightarrow \mathrm{F_1R} = \frac{\sqrt{\mathrm{a}^2\cos^2(\theta°) + \mathrm{b}^2}}{\cos(\theta°)} \quad \ldots(27.1)$$

$$\text{Eqn. (C.1)} \Longrightarrow \mathrm{OT} = \mathrm{a}\cos(\theta°) \quad \ldots(27.2)$$

Multiplying eqn. (27.1) by (27.2),

$$\mathrm{F_1R \times OT} = \left(\frac{\sqrt{\mathrm{a}^2\cos^2(\theta°) + \mathrm{b}^2}}{\cancel{\cos(\theta°)}}\right) \times \mathrm{a}\cancel{\cos(\theta°)}$$

$$\therefore \mathrm{F_1R \times OT} = \mathrm{a}\sqrt{\mathrm{a}^2\cos^2(\theta°) + \mathrm{b}^2} \quad \ldots(27.3)$$

$$\text{Eqn. (C.3)} \Longrightarrow \mathrm{BT} = \sqrt{\mathrm{a}^2\cos^2(\theta°) + \mathrm{b}^2}$$

$$\therefore \mathrm{BT \times a} = \mathrm{a}\sqrt{\mathrm{a}^2\cos^2(\theta°) + \mathrm{b}^2} \quad \ldots(27.4)$$

Equating eqns. (27.3) & (27.4),

$$\mathrm{F_1R \times OT = BT \times a} \quad \ldots(27.5)$$

Eqn. (27.5) is mathematical expression of the theorem.

THEOREM- 28:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{OZ \times OT^2 = OW \times a^2}$$

Proof of the theorem

$$\text{Eqn. (D.12)} \Longrightarrow \mathrm{OZ} = \frac{\mathrm{a}^2\sin(\theta°)}{\mathrm{b}\cos(\theta°)} \quad \ldots(28.1)$$

$$\text{Eqn. (C.1)} \Longrightarrow \mathrm{OT} = \mathrm{a}\cos(\theta°) \quad \ldots(28.2)$$

Multiplying the eqns. (28.1) & (28.2),

$$\mathrm{OZ \times OT^2} = \left(\frac{\mathrm{a}^2\sin(\theta°)}{\mathrm{b}\cancel{\cos(\theta°)}}\right) \times \mathrm{a}^2\cos^{\cancel{2}}(\theta°)$$

$$\therefore \mathrm{OZ \times OT^2} = \frac{\mathrm{a}^4\sin(\theta°)\cos(\theta°)}{\mathrm{b}} \quad \ldots(28.3)$$

$$\text{Eqn. (D.9)} \Longrightarrow \mathrm{OW} = \frac{\mathrm{a}^2\sin(\theta°)\cos(\theta°)}{\mathrm{b}}$$

Multiplying the above eqn. by a^2

$$\mathrm{OW \times a^2} = \frac{\mathrm{a}^4\sin(\theta°)\cos(\theta°)}{\mathrm{b}} \quad \ldots(28.4)$$

Equating eqns. (28.3) & (28.4),

$$\mathrm{OZ \times OT^2 = OW \times a^2} \quad \ldots(28.5)$$

Eqn. (28.5) is mathematical expression of the theorem.

THEOREM- 29:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{OJ \times {OF_1}^2 = OM \times OS^2}$$

Proof of the theorem

$$\text{Eqn. (D. 13)} \Longrightarrow \mathrm{OJ} = \frac{b^2\cos(\theta°)}{a\sin^2(\theta°)} \qquad \ldots(29.1)$$

$$\text{Eqn. (A. 3)} \Longrightarrow \mathrm{OF_1} = \sqrt{a^2 + b^2}$$

Squaring above eqn.,

$$\therefore {\mathrm{OF_1}}^2 = a^2 + b^2 \qquad \ldots(29.2)$$

Multiplying the eqns. (29.1) & (29.2),

$$\mathrm{OJ} \times {\mathrm{OF_1}}^2 = \left(\frac{b^2\cos(\theta°)}{a\sin^2(\theta°)}\right) \times (a^2 + b^2)$$

$$\therefore \mathrm{OJ} \times {\mathrm{OF_1}}^2 = \frac{b^2(a^2 + b^2)\cos(\theta°)}{a\sin^2(\theta°)} \qquad \ldots(29.3)$$

$$\text{Eqn. (C. 6)} \Longrightarrow \mathrm{OM} = \frac{a^2 + b^2}{a\cos(\theta°)} \qquad \ldots(29.4)$$

$$\text{Eqn. (C. 2)} \Longrightarrow \mathrm{OS} = \frac{b\cos(\theta°)}{\sin(\theta°)}$$

$$\therefore \mathrm{OS}^2 = \frac{b^2\cos^2(\theta°)}{\sin^2(\theta°)} \qquad \ldots(29.5)$$

Multiplying the eqns. (29.4) & (29.5),

$$\mathrm{OM} \times \mathrm{OS}^2 = \left(\frac{a^2 + b^2}{\cancel{a\cos(\theta°)}}\right) \times \left(\frac{b^2\cos^{\cancel{2}}(\theta°)}{\sin^2(\theta°)}\right)$$

$$\therefore \mathrm{OM} \times \mathrm{OS}^2 = \frac{b^2(a^2 + b^2)\cos(\theta°)}{a\sin^2(\theta°)} \qquad \ldots(29.6)$$

Equating eqns. (29.4) & (29.5),

$$\mathrm{OJ} \times {\mathrm{OF_1}}^2 = \mathrm{OM} \times \mathrm{OS}^2 \qquad \ldots(29.7)$$

Eqn. (29.7) is mathematical expression of the theorem.

THEOREM- 30:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{\frac{F_1T \times F_2T}{OT^2} = \frac{F_1M \times F_2M}{{OF_1}^2}}$$

Proof of the theorem

$$\text{Eqn. (E. 10)} \Longrightarrow \mathrm{F_1T} \times \mathrm{F_2T} = a^2\sin^2(\theta°) + b^2 \qquad \ldots(30.1)$$

$$\text{Eqn. (C.1)} \Longrightarrow \mathrm{OT} = a\cos(\theta°)$$

$$\therefore \mathrm{OT}^2 = a^2\cos^2(\theta°) \qquad \ldots(30.2)$$

Dividing eqn. (30.1) by (30.2),

$$\frac{F_1T \times F_2T}{\mathrm{OT}^2} = \frac{(a^2\sin^2(\theta°) + b^2)}{\left(a^2\cos^2(\theta°)\right)} \qquad \ldots(30.3)$$

$$\text{Eqn. (E.11)} \Longrightarrow F_1M \times F_2M = \frac{(a^2 + b^2)(a^2\sin^2(\theta°) + b^2)}{a^2\cos^2(\theta°)} \qquad \ldots(30.4)$$

$$\text{Eqn. (A.3)} \Longrightarrow \mathrm{OF_1} = \sqrt{a^2 + b^2}$$

$$\therefore \mathrm{OF_1}^2 = a^2 + b^2 \qquad \ldots(30.5)$$

Dividing eqn. (30.4) by (30.5),

$$\frac{F_1M \times F_2M}{\mathrm{OF_1}^2} = \frac{\cancel{(a^2 + b^2)}\left(a^2 + b^2 - a^2\cos^2(\theta°)\right)}{a^2\cos^2(\theta°)} \times \frac{1}{\cancel{a^2 + b^2}}$$

$$\therefore \frac{F_1M \times F_2M}{\mathrm{OF_1}^2} = \frac{a^2[1 - \cos^2(\theta°)] + b^2}{a^2\cos^2(\theta°)}$$

$$\therefore \frac{F_1M \times F_2M}{\mathrm{OF_1}^2} = \frac{a^2\sin^2(\theta°) + b^2}{a^2\cos^2(\theta°)} \qquad \ldots(30.6)$$

Equating eqns. (30.3) & (30.6),

$$\frac{F_1T \times F_2T}{\mathrm{OT}^2} = \frac{F_1M \times F_2M}{\mathrm{OF_1}^2} \qquad \ldots(30.7)$$

Eqn. (30.7) is mathematical expression of the theorem.

THEOREM- 31:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1},$

$\boldsymbol{VQ \times OT = VT \times a}$

Proof of the theorem

$$\text{Eqn. (B.4)} \Longrightarrow \mathrm{VQ} = \frac{a[1 - \cos(\theta°)]}{\cos(\theta°)} \qquad \ldots(31.1)$$

$$\text{Eqn. (C.1)} \Longrightarrow \mathrm{OT} = a\cos(\theta°) \qquad \ldots(31.2)$$

Multiplying the eqns. (31.1) & (31.2),

$$\mathrm{VQ} \times \mathrm{OT} = \left(\frac{a[1 - \cos(\theta°)]}{\cancel{\cos(\theta°)}}\right) \times a\cancel{\cos(\theta°)}$$

$$\therefore \mathrm{VQ} \times \mathrm{OT} = a^2[1 - \cos(\theta°)] \qquad \ldots(31.3)$$

$$\text{Eqn. (C.18)} \Longrightarrow \mathrm{VT} = a[1 - \cos(\theta°)]$$

Multiplying above eqn. by $'a'$

$$\mathrm{VT} \times a = a^2[1 - \cos(\theta°)] \qquad \ldots(31.4)$$

Equating eqns. (31.3) & (31.4),

$$\mathrm{VQ} \times \mathrm{OT} = \mathrm{VT} \times a \qquad \ldots(31.5)$$

Eqn. (31.5) is mathematical expression of the theorem.

THEOREM- 32:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{OX \times OI = PQ \times a}$

Proof of the theorem

$$\text{Eqn. (D. 10)} \Longrightarrow \text{OX} = \frac{a^2\sin(\theta°)}{b} \qquad \ldots(32.1)$$

$$\text{Eqn. (D. 7)} \Longrightarrow \text{OI} = \frac{b^2}{a\cos(\theta°)} \qquad \ldots(32.2)$$

Multiplying the eqns. (32.1) & (32.2),

$$\text{OX} \times \text{OI} = \left(\frac{a^{\cancel{2}}\sin(\theta°)}{\cancel{b}}\right) \times \left(\frac{b^{\cancel{2}}}{\cancel{a}\cos(\theta°)}\right)$$

$$\therefore \text{OX} \times \text{OI} = \frac{ab\sin(\theta°)}{\cos(\theta°)} \qquad \ldots(32.3)$$

$$\text{Eqn. (B. 2)} \Longrightarrow \text{PQ} = \frac{b\sin(\theta°)}{\cos(\theta°)}$$

Multiplying above eqn. by 'a'

$$\text{PQ} \times a = \frac{ab\sin(\theta°)}{\cos(\theta°)} \qquad \ldots(32.4)$$

Equating eqns. (32.3) & (32.4),

$$\text{OX} \times \text{OI} = \text{PQ} \times a \qquad \ldots(32.5)$$

Eqn. (32.5) is mathematical expression of the theorem.

THEOREM- 33:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{OF_1{}^2 - OB^2 = OV^2}$

Proof of the theorem

$$\text{Eqn. (A. 3)} \Longrightarrow \text{OF}_1 = \sqrt{a^2 + b^2}$$

$$\therefore \text{OF}_1{}^2 = a^2 + b^2 \qquad \ldots(33.1)$$

$$\text{Eqn. (A. 2)} \Longrightarrow \text{OB} = b$$

$$\therefore \text{OB}^2 = b^2 \qquad \ldots(33.2)$$

$$\text{Eqn. (A. 1)} \Longrightarrow \text{OV} = a \qquad \ldots(33.3)$$

Subtracting the eqn. (33.2) & (33.1),

$$\text{OF}_1{}^2 - \text{OB}^2 = a^2 \cancel{+ b^2 - b^2}$$

$$\therefore \text{OF}_1{}^2 - \text{OB}^2 = a^2$$

$$\therefore \text{OF}_1{}^2 - \text{OB}^2 = \text{OV}^2 \qquad \ldots(33.4)$$

Eqn. (33.4) is mathematical expression of the theorem.

THEOREM- 34:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{UF_2 \times OF_2 = 2a^2 + b^2}$$

Proof of the theorem

$$\text{Eqn. (A. 10)} \Longrightarrow \mathrm{UF_2} = \frac{2a^2 + b^2}{\sqrt{a^2 + b^2}} \qquad \ldots(34.1)$$

$$\text{Eqn. (A. 3)} \Longrightarrow \mathrm{OF_2} = \sqrt{a^2 + b^2} \qquad \ldots(34.2)$$

Multiplying the eqns. (34.1) & (34.2),

$$\mathrm{UF_2 \times OF_2} = \left(\frac{2a^2 + b^2}{\cancel{\sqrt{a^2 + b^2}}}\right) \times \cancel{\sqrt{a^2 + b^2}}$$

$$\therefore \mathrm{UF_2 \times OF_2} = 2a^2 + b^2 \qquad \ldots(34.3)$$

Eqn. (34.3) is mathematical expression of the theorem.

THEOREM- 35:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{BT^2 + (F_1T \times F_2T) = a^2 + 2b^2}$$

Proof of the theorem

$$\text{Eqn. (C. 3)} \Longrightarrow \mathrm{BT} = \sqrt{a^2\cos^2(\theta°) + b^2}$$

$$\therefore \mathrm{BT}^2 = a^2\cos^2(\theta°) + b^2 \qquad \ldots(35.1)$$

$$\text{Eqn. (E. 10)} \Longrightarrow \mathrm{F_1T \times F_2T} = a^2\sin^2(\theta°) + b^2 \qquad \ldots(35.2)$$

Adding eqns. (35.1) & (35.2)

$$\mathrm{BT}^2 + (\mathrm{F_1T \times F_2T}) = a^2\cos^2(\theta°) + b^2 + a^2\sin^2(\theta°) + b^2$$

$$\therefore \mathrm{BT}^2 + (\mathrm{F_1T \times F_2T}) = a^2[\cos^2(\theta°) + \sin^2(\theta°)] + b^2 + b^2$$

$$\therefore \mathrm{BT}^2 + (\mathrm{F_1T \times F_2T}) = a^2 + 2b^2 \qquad \ldots(35.3)$$

Eqn. (35.3) is mathematical expression of the theorem.

THEOREM- 36:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{(PF_1 + PF_2)a = F_1F_2 \times OQ}$$

Proof of the theorem

$$\text{Eqn. (B. 12)} \Longrightarrow \mathrm{PF_1} = \frac{\sqrt{a^2 + b^2} - a\cos(\theta°)}{\cos(\theta°)} \qquad \ldots(36.1)$$

$$\text{Eqn. (B. 13)} \Longrightarrow \mathrm{PF_2} = \frac{\sqrt{a^2 + b^2} + a\cos(\theta°)}{\cos(\theta°)} \qquad \ldots(36.2)$$

Adding eqns. (36.1) & (36.2),

$$PF_1 + PF_2 = \left(\frac{\sqrt{a^2+b^2} - a\cos(\theta°)}{\cos(\theta°)}\right) + \left(\frac{\sqrt{a^2+b^2} + a\cos(\theta°)}{\cos(\theta°)}\right)$$

$$\therefore PF_1 + PF_2 = \frac{\sqrt{a^2+b^2} \cancel{- a\cos(\theta°)} + \sqrt{a^2+b^2} \cancel{+ a\cos(\theta°)}}{\cos(\theta°)}$$

$$\therefore PF_1 + PF_2 = \frac{2\sqrt{a^2+b^2}}{\cos(\theta°)} \qquad \ldots(36.3)$$

$$\text{Eqn. (A.4)} \Longrightarrow F_1F_2 = 2\sqrt{a^2+b^2} \qquad \ldots(36.4)$$

$$\text{Eqn. (B.1)} \Longrightarrow OQ = \frac{a}{\cos(\theta°)} \qquad \ldots(36.5)$$

Multiplying eqns. (36.4) & (36.5),

$$F_1F_2 \times OQ = \left(2\sqrt{a^2+b^2}\right) \times \frac{a}{\cos(\theta°)}$$

$$\therefore \frac{F_1F_2 \times OQ}{a} = \frac{2\sqrt{a^2+b^2}}{\cos(\theta°)} \qquad \ldots(36.6)$$

Equating Eqns. (36.3) & (36.6),

$$PF_1 + PF_2 = \frac{F_1F_2 \times OQ}{a}$$

$$\therefore (PF_1 + PF_2)a = F_1F_2 \times OQ \qquad \ldots(36.7)$$

Eqn. (36.7) is mathematical expression of the theorem.

THEOREM- 37:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{PL + TW = PN}$

Proof of the theorem

$$\text{Eqn. (D.2)} \Longrightarrow PL = \frac{a\sin^2(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)} \qquad \ldots(37.1)$$

$$\text{Eqn. (D.3)} \Longrightarrow TW = \frac{a\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{b} \qquad \ldots(37.2)$$

Adding eqns. (37.1) & (37.2),

$$PL + TW = \left(\frac{a\sin^2(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}\right) + \left(\frac{a\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{b}\right)$$

$$\therefore PL + TW = \frac{a\sin^2(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2} + a\cos^2(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}$$

$$\therefore PL + TW = \frac{\sqrt{a^2\sin^2(\theta°)+b^2} \times [a\sin^2(\theta°) + a\cos^2(\theta°)]}{b\cos(\theta°)}$$

$$\therefore \mathrm{PL} + \mathrm{TW} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2} \times [\sin^2(\theta°) + \cos^2(\theta°)]}{\mathrm{b}\cos(\theta°)}$$

$$\therefore \mathrm{PL} + \mathrm{TW} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \qquad \ldots(37.3)$$

$$\text{Eqn. (C. 20)} \Longrightarrow \mathrm{PN} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \qquad \ldots(37.4)$$

Equating Eqns. (37.3) & (37.4),

$$\mathrm{PL} + \mathrm{TW} = \mathrm{PN} \qquad \ldots(37.5)$$

Eqn. (37.5) is mathematical expression of the theorem.

THEOREM- 38:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{RA' \times VH' = RS \times OX = a \times OQ}$

Proof of the theorem

$$\text{Eqn. (E. 3)} \Longrightarrow \mathrm{RA'} = \frac{\mathrm{a}^2\sin(\theta°)}{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}} \qquad \ldots(38.1)$$

$$\text{Eqn. (E. 4)} \Longrightarrow \mathrm{VH'} = \frac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)} \qquad \ldots(38.2)$$

Multiplying eqns. (38.1) & (38.2),

$$\mathrm{RA'} \times \mathrm{VH'} = \left(\frac{\mathrm{a}^2\cancel{\sin(\theta°)}}{\cos(\theta°)\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}\right) \times \left(\frac{\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}{\cancel{\sin(\theta°)}}\right)$$

$$\therefore \mathrm{RA'} \times \mathrm{VH'} = \frac{\mathrm{a}^2}{\cos(\theta°)} \qquad \ldots(38.3)$$

$$\text{Eqn. (C. 18)} \Longrightarrow \mathrm{RS} = \frac{\mathrm{b}}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(38.4)$$

$$\text{Eqn. (D. 10)} \Longrightarrow \mathrm{OX} = \frac{\mathrm{a}^2\sin(\theta°)}{\mathrm{b}} \qquad \ldots(38.5)$$

Multiplying eqns. (38.4) & (38.5),

$$\mathrm{RS} \times \mathrm{OX} = \left(\frac{\cancel{\mathrm{b}}}{\cancel{\sin(\theta°)}\cos(\theta°)}\right) \times \left(\frac{\mathrm{a}^2\cancel{\sin(\theta°)}}{\cancel{\mathrm{b}}}\right)$$

$$\therefore \mathrm{RS} \times \mathrm{OX} = \frac{\mathrm{a}^2}{\cos(\theta°)} \qquad \ldots(38.6)$$

$$\text{Eqn. (B. 1)} \Longrightarrow \mathrm{OQ} = \frac{\mathrm{a}}{\cos(\theta°)}$$

Multiplying above eqn. by ‘a’,

$$\mathrm{OQ} \times \mathrm{a} = \frac{\mathrm{a}^2}{\cos(\theta°)} \qquad \ldots(38.7)$$

Equating Eqns. (38.3), (38.6) & (38.7),

$$\mathrm{RA'} \times \mathrm{VH'} = \mathrm{RS} \times \mathrm{OX} = \mathrm{a} \times \mathrm{OQ} \qquad \ldots(38.8)$$

Eqn. (38.8) is mathematical expression of the theorem.

THEOREM- 39:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{PF_1 \times F_2T = PF_2 \times F_1T}$

Proof of the theorem

$$\text{Eqn. (B.12)} \Longrightarrow \mathrm{PF_1} = \frac{\sqrt{a^2 + b^2} - a\cos(\theta°)}{\cos(\theta°)} \qquad \ldots(39.1)$$

$$\text{Eqn. (C.12)} \Longrightarrow \mathrm{F_2T} = \sqrt{a^2 + b^2} + a\cos(\theta°) \qquad \ldots(39.2)$$

Multiplying eqns. (39.1) & (39.2),

$$\mathrm{PF_1} \times \mathrm{F_2T} = \left(\frac{\sqrt{a^2 + b^2} - a\cos(\theta°)}{\cos(\theta°)}\right) \times \left(\sqrt{a^2 + b^2} + a\cos(\theta°)\right)$$

$$\therefore \mathrm{PF_1} \times \mathrm{F_2T} = \frac{a^2 + b^2 - a^2\cos^2(\theta°)}{\cos(\theta°)}$$

$$\therefore \mathrm{PF_1} \times \mathrm{F_2T} = \frac{a^2[1 - \cos^2(\theta°)] + b^2}{\cos(\theta°)}$$

$$\therefore \mathrm{PF_1} \times \mathrm{F_2T} = \frac{a^2\sin^2(\theta°) + b^2}{\cos(\theta°)} \qquad \ldots(39.3)$$

$$\text{Eqn. (B.13)} \Longrightarrow \mathrm{PF_2} = \frac{\sqrt{a^2 + b^2} + a\cos(\theta°)}{\cos(\theta°)} \qquad \ldots(39.4)$$

$$\text{Eqn. (C.11)} \Longrightarrow \mathrm{F_1T} = \sqrt{a^2 + b^2} - a\cos(\theta°) \qquad \ldots(39.5)$$

Multiplying eqns. (39.4) & (39.5),

$$\mathrm{PF_2} \times \mathrm{F_1T} = \left(\frac{\sqrt{a^2 + b^2} + a\cos(\theta°)}{\cos(\theta°)}\right) \times \left(\sqrt{a^2 + b^2} - a\cos(\theta°)\right)$$

$$\therefore \mathrm{PF_2} \times \mathrm{F_1T} = \frac{a^2 + b^2 - a^2\cos^2(\theta°)}{\cos(\theta°)}$$

$$\therefore \mathrm{PF_2} \times \mathrm{F_1T} = \frac{a^2[1 - \cos^2(\theta°)] + b^2}{\cos(\theta°)}$$

$$\therefore \mathrm{PF_2} \times \mathrm{F_1T} = \frac{a^2\sin^2(\theta°) + b^2}{\cos(\theta°)} \qquad \ldots(39.6)$$

Equating eqns. (39.5) & (39.6),

$$\mathrm{PF_1} \times \mathrm{F_2T} = \mathrm{PF_2} \times \mathrm{F_1T} \qquad \ldots(39.7)$$

Eqn. (39.7) is mathematical expression of the theorem.

THEOREM- 40:

$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1,}$

$\boldsymbol{PM \times VX = PT \times VH'}$

Proof of the theorem

$$\text{Eqn. (C. 14)} \Longrightarrow \mathrm{PM} = \frac{\mathrm{b}\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{a}\cos(\theta°)} \quad \ldots(40.1)$$

$$\text{Eqn. (D. 4)} \Longrightarrow \mathrm{VX} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{b}} \quad \ldots(40.2)$$

Multiplying eqns. (40.1) & (40.2),

$$\mathrm{PM} \times \mathrm{VX} = \left(\frac{\cancel{\mathrm{b}}\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\cancel{\mathrm{a}}\cos(\theta°)}\right) \times \left(\frac{\cancel{\mathrm{a}}\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\cancel{\mathrm{b}}}\right)$$

$$\therefore \mathrm{PM} \times \mathrm{VX} = \frac{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}{\cos(\theta°)} \quad \ldots(40.3)$$

$$\text{Eqn. (C. 17)} \Longrightarrow \mathrm{PT} = \frac{\sin(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\cos(\theta°)} \quad \ldots(40.4)$$

$$\text{Eqn. (E. 4)} \Longrightarrow \mathrm{VH'} = \frac{\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\sin(\theta°)} \quad \ldots(40.5)$$

Multiplying eqns. (40.4) & (40.5),

$$\mathrm{PT} \times \mathrm{VH'} = \left(\frac{\cancel{\sin(\theta°)}\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\cos(\theta°)}\right) \times \left(\frac{\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\cancel{\sin(\theta°)}}\right)$$

$$\therefore \mathrm{PT} \times \mathrm{VH'} = \frac{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}{\cos(\theta°)} \quad \ldots(40.6)$$

Equating Eqns. (40.5) & (40.6),

$$\mathrm{PM} \times \mathrm{VX} = \mathrm{PT} \times \mathrm{VH'} \quad \ldots(40.7)$$

Eqn. (40.7) is mathematical expression of the theorem.

THEOREM- 41:

$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1,}$

$\boldsymbol{F_1N \times TW = F_1Y \times VX}$

Proof of the theorem

$$\text{Eqn. (B. 10)} \Longrightarrow \mathrm{F_1N} = \frac{\sqrt{\mathrm{a}^2+\mathrm{b}^2} \times \sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \quad \ldots(41.1)$$

$$\text{Eqn. (D. 3)} \Longrightarrow \mathrm{TW} = \frac{\mathrm{a}\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{b}} \quad \ldots(41.2)$$

Multiplying eqns. (41.1) & (41.2),

$$F_1N \times TW = \left(\frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b\cancel{\cos(\theta°)}}\right) \times \left(\frac{a\cancel{\cos(\theta°)}\sqrt{a^2\sin^2(\theta°)+b^2}}{b}\right)$$

$$\therefore F_1N \times TW = \frac{a\sqrt{a^2+b^2} \times (a^2\sin^2(\theta°)+b^2)}{b^2} \qquad \ldots(41.3)$$

$$\text{Eqn. (D. 5)} \Longrightarrow F_1Y = \frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b} \qquad \ldots(41.4)$$

$$\text{Eqn. (D. 4)} \Longrightarrow VX = \frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b} \qquad \ldots(41.5)$$

Multiplying eqns. (41.3) & (41.4),

$$F_1Y \times VX = \left(\frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b}\right) \times \left(\frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b}\right)$$

$$\therefore F_1Y \times VX = \frac{a\sqrt{a^2+b^2} \times (a^2\sin^2(\theta°)+b^2)}{b^2} \qquad \ldots(41.6)$$

Equating Eqns. (24.5) & (24.6),

$$F_1N \times TW = F_1Y \times VX \qquad \ldots(41.7)$$

Eqn. (41.7) is mathematical expression of the theorem.

THEOREM- 42:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{F_1N \times TS = F_1Y \times F_1S = F_1I' \times VX}$

Proof of the theorem

$$\text{Eqn. (B. 10)} \Longrightarrow F_1N = \frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)} \qquad \ldots(42.1)$$

$$\text{Eqn. (C. 10)} \Longrightarrow TS = \frac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)} \qquad \ldots(42.2)$$

Multiplying eqns. (42.1) & (42.2),

$$F_1N \times TS = \left(\frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b\cancel{\cos(\theta°)}}\right) \times \left(\frac{\cancel{\cos(\theta°)}\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)}\right)$$

$$\therefore F_1N \times TS = \frac{\sqrt{a^2+b^2}(a^2\sin^2(\theta°)+b^2)}{b\sin(\theta°)} \qquad \ldots(42.3)$$

$$\text{Eqn. (D. 5)} \Longrightarrow F_1Y = \frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b} \qquad \ldots(42.4)$$

$$\text{Eqn. (C. 13)} \Longrightarrow F_1S = \frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)} \qquad \ldots(42.5)$$

Multiplying eqns. (42.4) & (42.5),

$$F_1Y \times F_1S = \left(\frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b}\right) \times \left(\frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)}\right)$$

$$\therefore\ F_1Y \times F_1S = \frac{\sqrt{a^2+b^2}\left(a^2\sin^2(\theta°)+b^2\right)}{b\sin(\theta°)} \qquad \ldots(42.6)$$

$$\text{Eqn. (E.5)} \Longrightarrow F_1I' = \frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{a\sin(\theta°)} \qquad \ldots(42.7)$$

$$\text{Eqn. (D.4)} \Longrightarrow VX = \frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b} \qquad \ldots(42.8)$$

Multiplying eqns. (42.7) & (42.8),

$$F_1I' \times VX = \left(\frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{\cancel{a}\sin(\theta°)}\right) \times \left(\frac{\cancel{a}\sqrt{a^2\sin^2(\theta°)+b^2}}{b}\right)$$

$$\therefore\ F_1I' \times VX = \frac{\sqrt{a^2+b^2}\left(a^2\sin^2(\theta°)+b^2\right)}{b\sin(\theta°)} \qquad \ldots(42.9)$$

Equating Eqns. (42.3), (42.6) & (42.9),

$$F_1N \times TS = F_1Y \times F_1S = F_1I' \times VX \qquad \ldots(42.10)$$

Eqn. (42.10) is mathematical expression of the theorem.

THEOREM- 43:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{F_1N \times F_1I' = F_1S \times MN = VX \times ML'}$

Proof of the theorem

$$\text{Eqn. (B.10)} \Longrightarrow F_1N = \frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)} \qquad \ldots(43.1)$$

$$\text{Eqn. (E.5)} \Longrightarrow F_1I' = \frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{\cancel{a}\sin(\theta°)} \qquad \ldots(43.2)$$

Multiplying eqns. (43.1) & (43.2),

$$F_1N \times F_1I' = \left(\frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}\right) \times \left(\frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{a\sin(\theta°)}\right)$$

$$\therefore\ F_1N \times F_1I' = \frac{(a^2+b^2)\left(a^2\sin^2(\theta°)+b^2\right)}{ab\sin(\theta°)\cos(\theta°)} \qquad \ldots(43.3)$$

$$\text{Eqn. (C.13)} \Longrightarrow F_1S = \frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)} \qquad \ldots(43.4)$$

$$\text{Eqn. (C.21)} \Longrightarrow MN = \frac{(a^2+b^2)\sqrt{a^2\sin^2(\theta°)+b^2}}{ab\cos(\theta°)} \qquad \ldots(43.5)$$

Multiplying eqns. (43.4) & (43.5),

$$\mathrm{F_1S \times MN} = \left(\frac{\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{sin(\theta°)}}\right) \times \left(\frac{\mathrm{(a^2 + b^2)}\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{abcos(\theta°)}}\right)$$

$$\therefore \mathrm{F_1S \times MN} = \frac{\mathrm{(a^2 + b^2)(a^2 sin^2(\theta°) + b^2)}}{\mathrm{absin(\theta°)cos(\theta°)}} \qquad \ldots(43.6)$$

$$\mathrm{Eqn. (D.4)} \Longrightarrow \mathrm{VX} = \frac{\mathrm{a}\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{b}} \qquad \ldots(43.7)$$

$$\mathrm{Eqn. (E.7)} \Longrightarrow \mathrm{ML'} = \frac{\mathrm{(a^2 + b^2)}\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{a^2 sin(\theta°)cos(\theta°)}} \qquad \ldots(43.8)$$

Multiplying eqns. (43.7) & (43.8),

$$\mathrm{VX \times ML'} = \left(\frac{\cancel{\mathrm{a}}\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{b}}\right) \times \left(\frac{\mathrm{(a^2 + b^2)}\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{a^{\cancel{2}} sin(\theta°)cos(\theta°)}}\right)$$

$$\therefore \mathrm{VX \times ML'} = \frac{\mathrm{(a^2 + b^2)(a^2 sin^2(\theta°) + b^2)}}{\mathrm{absin(\theta°)cos(\theta°)}} \qquad \ldots(43.9)$$

Equating Eqns. (43.3), (43.6) & (43.9),

$$\mathrm{F_1N \times F_1I' = F_1S \times MN = VX \times ML'} \qquad \ldots(43.10)$$

Eqn. (43.10) is mathematical expression of the theorem.

THEOREM- 44:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{PS \times PL = PT \times QZ = TL \times OQ = TM \times OZ}$

Proof of the theorem

$$\mathrm{Eqn. (C.16)} \Longrightarrow \mathrm{PS} = \frac{\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{sin(\theta°)cos(\theta°)}} \qquad \ldots(44.1)$$

$$\mathrm{Eqn. (D.2)} \Longrightarrow \mathrm{PL} = \frac{\mathrm{asin^2(\theta°)}\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{bcos(\theta°)}} \qquad \ldots(44.2)$$

Multiplying eqns. (44.1) & (44.2),

$$\mathrm{PS \times PL} = \left(\frac{\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\cancel{\mathrm{sin(\theta°)}}\mathrm{cos(\theta°)}}\right) \times \left(\frac{\mathrm{asin^{\cancel{2}}(\theta°)}\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{bcos(\theta°)}}\right)$$

$$\therefore \mathrm{PS \times PL} = \frac{\mathrm{asin(\theta°)(a^2 sin^2(\theta°) + b^2)}}{\mathrm{bcos^2(\theta°)}} \qquad \ldots(44.3)$$

$$\mathrm{Eqn. (C.15)} \Longrightarrow \mathrm{PT} = \frac{\mathrm{sin(\theta°)}\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{cos(\theta°)}} \qquad \ldots(44.4)$$

$$\mathrm{Eqn. (D.6)} \Longrightarrow \mathrm{QZ} = \frac{\mathrm{a}\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{bcos(\theta°)}} \qquad \ldots(44.5)$$

Multiplying eqns. (44.4) & (44.5),

$$\mathrm{PT} \times \mathrm{QZ} = \left(\frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)}\right) \times \left(\frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)}\right)$$

$$\therefore \mathrm{PT} \times \mathrm{QZ} = \frac{a\sin(\theta°)(a^2\sin^2(\theta°) + b^2)}{b\cos^2(\theta°)} \quad \ldots(44.6)$$

$$\text{Eqn. (D. 1)} \Longrightarrow \mathrm{TL} = \frac{\sin(\theta°)[a^2\sin^2(\theta°) + b^2]}{b\cos(\theta°)} \quad \ldots(44.7)$$

$$\text{Eqn. (B. 1)} \Longrightarrow \mathrm{OQ} = \frac{a}{\cos(\theta°)} \quad \ldots(44.8)$$

Multiplying eqns. (44.7) & (44.8),

$$\mathrm{TL} \times \mathrm{OQ} = \left(\frac{\sin(\theta°)[a^2\sin^2(\theta°) + b^2]}{b\cos(\theta°)}\right) \times \left(\frac{a}{\cos(\theta°)}\right)$$

$$\therefore \mathrm{TL} \times \mathrm{OQ} = \frac{a\sin(\theta°)[a^2\sin^2(\theta°) + b^2]}{b\cos^2(\theta°)} \quad \ldots(44.9)$$

$$\text{Eqn. (C. 9)} \Longrightarrow \mathrm{TM} = \frac{a^2\sin^2(\theta°) + b^2}{a\cos(\theta°)} \quad \ldots(44.10)$$

$$\text{Eqn. (D. 12)} \Longrightarrow \mathrm{OZ} = \frac{a^2\sin(\theta°)}{b\cos(\theta°)} \quad \ldots(44.11)$$

Multiplying eqns. (44.10) & (44.11),

$$\mathrm{TM} \times \mathrm{OZ} = \left(\frac{a^2\sin^2(\theta°) + b^2}{\cancel{a}\cos(\theta°)}\right) \times \left(\frac{a^{\cancel{2}}\sin(\theta°)}{b\cos(\theta°)}\right)$$

$$\therefore \mathrm{TM} \times \mathrm{OZ} = \frac{a\sin(\theta°)[a^2\sin^2(\theta°) + b^2]}{b\cos^2(\theta°)} \quad \ldots(44.12)$$

Equating eqns. (44.3), (44.6), (44.9) & (44.12),

$$\mathrm{PS} \times \mathrm{PL} = \mathrm{PT} \times \mathrm{QZ} = \mathrm{TL} \times \mathrm{OQ} = \mathrm{TM} \times \mathrm{OZ} \quad \ldots(44.13)$$

Eqn. (44.7) is mathematical expression of the theorem.

THEOREM- 45:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{TM \times PN = PT \times SN = PS \times TL}$

Proof of the theorem

$$\text{Eqn. (C. 9)} \Longrightarrow \mathrm{TM} = \frac{a^2\sin^2(\theta°) + b^2}{a\cos(\theta°)} \quad \ldots(45.1)$$

$$\text{Eqn. (C. 19)} \Longrightarrow \mathrm{PN} = \frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)} \quad \ldots(45.2)$$

Multiplying eqns. (45.1) & (45.2),

$$\mathrm{TM} \times \mathrm{PN} = \left(\frac{a^2\sin^2(\theta°) + b^2}{a\cos(\theta°)}\right) \times \left(\frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)}\right)$$

$$\therefore \mathrm{TM} \times \mathrm{PN} = \frac{\left(a^2\sin^2(\theta°) + b^2\right)^{3/2}}{b\cos^2(\theta°)} \qquad \ldots(45.3)$$

$$\text{Eqn. (C.15)} \Longrightarrow \mathrm{PT} = \frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)} \qquad \ldots(45.4)$$

$$\text{Eqn. (C.20)} \Longrightarrow \mathrm{SN} = \frac{a^2\sin^2(\theta°) + b^2}{b\sin(\theta°)\cos(\theta°)} \qquad \ldots(45.5)$$

Multiplying eqns. (45.4) & (45.5),

$$\mathrm{PT} \times \mathrm{SN} = \left(\frac{\cancel{\sin(\theta°)}\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)}\right) \times \left(\frac{a^2\sin^2(\theta°) + b^2}{b\cancel{\sin(\theta°)}\cos(\theta°)}\right)$$

$$\therefore \mathrm{PT} \times \mathrm{SN} = \frac{\left(a^2\sin^2(\theta°) + b^2\right)^{3/2}}{b\cos^2(\theta°)} \qquad \ldots(45.6)$$

$$\text{Eqn. (C.18)} \Longrightarrow \mathrm{PS} = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(45.7)$$

$$\text{Eqn. (D.1)} \Longrightarrow \mathrm{TL} = \frac{\sin(\theta°)\left[a^2\sin^2(\theta°) + b^2\right]}{b\cos(\theta°)} \qquad \ldots(45.8)$$

Multiplying eqns. (45.7) & (45.8),

$$\mathrm{PS} \times \mathrm{TL} = \left(\frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\cancel{\sin(\theta°)}\cos(\theta°)}\right) \times \left(\frac{\cancel{\sin(\theta°)}\left[a^2\sin^2(\theta°) + b^2\right]}{b\cos(\theta°)}\right)$$

$$\therefore \mathrm{PS} \times \mathrm{TL} = \frac{\left(a^2\sin^2(\theta°) + b^2\right)^{3/2}}{b\cos^2(\theta°)} \qquad \ldots(45.9)$$

Equating eqns. (45.3), (45.6) & (45.9),

$$\mathrm{TM} \times \mathrm{PN} = \mathrm{PT} \times \mathrm{SN} \ \times \mathrm{PS} \times \mathrm{TL} \qquad \ldots(45.10)$$

Eqn. (45.10) is mathematical expression of the theorem.

THEOREM- 46:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$

$\boldsymbol{PT \times TM = PM \times TL}$

Proof of the theorem

$$\text{Eqn. (C.15)} \Longrightarrow \mathrm{PT} = \frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)} \qquad \ldots(46.1)$$

$$\text{Eqn. (C.9)} \Longrightarrow \mathrm{TM} = \frac{a^2\sin^2(\theta°) + b^2}{a\cos(\theta°)} \qquad \ldots(46.2)$$

Multiplying eqns. (46.1) & (46.2),

$$\mathrm{PT} \times \mathrm{TM} = \left(\frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)}\right) \times \left(\frac{a^2\sin^2(\theta°) + b^2}{a\cos(\theta°)}\right)$$

$$\therefore \text{PT} \times \text{TM} = \frac{\sin(\theta°)(a^2\sin^2(\theta°) + b^2)^{3/2}}{a\cos^2(\theta°)} \qquad \ldots(46.3)$$

$$\text{Eqn. (C.14)} \Longrightarrow \text{PM} = \frac{b\sqrt{a^2\sin^2(\theta°) + b^2}}{a\cos(\theta°)} \qquad \ldots(46.4)$$

$$\text{Eqn. (D.1)} \Longrightarrow \text{TL} = \frac{\sin(\theta°)[a^2\sin^2(\theta°) + b^2]}{b\cos(\theta°)} \qquad \ldots(46.5)$$

Multiplying eqns. (46.4) & (46.5),

$$\text{PM} \times \text{TL} = \left(\frac{\cancel{b}\sqrt{a^2\sin^2(\theta°) + b^2}}{a\cos(\theta°)}\right) \times \frac{\sin(\theta°)[a^2\sin^2(\theta°) + b^2]}{\cancel{b}\cos(\theta°)}$$

$$\therefore \text{PM} \times \text{TL} = \frac{\sin(\theta°)(a^2\sin^2(\theta°) + b^2)^{3/2}}{a\cos^2(\theta°)} \qquad \ldots(46.6)$$

Equating eqns. (46.3) & (46.6),

$$\text{PT} \times \text{TM} = \text{PM} \times \text{TL} \qquad \ldots(46.7)$$

Eqn. (46.7) is mathematical expression of the theorem.

THEOREM- 47:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{TL \times ML' = MN \times TM}$

Proof of the theorem

$$\text{Eqn. (D.1)} \Longrightarrow \text{TL} = \frac{\sin(\theta°)[a^2\sin^2(\theta°) + b^2]}{b\cos(\theta°)} \qquad \ldots(47.1)$$

$$\text{Eqn. (E.7)} \Longrightarrow \text{ML}' = \frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{a^2\sin(\theta°)\cos(\theta°)} \qquad \ldots(47.2)$$

Multiplying eqns. (47.1) & (47.2),

$$\text{TL} \times \text{ML}' = \left(\frac{\cancel{\sin(\theta°)}[a^2\sin^2(\theta°) + b^2]}{b\cos(\theta°)}\right) \times \left(\frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{a^2\cancel{\sin(\theta°)}\cos(\theta°)}\right)$$

$$\therefore \text{TL} \times \text{ML}' = \frac{(a^2 + b^2) \times (a^2\sin^2(\theta°) + b^2)^{3/2}}{a^2b\cos^2(\theta°)} \qquad \ldots(47.3)$$

$$\text{Eqn. (C.21)} \Longrightarrow \text{MN} = \frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{ab\cos(\theta°)} \qquad \ldots(47.4)$$

$$\text{Eqn. (C.9)} \Longrightarrow \text{TM} = \frac{a^2\sin^2(\theta°) + b^2}{a\cos(\theta°)} \qquad \ldots(47.5)$$

Multiplying eqns. (47.4) & (47.5),

$$\text{MN} \times \text{TM} = \left(\frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{ab\cos(\theta°)}\right) \times \left(\frac{a^2\sin^2(\theta°) + b^2}{a\cos(\theta°)}\right)$$

$$\therefore \mathrm{MN} \times \mathrm{TM} = \frac{(a^2 + b^2) \times (a^2 \sin^2(\theta°) + b^2)^{3/2}}{a^2 b \cos^2(\theta°)} \quad \ldots(47.6)$$

Equating eqns. (47.3) & (47.6),

$$\mathrm{TL} \times \mathrm{ML}' = \mathrm{MN} \times \mathrm{TM} \quad \ldots(47.7)$$

Eqn. (47.7) is mathematical expression of the theorem.

THEOREM- 48:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{TL \times TS = TM \times TW}$$

Proof of the theorem

$$\text{Eqn. (D.1)} \Longrightarrow \mathrm{TL} = \frac{\sin(\theta°)[a^2 \sin^2(\theta°) + b^2]}{b\cos(\theta°)} \quad \ldots(48.1)$$

$$\text{Eqn. (C.10)} \Longrightarrow \mathrm{TS} = \frac{\cos(\theta°)\sqrt{a^2 \sin^2(\theta°) + b^2}}{\sin(\theta°)} \quad \ldots(48.2)$$

Multiplying eqns. (48.1) & (48.2),

$$\mathrm{TL} \times \mathrm{TS} = \left(\frac{\cancel{\sin(\theta°)}[a^2 \sin^2(\theta°) + b^2]}{b\cancel{\cos(\theta°)}}\right) \times \left(\frac{\cancel{\cos(\theta°)}\sqrt{a^2 \sin^2(\theta°) + b^2}}{\cancel{\sin(\theta°)}}\right)$$

$$\therefore \mathrm{TL} \times \mathrm{TS} = \frac{(a^2 \sin^2(\theta°) + b^2)^{3/2}}{b} \quad \ldots(48.3)$$

$$\text{Eqn. (C.9)} \Longrightarrow \mathrm{TM} = \frac{a^2 \sin^2(\theta°) + b^2}{a\cos(\theta°)} \quad \ldots(48.4)$$

$$\text{Eqn. (D.3)} \Longrightarrow \mathrm{TW} = \frac{a\cos(\theta°)\sqrt{a^2 \sin^2(\theta°) + b^2}}{b} \quad \ldots(48.5)$$

Multiplying eqns. (48.4) & (48.5),

$$\mathrm{TM} \times \mathrm{TW} = \left(\frac{a^2 \sin^2(\theta°) + b^2}{\cancel{a\cos(\theta°)}}\right) \times \left(\frac{\cancel{a\cos(\theta°)}\sqrt{a^2 \sin^2(\theta°) + b^2}}{b}\right)$$

$$\therefore \mathrm{TM} \times \mathrm{TW} = \frac{(a^2 \sin^2(\theta°) + b^2)^{3/2}}{b} \quad \ldots(48.6)$$

Equating eqns. (48.3) & (48.6),

$$\mathrm{TL} \times \mathrm{TS} = \mathrm{TM} \times \mathrm{TW} \quad \ldots(48.7)$$

Eqn. (48.7) is mathematical expression of the theorem.

THEOREM- 49:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{PS \times VH' = F_1S \times QJ'}$$

Proof of the theorem

$$\text{Eqn. (C.16)} \Longrightarrow PS = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(49.1)$$

$$\text{Eqn. (E.4)} \Longrightarrow VH' = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)} \qquad \ldots(49.2)$$

Multiplying eqns. (49.1) & (49.2),

$$PS \times VH' = \left(\frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)\cos(\theta°)}\right) \times \left(\frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)}\right)$$

$$\therefore PS \times VH' = \frac{a^2\sin^2(\theta°) + b^2}{\sin^2(\theta°)\cos(\theta°)} \qquad \ldots(49.3)$$

$$\text{Eqn. (C.13)} \Longrightarrow F_1S = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)} \qquad \ldots(49.4)$$

$$\text{Eqn. (E.6)} \Longrightarrow QJ' = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(49.5)$$

Multiplying eqns. (49.4) & (49.5),

$$F_1S \times QJ' = \left(\frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)}\right) \times \left(\frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)\cos(\theta°)}\right)$$

$$\therefore F_1S \times QJ' = \frac{a^2\sin^2(\theta°) + b^2}{\sin^2(\theta°)\cos(\theta°)} \qquad \ldots(49.6)$$

Equating eqns. (49.3) & (49.6),

$$PS \times VH' = F_1S \times QJ' \qquad \ldots(49.7)$$

Eqn. (49.7) is mathematical expression of the theorem.

THEOREM- 50:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{F_1S \times QZ = PN \times VH'}$

Proof of the theorem

$$\text{Eqn. (C.13)} \Longrightarrow F_1S = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)} \qquad \ldots(50.1)$$

$$\text{Eqn. (D.6)} \Longrightarrow QZ = \frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)} \qquad \ldots(50.2)$$

Multiplying eqns. (50.1) & (50.2),

$$F_1S \times QZ = \left(\frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)}\right) \times \left(\frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)}\right)$$

$$\therefore F_1S \times QZ = \frac{a(a^2\sin^2(\theta°) + b^2)}{b\sin(\theta°)\cos(\theta°)} \qquad \ldots(50.3)$$

$$\text{Eqn. (C.19)} \Longrightarrow \mathrm{PN} = \frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)} \qquad \ldots(50.4)$$

$$\text{Eqn. (E.4)} \Longrightarrow \mathrm{VH}' = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)} \qquad \ldots(50.5)$$

Multiplying eqns. (50.3) & (50.4),

$$\mathrm{PN} \times \mathrm{VH}' = \left(\frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)}\right) \times \left(\frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)}\right)$$

$$\therefore \mathrm{PN} \times \mathrm{VH}' = \frac{a(a^2\sin^2(\theta°) + b^2)}{b\sin(\theta°)\cos(\theta°)} \qquad \ldots(50.6)$$

Equating eqns. (50.3) & (50.6),

$$\mathrm{F_1S} \times \mathrm{QZ} = \mathrm{PN} \times \mathrm{VH}' \qquad \ldots(50.7)$$

Eqn. (50.7) is mathematical expression of the theorem.

THEOREM- 51:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{PT \times F_1Y = PL \times F_1I'}$

Proof of the theorem

$$\text{Eqn. (C.15)} \Longrightarrow \mathrm{PT} = \frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)} \qquad \ldots(51.1)$$

$$\text{Eqn. (D.5)} \Longrightarrow \mathrm{F_1Y} = \frac{\sqrt{a^2 + b^2} \times \sqrt{a^2\sin^2(\theta°) + b^2}}{b} \qquad \ldots(51.2)$$

Multiplying eqns. (51.1) & (51.2),

$$\mathrm{PT} \times \mathrm{F_1Y} = \left(\frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)}\right) \times \left(\frac{\sqrt{a^2 + b^2} \times \sqrt{a^2\sin^2(\theta°) + b^2}}{b}\right)$$

$$\therefore \mathrm{PT} \times \mathrm{F_1Y} = \frac{\sqrt{a^2 + b^2}\sin(\theta°)(a^2\sin^2(\theta°) + b^2)}{b\cos(\theta°)} \qquad \ldots(51.3)$$

$$\text{Eqn. (D.2)} \Longrightarrow \mathrm{PL} = \frac{a\sin^2(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)} \qquad \ldots(51.4)$$

$$\text{Eqn. (E.5)} \Longrightarrow \mathrm{F_1I}' = \frac{\sqrt{a^2 + b^2} \times \sqrt{a^2\sin^2(\theta°) + b^2}}{\cancel{a}\sin(\theta°)} \qquad \ldots(51.5)$$

Multiplying eqns. (51.4) & (51.5),

$$\mathrm{PL} \times \mathrm{F_1I}' = \left(\frac{\cancel{a}\sin^{\cancel{2}}(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)}\right) \times \left(\frac{\sqrt{a^2 + b^2} \times \sqrt{a^2\sin^2(\theta°) + b^2}}{\cancel{a\sin(\theta°)}}\right)$$

$$\therefore \mathrm{PL} \times \mathrm{F_1I}' = \frac{\sqrt{a^2 + b^2}\sin(\theta°)(a^2\sin^2(\theta°) + b^2)}{b\cos(\theta°)} \qquad \ldots(51.6)$$

Equating eqns. (51.3) & (51.6),

$$PT \times F_1Y = PL \times F_1I'$$

Eqn. (51.7) is mathematical expression of the theorem.

THEOREM- 52:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{(F_1M \times F_2M) \times OT = (F_1T \times F_2T) \times OM}$

Proof of the theorem

$$\text{Eqn. (E. 11)} \Longrightarrow F_1M \times F_2M = \frac{(a^2+b^2)(a^2\sin^2(\theta°)+b^2)}{a^2\cos^2(\theta°)} \qquad \ldots(52.1)$$

$$\text{Eqn. (C. 1)} \Longrightarrow OT = a\cos(\theta°) \qquad \ldots(52.2)$$

Multiplying eqns. (52.1) & (52.2),

$$F_1M \times F_2M \times OT = \left(\frac{(a^2+b^2)(a^2\sin^2(\theta°)+b^2)}{a^2\cos^2(\theta°)}\right) \times \left(a\cos(\theta°)\right)$$

$$\therefore F_1M \times F_2M \times OT = \frac{(a^2+b^2)(a^2\sin^2(\theta°)+b^2)}{a\cos(\theta°)} \qquad \ldots(52.3)$$

$$\text{Eqn. (E. 10)} \Longrightarrow F_1T \times F_2T = a^2\sin^2(\theta°) + b^2 \qquad \ldots(52.4)$$

$$\text{Eqn. (C. 6)} \Longrightarrow OM = \frac{a^2+b^2}{a\cos(\theta°)} \qquad \ldots(52.5)$$

Multiplying eqns. (52.4) & (52.5),

$$F_1T \times F_2T \times OM = (a^2\sin^2(\theta°)+b^2) \times \left(\frac{a^2+b^2}{a\cos(\theta°)}\right)$$

$$\therefore F_1T \times F_2T \times OM = \frac{(a^2+b^2)(a^2\sin^2(\theta°)+b^2)}{a\cos(\theta°)} \qquad \ldots(52.6)$$

Equating eqns. (52.3) & (52.6),

$$F_1M \times F_2M \times OT = F_1T \times F_2T \times OM \qquad \ldots(52.7)$$

Eqn. (52.7) is mathematical expression of the theorem.

THEOREM- 53:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{F_1S \times TW = TS \times VX}$

Proof of the theorem

$$\text{Eqn. (C. 13)} \Longrightarrow F_1S = \frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)} \qquad \ldots(53.1)$$

$$\text{Eqn. (D. 3)} \Longrightarrow TW = \frac{a\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{b} \qquad \ldots(53.2)$$

Multiplying eqns. (53.1) & (53.2),

$$F_1S \times TW = \left(\frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)}\right) \times \left(\frac{a\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{b}\right)$$

$$\therefore F_1S \times TW = \frac{a\cos(\theta°)(a^2\sin^2(\theta°) + b^2)}{b\sin(\theta°)} \qquad \ldots(53.3)$$

$$\text{Eqn. (C. 10)} \Longrightarrow TS = \frac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)} \qquad \ldots(53.4)$$

$$\text{Eqn. (D. 4)} \Longrightarrow VX = \frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b} \qquad \ldots(53.5)$$

Multiplying eqns. (53.4) & (53.5),

$$TS \times VX = \left(\frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)}\right) \times \left(\frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)}\right)$$

$$\therefore TS \times VX = \frac{a\cos(\theta°)(a^2\sin^2(\theta°) + b^2)}{b\sin(\theta°)} \qquad \ldots(53.6)$$

Equating eqns. (53.3) & (53.6),

$$F_1S \times TW = TS \times VX \qquad \ldots(53.7)$$

Eqn. (53.7) is mathematical expression of the theorem.

THEOREM- 54:

$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$

$$\boldsymbol{OQ \times TQ = PQ \times RN = PT \times RA' = PN \times QE' = \left(PQ \times \frac{a}{b}\right)^2}$$

Proof of the theorem

$$\text{Eqn. (B. 1)} \Longrightarrow OQ = \frac{a}{\cos(\theta°)} \qquad \ldots(54.1)$$

$$\text{Eqn. (C. 4)} \Longrightarrow TQ = \frac{a\sin^2(\theta°)}{\cos(\theta°)} \qquad \ldots(54.2)$$

Multiplying eqns. (54.1) & (54.2),

$$OQ \times TQ = \left(\frac{a}{\cos(\theta°)}\right) \times \left(\frac{a\sin^2(\theta°)}{\cos(\theta°)}\right)$$

$$\therefore OQ \times TQ = \frac{a^2\sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(54.3)$$

$$\text{Eqn. (B. 2)} \Longrightarrow PQ = \frac{b\sin(\theta°)}{\cos(\theta°)} \qquad \ldots(54.4)$$

$$\text{Eqn. (C. 17)} \Longrightarrow RN = \frac{a^2\sin(\theta°)}{b\cos(\theta°)} \qquad \ldots(54.5)$$

Multiplying eqns. (54.4) & (54.5),

$$PQ \times RN = \left(\frac{\cancel{b}\sin(\theta°)}{\cos(\theta°)}\right) \times \left(\frac{a^2\sin(\theta°)}{b\cos(\theta°)}\right)$$

$$\therefore \mathrm{PQ} \times \mathrm{RN} = \frac{a^2 \sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(54.6)$$

$$\text{Eqn. (C.15)} \Longrightarrow \mathrm{PT} = \frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)} \qquad \ldots(54.7)$$

$$\text{Eqn. (E.3)} \Longrightarrow \mathrm{RA}' = \frac{a^2 \sin(\theta°)}{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}} \qquad \ldots(54.8)$$

Multiplying eqns. (54.7) & (54.8),

$$\mathrm{PT} \times \mathrm{RA}' = \left(\frac{\sin(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{\cos(\theta°)}\right) \times \left(\frac{a^2 \sin(\theta°)}{\cos(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \mathrm{PT} \times \mathrm{RA}' = \frac{a^2 \sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(54.9)$$

$$\text{Eqn. (C.19)} \Longrightarrow \mathrm{PN} = \frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)} \qquad \ldots(54.10)$$

$$\text{Eqn. (E.2)} \Longrightarrow \mathrm{QE}' = \frac{ab\sin^2(\theta°)}{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}} \qquad \ldots(54.11)$$

Multiplying eqns. (54.10) & (54.11),

$$\mathrm{PN} \times \mathrm{QE}' = \left(\frac{a\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{\cancel{b}\cos(\theta°)}\right) \times \left(\frac{a\cancel{b}\sin^2(\theta°)}{\cos(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \mathrm{PN} \times \mathrm{QE}' = \frac{a^2 \sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(54.12)$$

$$\text{Eqn. (B.2)} \Longrightarrow \mathrm{PQ} = \frac{b\sin(\theta°)}{\cos(\theta°)}$$

$$\mathrm{PQ}^2 = \frac{b^2 \sin^2(\theta°)}{\cos^2(\theta°)}$$

Multiplying above eqn. by $\frac{a^2}{b^2}$

$$\therefore \mathrm{PQ}^2 \times \frac{a^2}{b^2} = \frac{\cancel{b^2}\sin^2(\theta°)}{\cos^2(\theta°)} \times \frac{a^2}{\cancel{b^2}}$$

$$\therefore \mathrm{PQ}^2 \times \frac{a^2}{b^2} = \frac{a^2 \sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(54.13)$$

Equating eqns. (54.3), (54.6), (54.9), (54.12) & (54.13),

$$\mathrm{OQ} \times \mathrm{TQ} = \mathrm{PQ} \times \mathrm{RN} = \mathrm{PT} \times \mathrm{RA}' = \mathrm{PN} \times \mathrm{QE}' = \left(\mathrm{PQ} \times \frac{a}{b}\right)^2 \qquad \ldots(54.14)$$

Eqn. (54.14) is mathematical expression of the theorem.

THEOREM- 55:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{OQ \times TW = TS \times RN}$$

Proof of the theorem

$$\text{Eqn. (B.1)} \Longrightarrow \text{OQ} = \frac{\text{a}}{\cos(\theta°)} \quad \ldots(55.1)$$

$$\text{Eqn. (D.3)} \Longrightarrow \text{TW} = \frac{\text{a}\cos(\theta°)\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\text{b}} \quad \ldots(55.2)$$

Multiplying eqns. (55.1) & (55.2),

$$\text{OQ} \times \text{TW} = \left(\frac{\text{a}}{\cancel{\cos(\theta°)}}\right) \times \left(\frac{\text{a}\cancel{\cos(\theta°)}\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\text{b}}\right)$$

$$\therefore \text{OQ} \times \text{TW} = \frac{\text{a}^2\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\text{b}} \quad \ldots(55.3)$$

$$\text{Eqn. (C.10)} \Longrightarrow \text{TS} = \frac{\cos(\theta°)\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\sin(\theta°)} \quad \ldots(55.4)$$

$$\text{Eqn. (C.17)} \Longrightarrow \text{RN} = \frac{\text{a}^2\sin(\theta°)}{\text{b}\cos(\theta°)} \quad \ldots(55.5)$$

Multiplying eqns. (55.4) & (55.5),

$$\text{TS} \times \text{RN} = \left(\frac{\cancel{\cos(\theta°)}\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\cancel{\sin(\theta°)}}\right) \times \left(\frac{\text{a}^2\cancel{\sin(\theta°)}}{\text{b}\cancel{\cos(\theta°)}}\right)$$

$$\therefore \text{TS} \times \text{RN} = \frac{\text{a}^2\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\text{b}} \quad \ldots(55.6)$$

Equating eqns. (55.3) & (55.6),

$$\text{OQ} \times \text{TW} = \text{TS} \times \text{RN} \quad \ldots(55.7)$$

Eqn. (55.7) is mathematical expression of the theorem.

THEOREM- 56:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{OV \times F_1N = PN \times OF_1}$$

Proof of the theorem

$$\text{Eqn. (A.1)} \Longrightarrow \text{OV} = \text{a} \quad \ldots(56.1)$$

$$\text{Eqn. (B.10)} \Longrightarrow \text{F}_1\text{N} = \frac{\sqrt{\text{a}^2 + \text{b}^2} \times \sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\text{b}\cos(\theta°)} \quad \ldots(56.2)$$

Multiplying eqns. (56.1) & (56.2),

$$\text{OV} \times \text{F}_1\text{N} = \text{a} \times \left(\frac{\sqrt{\text{a}^2 + \text{b}^2} \times \sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\text{b}\cos(\theta°)}\right)$$

$$\therefore \mathrm{OV} \times \mathrm{F_1N} = \frac{\mathrm{a}\sqrt{(\mathrm{a}^2+\mathrm{b}^2)\times(\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2)}}{\mathrm{b}\cos(\theta°)} \qquad \ldots(56.3)$$

$$\text{Eqn. (C. 19)} \Longrightarrow \mathrm{PN} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \qquad \ldots(56.4)$$

$$\text{Eqn. (A. 3)} \Longrightarrow \mathrm{OF_1} = \sqrt{\mathrm{a}^2+\mathrm{b}^2} \qquad \ldots(56.5)$$

Multiplying eqns. (56.4) & (56.5),

$$\mathrm{PN} \times \mathrm{OF_1} = \left(\frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{b}\cos(\theta°)}\right) \times \sqrt{\mathrm{a}^2+\mathrm{b}^2}$$

$$\therefore \mathrm{PN} \times \mathrm{OF_1} = \frac{\mathrm{a}\sqrt{(\mathrm{a}^2+\mathrm{b}^2)\times(\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2)}}{\mathrm{b}\cos(\theta°)} \qquad \ldots(56.6)$$

Equating eqns. (56.3) & (56.6),

$$\mathrm{OV} \times \mathrm{F_1N} = \mathrm{PN} \times \mathrm{OF_1} \qquad \ldots(56.7)$$

Eqn. (56.7) is mathematical expression of the theorem.

THEOREM- 57:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{OF_1 \times F_1N = MN \times OV}$

Proof of the theorem

$$\text{Eqn. (A. 3)} \Longrightarrow \mathrm{OF_1} = \sqrt{\mathrm{a}^2+\mathrm{b}^2} \qquad \ldots(57.1)$$

$$\text{Eqn. (B. 10)} \Longrightarrow \mathrm{F_1N} = \frac{\sqrt{\mathrm{a}^2+\mathrm{b}^2}\times\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \qquad \ldots(57.2)$$

Multiplying eqns. (57.1) & (57.2),

$$\mathrm{OF_1} \times \mathrm{F_1N} = \sqrt{\mathrm{a}^2+\mathrm{b}^2} \times \left(\frac{\sqrt{\mathrm{a}^2+\mathrm{b}^2}\times\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{b}\cos(\theta°)}\right)$$

$$\therefore \mathrm{OF_1} \times \mathrm{F_1N} = \frac{(\mathrm{a}^2+\mathrm{b}^2)\times\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \qquad \ldots(57.3)$$

$$\text{Eqn. (C. 21)} \Longrightarrow \mathrm{MN} = \frac{(\mathrm{a}^2+\mathrm{b}^2)\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{ab}\cos(\theta°)} \qquad \ldots(57.4)$$

$$\text{Eqn. (A. 1)} \Longrightarrow \mathrm{OV} = \mathrm{a} \qquad \ldots(57.5)$$

Multiplying eqns. (57.4) & (57.5),

$$\mathrm{MN} \times \mathrm{OV} = \left(\frac{(\mathrm{a}^2+\mathrm{b}^2)\times\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\cancel{\mathrm{a}}\mathrm{b}\cos(\theta°)}\right) \times \cancel{\mathrm{a}}$$

$$\therefore \mathrm{MN} \times \mathrm{OV} = \frac{(\mathrm{a}^2+\mathrm{b}^2)\times\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \qquad \ldots(57.6)$$

Equating eqns. (57.5) & (57.6),

$$OF_1 \times F_1N = MN \times OV \qquad \ldots(57.7)$$

Eqn. (57.7) is mathematical expression of the theorem.

THEOREM- 58:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{OM \times OB = ON \times OI}$

Proof of the theorem

$$\text{Eqn. (C.6)} \Longrightarrow \mathrm{OM} = \frac{a^2 + b^2}{a\cos(\theta°)} \qquad \ldots(58.1)$$

$$\text{Eqn. }(B.2) \Longrightarrow \mathrm{PQ} = \frac{b\sin(\theta°)}{\cos(\theta°)} \qquad \ldots(58.2)$$

Multiplying eqns. (58.1) & (58.2),

$$\mathrm{OM} \times \mathrm{PQ} = \left(\frac{a^2 + b^2}{a\cos(\theta°)}\right) \times \left(\frac{b\sin(\theta°)}{\cos(\theta°)}\right)$$

$$\therefore \mathrm{OM} \times \mathrm{PQ} = \frac{b(a^2 + b^2)\sin(\theta°)}{a\cos^2(\theta°)} \qquad \ldots(58.3)$$

$$\text{Eqn. (C.7)} \Longrightarrow \mathrm{ON} = \frac{(a^2 + b^2)\sin(\theta°)}{b\cos(\theta°)} \qquad \ldots(58.4)$$

$$\text{Eqn. (D.7)} \Longrightarrow \mathrm{OI} = \frac{b^2}{a\cos(\theta°)} \qquad \ldots(58.5)$$

Multiplying eqns. (58.4) & (58.5),

$$\mathrm{ON} \times \mathrm{OI} = \left(\frac{(a^2 + b^2)\sin(\theta°)}{\cancel{b}\cos(\theta°)}\right) \times \left(\frac{b^{\cancel{2}}}{a\cos(\theta°)}\right)$$

$$\therefore \mathrm{ON} \times \mathrm{OI} = \frac{b(a^2 + b^2)\sin(\theta°)}{a\cos^2(\theta°)} \qquad \ldots(58.6)$$

Equating eqns. (58.3) & (58.6),

$$\mathrm{OM} \times \mathrm{PQ} = \mathrm{ON} \times \mathrm{OI} \qquad \ldots(58.7)$$

Eqn. (58.7) is mathematical expression of the theorem.

THEOREM- 59:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{PM \times OV = PQ \times F_1S}$

Proof of the theorem

$$\text{Eqn. (C.14)} \Longrightarrow \mathrm{PM} = \frac{b\sqrt{a^2\sin^2(\theta°) + b^2}}{a\cos(\theta°)} \qquad \ldots(59.1)$$

$$\text{Eqn. (A.1)} \Longrightarrow \mathrm{OV} = a \qquad \ldots(59.2)$$

Multiplying eqns. (59.1) & (59.2),

$$\mathrm{PM} \times \mathrm{OV} = \left(\frac{\mathrm{b}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\cancel{\mathrm{a}}\cos(\theta°)}\right) \times \cancel{\mathrm{a}}$$

$$\therefore \mathrm{PM} \times \mathrm{OV} = \frac{\mathrm{b}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\cos(\theta°)} \quad \ldots(59.3)$$

$$\mathrm{Eqn.}\,(\mathrm{B}.2) \Longrightarrow \mathrm{PQ} = \frac{\mathrm{b}\sin(\theta°)}{\cos(\theta°)} \quad \ldots(59.4)$$

$$\mathrm{Eqn.}\,(\mathrm{C}.13) \Longrightarrow \mathrm{F_1S} = \frac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)} \quad \ldots(59.5)$$

Multiplying eqns. (59.4) & (59.5),

$$\mathrm{PQ} \times \mathrm{F_1S} = \left(\frac{\mathrm{b}\cancel{\sin(\theta°)}}{\cos(\theta°)}\right) \times \left(\frac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\cancel{\sin(\theta°)}}\right)$$

$$\therefore \mathrm{PQ} \times \mathrm{F_1S} = \frac{\mathrm{b}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\cos(\theta°)} \quad \ldots(59.6)$$

Equating eqns. (59.3) & (59.6),

$$\mathrm{PM} \times \mathrm{OV} = \mathrm{PQ} \times \mathrm{F_1S} \quad \ldots(59.7)$$

Eqn. (59.7) is mathematical expression of the theorem.

THEOREM- 60:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$**,**

$\boldsymbol{OL' \times OX = OM \times OV}$

Proof of the theorem

$$\mathrm{Eqn.}\,(\mathrm{E}.8) \Longrightarrow \mathrm{OL'} = \frac{\mathrm{b}(\mathrm{a}^2 + \mathrm{b}^2)}{\mathrm{a}^2\sin(\theta°)\cos(\theta°)} \quad \ldots(60.1)$$

$$\mathrm{Eqn.}\,(\mathrm{D}.10) \Longrightarrow \mathrm{OX} = \frac{\mathrm{a}^2\sin(\theta°)}{\mathrm{b}} \quad \ldots(60.2)$$

Multiplying eqns. (60.1) & (60.2),

$$\mathrm{OL'} \times \mathrm{OX} = \left(\frac{\cancel{\mathrm{b}}(\mathrm{a}^2 + \mathrm{b}^2)}{\cancel{\mathrm{a}^2\sin(\theta°)}\cos(\theta°)}\right) \times \left(\frac{\cancel{\mathrm{a}^2\sin(\theta°)}}{\cancel{\mathrm{b}}}\right)$$

$$\therefore \mathrm{OL'} \times \mathrm{OX} = \frac{\mathrm{a}^2 + \mathrm{b}^2}{\cos(\theta°)} \quad \ldots(60.3)$$

$$\mathrm{Eqn.}\,(\mathrm{C}.6) \Longrightarrow \mathrm{OM} = \frac{\mathrm{a}^2 + \mathrm{b}^2}{\mathrm{a}\cos(\theta°)} \quad \ldots(60.4)$$

$$\mathrm{Eqn.}\,(\mathrm{A}.1) \Longrightarrow \mathrm{OV} = \mathrm{a} \quad \ldots(60.5)$$

Multiplying eqns. (60.4) & (60.5),

$$\mathrm{OM} \times \mathrm{OV} = \left(\frac{\mathrm{a}^2 + \mathrm{b}^2}{\cancel{\mathrm{a}}\cos(\theta°)}\right) \times \cancel{\mathrm{a}}$$

$$\therefore \mathrm{OM} \times \mathrm{OV} = \frac{a^2 + b^2}{\cos(\theta°)} \qquad \ldots(60.6)$$

Equating eqns. (60.3) & (60.4),

$$\mathrm{OL'} \times \mathrm{OX} = \mathrm{OM} \times \mathrm{OV} \qquad \ldots(60.7)$$

Eqn. (60.7) is mathematical expression of the theorem.

THEOREM- 61:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{PF_1 \times OF_1 = a \times F_1M}$

Proof of the theorem

$$\text{Eqn. (B.12)} \Longrightarrow \mathrm{PF_1} = \frac{\sqrt{a^2 + b^2} - a\cos(\theta°)}{\cos(\theta°)} \qquad \ldots(61.1)$$

$$\text{Eqn. (A.3)} \Longrightarrow \mathrm{OF_1} = \sqrt{a^2 + b^2} \qquad \ldots(61.2)$$

Multiplying eqns. (61.1) & (61.2),

$$\mathrm{PF_1} \times \mathrm{OF_1} = \left(\frac{\sqrt{a^2 + b^2} - a\cos(\theta°)}{\cos(\theta°)}\right) \times \sqrt{a^2 + b^2}$$

$$\therefore \mathrm{PF_1} \times \mathrm{OF_1} = \frac{\sqrt{a^2 + b^2}\left(\sqrt{a^2 + b^2} - a\cos(\theta°)\right)}{\cos(\theta°)} \qquad \ldots(61.3)$$

$$\text{Eqn. (B.8)} \Longrightarrow \mathrm{F_1M} = \frac{\sqrt{a^2 + b^2}\left(\sqrt{a^2 + b^2} - a\cos(\theta°)\right)}{a\cos(\theta°)}$$

Multiplying above eqn. by ‘a’

$$a \times \mathrm{F_1M} = \cancel{a} \times \left(\frac{\sqrt{a^2 + b^2}\left(\sqrt{a^2 + b^2} - a\cos(\theta°)\right)}{\cancel{a}\cos(\theta°)}\right)$$

$$a \times \mathrm{F_1M} = \frac{\sqrt{a^2 + b^2}\left(\sqrt{a^2 + b^2} - a\cos(\theta°)\right)}{\cos(\theta°)} \qquad \ldots(61.4)$$

Equating eqns. (61.3) & (61.4),

$$\mathrm{PF_1} \times \mathrm{OF_1} = a \times \mathrm{F_1M} \qquad \ldots(61.5)$$

Eqn. (61.5) is mathematical expression of the theorem.

THEOREM- 62:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{PF_2 \times OF_2 = a \times F_2M}$

Proof of the theorem

$$\text{Eqn. (B.13)} \Longrightarrow \mathrm{PF_2} = \frac{\sqrt{a^2 + b^2} + a\cos(\theta°)}{\cos(\theta°)} \qquad \ldots(62.1)$$

$$\text{Eqn. (A.3)} \Longrightarrow \mathrm{OF_2} = \sqrt{\mathrm{a^2+b^2}} \qquad \ldots(62.2)$$

Multiplying eqns. (62.1) & (62.2),

$$\mathrm{PF_2 \times OF_2} = \left(\frac{\sqrt{\mathrm{a^2+b^2}} + \mathrm{a}\cos(\theta°)}{\cos(\theta°)}\right) \times \sqrt{\mathrm{a^2+b^2}}$$

$$\therefore \mathrm{PF_2 \times OF_2} = \frac{\sqrt{\mathrm{a^2+b^2}}\left(\sqrt{\mathrm{a^2+b^2}} + \mathrm{a}\cos(\theta°)\right)}{\cos(\theta°)} \qquad \ldots(62.3)$$

$$\text{Eqn. (B.9)} \Longrightarrow \mathrm{F_2M} = \frac{\sqrt{\mathrm{a^2+b^2}}\left(\sqrt{\mathrm{a^2+b^2}} + \mathrm{a}\cos(\theta°)\right)}{\mathrm{a}\cos(\theta°)}$$

Multiplying above eqn. by ‘a’

$$\mathrm{a \times F_2M} = \frac{\sqrt{\mathrm{a^2+b^2}}\left(\sqrt{\mathrm{a^2+b^2}} + \mathrm{a}\cos(\theta°)\right)}{\cos(\theta°)} \qquad \ldots(62.4)$$

Equating eqns. (62.3) & (62.4),

$$\mathrm{PF_2 \times OF_2 = a \times F_2M} \qquad \ldots(62.5)$$

Eqn. (62.5) is mathematical expression of the theorem.

THEOREM- 63:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{OV \times OY = OX \times OF_1}$

Proof of the theorem

$$\text{Eqn. (A.1)} \Longrightarrow \mathrm{OV = a} \qquad \ldots(63.1)$$

$$\text{Eqn. (D.11)} \Longrightarrow \mathrm{OY} = \frac{\mathrm{a}\sin(\theta°)\sqrt{\mathrm{a^2+b^2}}}{\mathrm{b}} \qquad \ldots(63.2)$$

Multiplying eqns. (63.1) & (63.2),

$$\mathrm{OV \times OY = a} \times \left(\frac{\mathrm{a}\sin(\theta°)\sqrt{\mathrm{a^2+b^2}}}{\mathrm{b}}\right)$$

$$\therefore \mathrm{OV \times OY} = \frac{\mathrm{a^2}\sqrt{\mathrm{a^2+b^2}}\sin(\theta°)}{\mathrm{b}} \qquad \ldots(63.3)$$

$$\text{Eqn. (D.10)} \Longrightarrow \mathrm{OX} = \frac{\mathrm{a^2}\sin(\theta°)}{\mathrm{b}} \qquad \ldots(63.4)$$

$$\text{Eqn. (A.3)} \Longrightarrow \mathrm{OF_1} = \sqrt{\mathrm{a^2+b^2}} \qquad \ldots(63.5)$$

Multiplying eqns. (63.4) & (63.5),

$$\mathrm{OX \times OF_1} = \left(\frac{\mathrm{a^2}\sin(\theta°)}{\mathrm{b}}\right) \times \sqrt{\mathrm{a^2+b^2}}$$

$$\therefore \mathrm{OX \times OF_1} = \frac{\mathrm{a^2}\sqrt{\mathrm{a^2+b^2}}\sin(\theta°)}{\mathrm{b}} \qquad \ldots(63.6)$$

Equating eqns. (63.3) & (63.6),

$$\mathrm{OV} \times \mathrm{OY} = \mathrm{OX} \times \mathrm{OF_1} \qquad \ldots(63.7)$$

Eqn. (63.7) is mathematical expression of the theorem.

THEOREM- 64:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{F_1M \times PF_1 = F_1N \times PM}$$

Proof of the theorem

$$\text{Eqn. (B.8)} \Longrightarrow \mathrm{F_1M} = \frac{\sqrt{a^2+b^2}\left(\sqrt{a^2+b^2} - a\cos(\theta°)\right)}{a\cos(\theta°)} \qquad \ldots(64.1)$$

$$\text{Eqn. (B.12)} \Longrightarrow \mathrm{PF_1} = \frac{\sqrt{a^2+b^2} - a\cos(\theta°)}{\cos(\theta°)} \qquad \ldots(64.2)$$

Multiplying eqns. (64.1) & (64.2),

$$\mathrm{F_1M \times PF_1} = \left(\frac{\sqrt{a^2+b^2}\left(\sqrt{a^2+b^2} - a\cos(\theta°)\right)}{a\cos(\theta°)}\right) \times \left(\frac{\sqrt{a^2+b^2} - a\cos(\theta°)}{\cos(\theta°)}\right)$$

$$\therefore \mathrm{F_1M \times PF_1} = \frac{\sqrt{a^2+b^2}\left(a^2+b^2 - a^2\cos^2(\theta°)\right)}{a\cos^2(\theta°)}$$

$$\therefore \mathrm{F_1M \times PF_1} = \frac{\sqrt{a^2+b^2}\left(a^2[1-\cos^2(\theta°)] + b^2\right)}{a\cos^2(\theta°)}$$

$$\therefore \mathrm{F_1M \times PF_1} = \frac{\sqrt{a^2+b^2}\left(a^2\sin^2(\theta°) + b^2\right)}{a\cos^2(\theta°)} \qquad \ldots(64.3)$$

$$\text{Eqn. (B.10)} \Longrightarrow \mathrm{F_1N} = \frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)} \qquad \ldots(64.4)$$

$$\text{Eqn. (C.14)} \Longrightarrow \mathrm{PM} = \frac{b\sqrt{a^2\sin^2(\theta°) + b^2}}{a\cos(\theta°)} \qquad \ldots(64.5)$$

Multiplying eqns. (64.4) & (64.5),

$$\mathrm{F_1N \times PM} = \left(\frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°) + b^2}}{\cancel{b}\cos(\theta°)}\right) \times \left(\frac{\cancel{b}\sqrt{a^2\sin^2(\theta°) + b^2}}{a\cos(\theta°)}\right)$$

$$\therefore \mathrm{F_1N \times PM} = \frac{\sqrt{a^2+b^2}\left(a^2\sin^2(\theta°) + b^2\right)}{a\cos^2(\theta°)} \qquad \ldots(64.6)$$

Equating eqns. (64.3) & (64.6),

$$\mathrm{F_1M \times PF_1 = F_1N \times PM} \qquad \ldots(64.7)$$

Eqn. (64.7) is mathematical expression of the theorem.

THEOREM- 65:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{PT \times ML' = F_1M \times F_2M}$$

Proof of the theorem

$$\text{Eqn. (C.15)} \Longrightarrow \mathrm{PT} = \frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)} \qquad \ldots(65.1)$$

$$\text{Eqn. (E.7)} \Longrightarrow \mathrm{ML'} = \frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{a^2\sin(\theta°)\cos(\theta°)} \qquad \ldots(65.2)$$

Multiplying eqns. (65.1) & (65.2),

$$\mathrm{PT} \times \mathrm{ML'} = \left(\frac{\cancel{\sin(\theta°)}\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)}\right) \times \left(\frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{a^2\cancel{\sin(\theta°)}\cos(\theta°)}\right)$$

$$\therefore \mathrm{PT} \times \mathrm{ML'} = \frac{(a^2 + b^2)(a^2\sin^2(\theta°) + b^2)}{a^2\cos^2(\theta°)} \qquad \ldots(65.3)$$

$$\text{Eqn. (E.11)} \Longrightarrow \mathrm{F_1M} \times \mathrm{F_2M} = \frac{(a^2 + b^2)(a^2\sin^2(\theta°) + b^2)}{a^2\cos^2(\theta°)} \qquad \ldots(65.4)$$

Equating eqns. (65.3) & (65.4),

$$\mathrm{PT} \times \mathrm{ML'} = \mathrm{F_1M} \times \mathrm{F_2M} \qquad \ldots(65.5)$$

Eqn. (65.5) is mathematical expression of the theorem.

THEOREM- 66:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{TQ \times OT = PQ \times OW}$

Proof of the theorem

$$\text{Eqn. (C.4)} \Longrightarrow \mathrm{TQ} = \frac{a\sin^2(\theta°)}{\cos(\theta°)} \qquad \ldots(66.1)$$

$$\text{Eqn. (C.1)} \Longrightarrow \mathrm{OT} = a\cos(\theta°) \qquad \ldots(66.2)$$

Multiplying eqns. (66.1) & (66.2),

$$\mathrm{TQ} \times \mathrm{OT} = \left(\frac{a\sin^2(\theta°)}{\cancel{\cos(\theta°)}}\right) \times a\cancel{\cos(\theta°)}$$

$$\therefore \mathrm{TQ} \times \mathrm{OT} = a^2\sin^2(\theta°) \qquad \ldots(66.3)$$

$$\text{Eqn. (B.2)} \Longrightarrow \mathrm{PQ} = \frac{b\sin(\theta°)}{\cos(\theta°)} \qquad \ldots(66.4)$$

$$\text{Eqn. (D.9)} \Longrightarrow \mathrm{OW} = \frac{a^2\sin(\theta°)\cos(\theta°)}{b} \qquad \ldots(66.5)$$

Multiplying eqns. (66.4) & (66.5),

$$\mathrm{PQ} \times \mathrm{OW} = \left(\frac{\cancel{b}\sin(\theta°)}{\cancel{\cos(\theta°)}}\right) \times \left(\frac{a^2\sin(\theta°)\cancel{\cos(\theta°)}}{\cancel{b}}\right)$$

$$\therefore \mathrm{PQ} \times \mathrm{OW} = a^2\sin^2(\theta°) \qquad \ldots(66.6)$$

Equating eqns. (66.3) & (66.6),

$$\mathrm{TQ} \times \mathrm{OT} = \mathrm{PQ} \times \mathrm{OW} \qquad \ldots(66.7)$$

Eqn. (66.7) is mathematical expression of the theorem.

THEOREM- 67:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{TQ \times OS = QM \times OW = OK \times PT = TK \times PM}$

Proof of the theorem

$$\text{Eqn. (C.4)} \Longrightarrow \mathrm{TQ} = \frac{\mathrm{a}\sin^2(\theta°)}{\cos(\theta°)} \qquad \dots(67.1)$$

$$\text{Eqn. (C.2)} \Longrightarrow \mathrm{OS} = \frac{\mathrm{b}\cos(\theta°)}{\sin(\theta°)} \qquad \dots(67.2)$$

Multiplying eqns. (67.1) & (67.2),

$$\mathrm{TQ} \times \mathrm{OS} = \left(\frac{\mathrm{b}\cancel{\cos(\theta°)}}{\cancel{\sin(\theta°)}}\right) \times \left(\frac{\mathrm{a}\sin^{\cancel{2}}(\theta°)}{\cancel{\cos(\theta°)}}\right)$$

$$\therefore \mathrm{TQ} \times \mathrm{OS} = \mathrm{ab}\sin(\theta°) \qquad \dots(67.3)$$

$$\text{Eqn. (C.5)} \Longrightarrow \mathrm{QM} = \frac{\mathrm{b}^2}{\mathrm{a}\cos(\theta°)} \qquad \dots(67.4)$$

$$\text{Eqn. (D.9)} \Longrightarrow \mathrm{OW} = \frac{\mathrm{a}^2\sin(\theta°)\cos(\theta°)}{\mathrm{b}} \qquad \dots(67.5)$$

Multiplying eqns. (67.4) & (67.5),

$$\mathrm{TQ} \times \mathrm{OS} = \left(\frac{\mathrm{b}\cancel{\cos(\theta°)}}{\cancel{\sin(\theta°)}}\right) \times \left(\frac{\mathrm{a}\sin^{\cancel{2}}(\theta°)}{\cancel{\cos(\theta°)}}\right)$$

$$\therefore \mathrm{TQ} \times \mathrm{OS} = \mathrm{ab}\sin(\theta°) \qquad \dots(67.6)$$

$$\text{Eqn. (D.15)} \Longrightarrow \mathrm{OK} = \frac{\mathrm{ab}\cos(\theta°)}{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}} \qquad \dots(67.7)$$

$$\text{Eqn. (C.15)} \Longrightarrow \mathrm{PT} = \frac{\sin(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\cos(\theta°)} \qquad \dots(67.8)$$

Multiplying eqns. (67.7) & (67.8),

$$\mathrm{OK} \times \mathrm{PT} = \left(\frac{\mathrm{ab}\cancel{\cos(\theta°)}}{\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}\right) \times \left(\frac{\sin(\theta°)\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}{\cancel{\cos(\theta°)}}\right)$$

$$\therefore \mathrm{OK} \times \mathrm{PT} = \mathrm{ab}\sin(\theta°) \qquad \dots(67.9)$$

$$\text{Eqn. (D.16)} \Longrightarrow \mathrm{TK} = \frac{\mathrm{a}^2\sin(\theta°)\cos(\theta°)}{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}} \qquad \dots(67.10)$$

$$\text{Eqn. (C.14)} \Longrightarrow \mathrm{PM} = \frac{\mathrm{b}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\cos(\theta°)} \qquad \dots(67.11)$$

Multiplying eqns. (67.10) & (67.11),

$$\mathrm{TK} \times \mathrm{PM} = \left(\frac{\mathrm{a}^{\cancel{2}}\sin(\theta°)\cancel{\cos(\theta°)}}{\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}\right) \times \left(\frac{\mathrm{b}\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}{\cancel{\mathrm{a}\cos(\theta°)}}\right)$$

$$\therefore \mathrm{TK} \times \mathrm{PM} = \mathrm{ab}\sin(\theta°) \quad \ldots(67.12)$$

Equating eqns. (67.3), (67.6), (67.9) & (67.12),

$$\mathrm{TQ} \times \mathrm{OS} = \mathrm{QM} \times \mathrm{OW} = \mathrm{OK} \times \mathrm{PT} = \mathrm{TK} \times \mathrm{PM} \quad \ldots(67.13)$$

Eqn. (67.13) is mathematical expression of the theorem.

THEOREM- 68:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{PN \times OS = OQ \times TS = OV \times VH'}$

Proof of the theorem

$$\text{Eqn. (C. 19)} \Longrightarrow \mathrm{PN} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \quad \ldots(68.1)$$

$$\text{Eqn. (C. 2)} \Longrightarrow \mathrm{OS} = \frac{\mathrm{b}\cos(\theta°)}{\sin(\theta°)} \quad \ldots(68.2)$$

Multiplying eqns. (68.1) & (68.2),

$$\mathrm{PN} \times \mathrm{OS} = \left(\frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\cancel{\mathrm{b}\cos(\theta°)}}\right) \times \left(\frac{\cancel{\mathrm{b}\cos(\theta°)}}{\sin(\theta°)}\right)$$

$$\therefore \mathrm{PN} \times \mathrm{OS} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)} \quad \ldots(68.3)$$

$$\text{Eqn. (B. 1)} \Longrightarrow \mathrm{OQ} = \frac{\mathrm{a}}{\cos(\theta°)} \quad \ldots(68.4)$$

$$\text{Eqn. (C. 10)} \Longrightarrow \mathrm{TS} = \frac{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)} \quad \ldots(68.5)$$

Multiplying eqns. (68.4) & (68.5),

$$\mathrm{OQ} \times \mathrm{TS} = \left(\frac{\mathrm{a}}{\cancel{\cos(\theta°)}}\right) \times \left(\frac{\cancel{\cos(\theta°)}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)}\right)$$

$$\therefore \mathrm{OQ} \times \mathrm{TS} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)} \quad \ldots(68.6)$$

$$\text{Eqn. (A. 1)} \Longrightarrow \mathrm{OV} = \mathrm{a} \quad \ldots(68.7)$$

$$\text{Eqn. (E. 4)} \Longrightarrow \mathrm{VH'} = \frac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)} \quad \ldots(68.8)$$

Multiplying eqns. (68.7) & (68.8),

$$\mathrm{OV} \times \mathrm{VH'} = \mathrm{a} \times \left(\frac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)}\right)$$

$$\therefore \mathrm{OV} \times \mathrm{VH'} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)} \quad \ldots(68.9)$$

Equating eqns. (68.3), (68.6) & (68.9),

$$\mathrm{PN} \times \mathrm{OS} = \mathrm{OQ} \times \mathrm{TS} = \mathrm{OV} \times \mathrm{VH'} \qquad \ldots(68.10)$$

Eqn. (68.10) is mathematical expression of the theorem.

THEOREM- 69:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{UF_1 \times UF_2 = \frac{b^2(2a^2 + b^2)}{a^2 + b^2}}$$

Proof of the theorem

$$\text{Eqn. (A. 9)} \Longrightarrow \mathrm{UF_1} = \frac{b^2}{\sqrt{a^2 + b^2}} \qquad \ldots(69.1)$$

$$\text{Eqn. (A. 10)} \Longrightarrow \mathrm{UF_2} = \frac{2a^2 + b^2}{\sqrt{a^2 + b^2}} \qquad \ldots(69.2)$$

Multiplying eqns. (69.1) & (69.2),

$$\mathrm{UF_1} \times \mathrm{UF_2} = \left(\frac{b^2}{\sqrt{a^2 + b^2}}\right) \times \left(\mathrm{UF_2} = \frac{2a^2 + b^2}{\sqrt{a^2 + b^2}}\right)$$

$$\therefore \mathrm{UF_1} \times \mathrm{UF_2} = \frac{b^2(2a^2 + b^2)}{a^2 + b^2} \qquad \ldots(69.3)$$

Eqn. (69.3) is mathematical expression of the theorem.

THEOREM- 70:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{OZ \times OI = OQ \times PQ = QM \times RN}$$

Proof of the theorem

$$\text{Eqn. (D. 12)} \Longrightarrow \mathrm{OZ} = \frac{a^2 \sin(\theta°)}{b\cos(\theta°)} \qquad \ldots(70.1)$$

$$\text{Eqn. (D. 7)} \Longrightarrow \mathrm{OI} = \frac{b^2}{a\cos(\theta°)} \qquad \ldots(70.2)$$

Multiplying eqns. (70.1) & (70.2),

$$\mathrm{OZ} \times \mathrm{OI} = \left(\frac{a^{\cancel{2}} \sin(\theta°)}{\cancel{b}\cos(\theta°)}\right) \times \left(\frac{b^{\cancel{2}}}{\cancel{a}\cos(\theta°)}\right)$$

$$\therefore \mathrm{OZ} \times \mathrm{OI} = \frac{ab\sin(\theta°)}{\cos^2(\theta°)} \qquad \ldots(70.3)$$

$$\text{Eqn. (B. 1)} \Longrightarrow \mathrm{OQ} = \frac{a}{\cos(\theta°)} \qquad \ldots(70.4)$$

$$\text{Eqn. (B. 2)} \Longrightarrow \mathrm{PQ} = \frac{b\sin(\theta°)}{\cos(\theta°)} \qquad \ldots(70.5)$$

Multiplying eqns. (70.4) & (70.5),

$$OQ \times PQ = \left(\frac{a}{\cos(\theta°)}\right) \times \left(\frac{b\sin(\theta°)}{\cos(\theta°)}\right)$$

$$\therefore OQ \times PQ = \frac{ab\sin(\theta°)}{\cos^2(\theta°)} \quad \ldots (70.6)$$

$$\text{Eqn. (C.5)} \Longrightarrow QM = \frac{b^2}{a\cos(\theta°)} \quad \ldots (70.7)$$

$$\text{Eqn. (C.17)} \Longrightarrow RN = \frac{a^2\sin(\theta°)}{b\cos(\theta°)} \quad \ldots (70.8)$$

Multiplying eqns. (70.7) & (70.8),

$$QM \times RN = \left(\frac{b^{\cancel{2}}}{\cancel{a}\cos(\theta°)}\right) \times \left(\frac{a^{\cancel{2}}\sin(\theta°)}{\cancel{b}\cos(\theta°)}\right)$$

$$\therefore QM \times RN = \frac{ab\sin(\theta°)}{\cos^2(\theta°)} \quad \ldots (70.9)$$

Equating eqns. (70.3), (70.6) & (70.9),

$$OZ \times OI = OQ \times PQ = QM \times RN \quad \ldots (70.10)$$

Eqn. (70.10) is mathematical expression of the theorem.

THEOREM- 71:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$**,**

$\boldsymbol{OL' \times MN^2 = ON \times ML'^2}$

Proof of the theorem

$$\text{Eqn. (E.8)} \Longrightarrow OL' = \frac{b(a^2+b^2)}{a^2\sin(\theta°)\cos(\theta°)} \quad \ldots (71.1)$$

$$\text{Eqn. (C.21)} \Longrightarrow MN = \frac{(a^2+b^2)\sqrt{a^2\sin^2(\theta°)+b^2}}{ab\cos(\theta°)}$$

$$\therefore MN^2 = \frac{(a^2+b^2)^2(a^2\sin^2(\theta°)+b^2)}{a^2b^2\cos^2(\theta°)} \quad \ldots (71.2)$$

Multiplying eqns. (71.1) & (71.2),

$$OL' \times MN^2 = \left(\frac{\cancel{b}(a^2+b^2)}{a^2\sin(\theta°)\cos(\theta°)}\right) \times \left(\frac{(a^2+b^2)^2(a^2\sin^2(\theta°)+b^2)}{a^2b^{\cancel{2}}\cos^2(\theta°)}\right)$$

$$\therefore OL' \times MN^2 = \frac{(a^2+b^2)^3 \times (a^2\sin^2(\theta°)+b^2)}{a^4b\sin(\theta°)\cos^3(\theta°)} \quad \ldots (71.3)$$

$$\text{Eqn. (C.7)} \Longrightarrow ON = \frac{(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)} \quad \ldots (71.4)$$

$$\text{Eqn. (E.7)} \Longrightarrow ML' = \frac{(a^2+b^2)\sqrt{a^2\sin^2(\theta°)+b^2}}{a^2\sin(\theta°)\cos(\theta°)}$$

$$\therefore ML'^2 = \frac{(a^2+b^2)^2(a^2\sin^2(\theta°)+b^2)}{a^4\sin^2(\theta°)\cos^2(\theta°)} \quad \ldots (71.5)$$

Multiplying eqns. (71.4) & (71.5),

$$\mathrm{ON} \times \mathrm{ML}'^2 = \left(\frac{(a^2+b^2)\cancel{\sin(\theta°)}}{b\cos(\theta°)}\right) \times \left(\frac{(a^2+b^2)^2(a^2\sin^2(\theta°)+b^2)}{a^4\sin^{\cancel{2}}(\theta°)\cos^2(\theta°)}\right)$$

$$\therefore \mathrm{ON} \times \mathrm{ML}'^2 = \frac{(a^2+b^2)^3 \times (a^2\sin^2(\theta°)+b^2)}{a^4 b\sin(\theta°)\cos^3(\theta°)} \quad \ldots(71.6)$$

Equating eqns. (71.3) & (71.6),

$$\mathrm{OL}' \times \mathrm{MN}^2 = \mathrm{ON} \times \mathrm{ML}'^2 \quad \ldots(71.7)$$

Eqn. (71.7) is mathematical expression of the theorem.

THEOREM- 72:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{F_1Q \times F_2Q = OP^2 - F_1R^2}$

Proof of the theorem

$$\text{Eqn. (E.12)} \Longrightarrow F_1Q \times F_2Q = \frac{a^2\sin^2(\theta°) - b^2\cos^2(\theta°)}{\cos^2(\theta°)} \quad \ldots(72.1)$$

$$\text{Eqn. (B.5)} \Longrightarrow \mathrm{OP} = \frac{\sqrt{a^2 + b^2\sin^2(\theta°)}}{\cos(\theta°)}$$

$$\therefore \mathrm{OP}^2 = \frac{a^2 + b^2\sin^2(\theta°)}{\cos^2(\theta°)} \quad \ldots(72.2)$$

$$\text{Eqn. (B.18)} \Longrightarrow F_1R = \frac{\sqrt{a^2\cos^2(\theta°) + b^2}}{\cos(\theta°)}$$

$$\therefore F_1R^2 = \frac{a^2\cos^2(\theta°) + b^2}{\cos^2(\theta°)} \quad \ldots(72.3)$$

Subtracting eqn. (72.3) from (72.2),

$$\mathrm{OP}^2 - F_1R^2 = \left(\frac{a^2 + b^2\sin^2(\theta°)}{\cos^2(\theta°)}\right) - \left(\frac{a^2\cos^2(\theta°) + b^2}{\cos^2(\theta°)}\right)$$

$$\therefore \mathrm{OP}^2 - F_1R^2 = \frac{a^2 + b^2\sin^2(\theta°) - [a^2\cos^2(\theta°) + b^2]}{\cos^2(\theta°)}$$

$$\therefore \mathrm{OP}^2 - F_1R^2 = \frac{a^2 + b^2\sin^2(\theta°) - a^2\cos^2(\theta°) - b^2}{\cos^2(\theta°)}$$

$$\therefore \mathrm{OP}^2 - F_1R^2 = \frac{a^2 - a^2\cos^2(\theta°) + b^2\sin^2(\theta°) - b^2}{\cos^2(\theta°)}$$

$$\therefore \mathrm{OP}^2 - F_1R^2 = \frac{a^2[1 - \cos^2(\theta°)] + b^2[\sin^2(\theta°) - 1]}{\cos^2(\theta°)}$$

$$\therefore \mathrm{OP}^2 - F_1R^2 = \frac{a^2\sin^2(\theta°) - b^2\cos^2(\theta°)}{\cos^2(\theta°)} \quad \ldots(72.4)$$

Equating Eqns. (72.1) & (72.4),

$$F_1Q \times F_2Q = \mathrm{OP}^2 - F_1R^2 \quad \ldots(72.5)$$

Eqn. (72.5) is mathematical expression of the theorem.

THEOREM- 73:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{OK \times F_1N = a^2}$

Proof of the theorem

$$\text{Eqn. (D.15)} \Longrightarrow \mathrm{OK} = \frac{\mathrm{abcos}(\theta°)}{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}} \qquad \ldots(73.1)$$

$$\text{Eqn. (B.10)} \Longrightarrow \mathrm{F_1N} = \frac{\sqrt{\mathrm{a}^2 + \mathrm{b}^2} \times \sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{bcos}(\theta°)} \qquad \ldots(73.2)$$

Multiplying eqns. (73.2) & (73.1),

$$\mathrm{OK} \times \mathrm{F_1N} = \left(\frac{\mathrm{a}\cancel{\mathrm{bcos}(\theta°)}}{\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}\right) \times \left(\frac{\sqrt{\mathrm{a}^2 + \mathrm{b}^2} \times \cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}{\cancel{\mathrm{bcos}(\theta°)}}\right)$$

$$\therefore \mathrm{OU} \times \mathrm{OF_1} = \mathrm{a}\sqrt{\mathrm{a}^2 + \mathrm{b}^2} \qquad \ldots(73.3)$$

Eqn. (73.3) is mathematical expression of the theorem.

THEOREM- 74:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{PQ \times OF_1{}^2 = ON \times b^2}$

Proof of the theorem

$$\text{Eqn. (B.2)} \Longrightarrow \mathrm{PQ} = \frac{\mathrm{bsin}(\theta°)}{\cos(\theta°)} \qquad \ldots(74.1)$$

$$\text{Eqn. (A.3)} \Longrightarrow \mathrm{OF_1} = \sqrt{\mathrm{a}^2 + \mathrm{b}^2}$$

$$\mathrm{OF_1}^2 = \mathrm{a}^2 + \mathrm{b}^2 \qquad \ldots(74.2)$$

Multiplying eqns. (74.1) & (74.2),

$$\mathrm{PQ} \times \mathrm{OF_1}^2 = \left(\frac{\mathrm{bsin}(\theta°)}{\cos(\theta°)}\right) \times (\mathrm{a}^2 + \mathrm{b}^2)$$

$$\therefore \mathrm{PQ} \times \mathrm{OF_1}^2 = \frac{\mathrm{b}(\mathrm{a}^2 + \mathrm{b}^2)\sin(\theta°)}{\cos(\theta°)} \qquad \ldots(74.3)$$

$$\text{Eqn. (C.7)} \Longrightarrow \mathrm{ON} = \frac{(\mathrm{a}^2 + \mathrm{b}^2)\sin(\theta°)}{\mathrm{bcos}(\theta°)}$$

$$\mathrm{ON} \times \mathrm{b}^2 = \frac{\mathrm{b}^{\cancel{2}}(\mathrm{a}^2 + \mathrm{b}^2)\sin(\theta°)}{\cancel{\mathrm{b}}\cos(\theta°)} \qquad \ldots(74.4)$$

$$\therefore \mathrm{ON} \times \mathrm{b}^2 = \frac{\mathrm{b}(\mathrm{a}^2 + \mathrm{b}^2)\sin(\theta°)}{\cos(\theta°)}$$

Equating eqns. (74.3) & (74.4),

$$\mathrm{PQ} \times \mathrm{OF_1}^2 = \mathrm{ON} \times \mathrm{b}^2 \qquad \ldots(74.5)$$

Eqn. (74.5) is mathematical expression of the theorem.

THEOREM- 75:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{PM \times {OF_1}^2 = MN \times b^2}$$

Proof of the theorem

$$\text{Eqn. (C. 14)} \Longrightarrow \mathrm{PM} = \frac{\mathrm{b}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\cos(\theta°)} \qquad \ldots(75.1)$$

$$\text{Eqn. (A. 3)} \Longrightarrow \mathrm{OF_1} = \sqrt{\mathrm{a}^2 + \mathrm{b}^2}$$

Squaring the above eqn.,

$$\mathrm{OF_1}^2 = \mathrm{a}^2 + \mathrm{b}^2 \qquad \ldots(75.2)$$

Multiplying eqns. (75.1) & (75.2),

$$\mathrm{PM} \times \mathrm{OF_1}^2 = \left(\frac{\mathrm{b}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\cos(\theta°)}\right) \times (\mathrm{a}^2 + \mathrm{b}^2)$$

$$\therefore \mathrm{PM} \times \mathrm{OF_1}^2 = \frac{\mathrm{b}^{\cancel{2}}(\mathrm{a}^2 + \mathrm{b}^2)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\cancel{\mathrm{b}}\cos(\theta°)}$$

$$\therefore \mathrm{PM} \times \mathrm{OF_1}^2 = \frac{\mathrm{b}(\mathrm{a}^2 + \mathrm{b}^2)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\cos(\theta°)} \qquad \ldots(75.3)$$

$$\text{Eqn. (C. 21)} \Longrightarrow \mathrm{MN} = \frac{(\mathrm{a}^2 + \mathrm{b}^2)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{ab}\cos(\theta°)}$$

Multiplying above eqn. by b^2,

$$\therefore \mathrm{MN} \times \mathrm{b}^2 = \frac{\mathrm{b}^2(\mathrm{a}^2 + \mathrm{b}^2)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{ab}\cos(\theta°)} \qquad \ldots(75.4)$$

Equating eqns. (75.3) & (75.4),

$$\mathrm{PM} \times \mathrm{OF_1}^2 = \mathrm{MN} \times \mathrm{b}^2 \qquad \ldots(75.5)$$

Eqn. (75.5) is mathematical expression of the theorem.

THEOREM- 76:

$$\boldsymbol{In\ a\ Hyperbola\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$$

$$\boldsymbol{OQ \times {OF_1}^2 = OM \times a^2}$$

Proof of the theorem

$$\text{Eqn. (B. 1)} \Longrightarrow \mathrm{OQ} = \frac{\mathrm{a}}{\cos(\theta°)} \qquad \ldots(76.1)$$

$$\text{Eqn. (A. 3)} \Longrightarrow \mathrm{OF_1} = \sqrt{\mathrm{a}^2 + \mathrm{b}^2}$$

Squaring the above eqn.,

$$\mathrm{OF_1}^2 = \mathrm{a}^2 + \mathrm{b}^2 \qquad \ldots(76.2)$$

Multiplying eqns. (76.1) & (76.2),

$$\mathrm{OQ} \times \mathrm{OF_1}^2 = \left(\frac{\mathrm{a}}{\cos(\theta°)}\right) \times (\mathrm{a}^2 + \mathrm{b}^2)$$

$$\therefore \mathrm{OQ} \times \mathrm{OF_1}^2 = \frac{\mathrm{a}(\mathrm{a}^2 + \mathrm{b}^2)}{\cos(\theta°)} \qquad \ldots(76.3)$$

$$\text{Eqn. (C. 6)} \Longrightarrow \mathrm{OM} = \frac{\mathrm{a}^2 + \mathrm{b}^2}{\mathrm{a}\cos(\theta°)}$$

Multiplying eqn. by a^2,

$$\mathrm{OM} \times \mathrm{a}^2 = \left(\frac{\mathrm{a}^2 + \mathrm{b}^2}{\cancel{\mathrm{a}}\cos(\theta°)}\right) \times \mathrm{a}^{\cancel{2}}$$

$$\therefore \mathrm{OM} \times \mathrm{a}^2 = \frac{\mathrm{a}(\mathrm{a}^2 + \mathrm{b}^2)}{\cos(\theta°)} \qquad \ldots(76.4)$$

Equating eqns. (76.3) & (76.4),

$$\mathrm{OM} \times \mathrm{a}^2 = \mathrm{OQ} \times \mathrm{OF_1}^2$$

Eqn. (76.4) is mathematical expression of the theorem.

THEOREM- 77:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$**,**

$\boldsymbol{PM \times TQ^2 = PL \times PQ^2}$

Proof of the theorem

$$\text{Eqn. (C. 14)} \Longrightarrow \mathrm{PM} = \frac{\mathrm{b}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\cos(\theta°)} \qquad \ldots(77.1)$$

$$\text{Eqn. (C. 4)} \Longrightarrow \mathrm{TQ} = \frac{\mathrm{a}\sin^2(\theta°)}{\cos(\theta°)}$$

Squaring above eqn.,

$$\mathrm{TQ}^2 = \left(\frac{\mathrm{a}\sin^2(\theta°)}{\cos(\theta°)}\right)^2$$

$$\therefore \mathrm{TQ}^2 = \frac{\mathrm{a}^2\sin^4(\theta°)}{\cos^2(\theta°)} \qquad \ldots(77.2)$$

$$\text{Eqn. (B. 2)} \Longrightarrow \mathrm{PQ} = \frac{\mathrm{b}sin(\theta°)}{\cos(\theta°)}$$

Multiplying eqns. (77.1) & (77.2),

$$\mathrm{PM} \times \mathrm{TQ}^2 = \left(\frac{\mathrm{b}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\cancel{\mathrm{a}}\cos(\theta°)}\right) \times \left(\frac{\mathrm{a}^{\cancel{2}}\sin^4(\theta°)}{\cos^2(\theta°)}\right)$$

$$\therefore \mathrm{PM} \times \mathrm{TQ}^2 = \frac{\mathrm{ab}\sin^4(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\cos^3(\theta°)} \qquad \ldots(77.3)$$

$$\text{Eqn. (D. 2)} \Longrightarrow \mathrm{PL} = \frac{\mathrm{a}\sin^2(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \quad \ldots(77.4)$$

$$\text{Eqn. (B. 2)} \Longrightarrow \mathrm{PQ} = \frac{\mathrm{b}\sin(\theta°)}{\cos(\theta°)}$$

Squaring above eqn.,

$$\mathrm{PQ}^2 = \frac{\mathrm{b}^2\sin^2(\theta°)}{\cos^2(\theta°)} \quad \ldots(77.5)$$

Multiplying eqns. (77.4) & (77.5),

$$\mathrm{PL} \times \mathrm{PQ}^2 = \left(\frac{\mathrm{a}\sin^2(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\cancel{\mathrm{b}}\cos(\theta°)}\right) \times \left(\frac{\mathrm{b}^{\cancel{2}}\sin^2(\theta°)}{\cos^2(\theta°)}\right)$$

$$\therefore \mathrm{PL} \times \mathrm{PQ}^2 = \frac{\mathrm{ab}\sin^4(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\cos^3(\theta°)} \quad \ldots(77.6)$$

Equating Eqns. (77.3) & (77.6),

$$\mathrm{PM} \times \mathrm{TQ}^2 = \mathrm{PL} \times \mathrm{PQ}^2 \quad \ldots(77.7)$$

Eqn. (77.7) is mathematical expression of the theorem.

THEOREM- 78:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{TW \times {OF_1}^2 = OL' \times TS^2}$

Proof of the theorem

$$\text{Eqn. (D. 3)} \Longrightarrow \mathrm{TW} = \frac{\mathrm{a}\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}} \quad \ldots(78.1)$$

$$\text{Eqn. (A. 3)} \Longrightarrow \mathrm{OF_1} = \sqrt{\mathrm{a}^2 + \mathrm{b}^2}$$

Squaring above eqn.,

$${\mathrm{OF_1}}^2 = \mathrm{a}^2 + \mathrm{b}^2 \quad \ldots(78.2)$$

Multiplying eqns. (82.1) & (82.2),

$$\mathrm{TW} \times {\mathrm{OF_1}}^2 = \left(\frac{\mathrm{a}\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}}\right) \times (\mathrm{a}^2 + \mathrm{b}^2)$$

$$\therefore \mathrm{TW} \times {\mathrm{OF_1}}^2 = \frac{(\mathrm{a}^2 + \mathrm{b}^2)\mathrm{a}\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}} \quad \ldots(78.3)$$

$$\text{Eqn. (E. 8)} \Longrightarrow \mathrm{OL'} = \frac{\mathrm{b}(\mathrm{a}^2 + \mathrm{b}^2)}{\mathrm{a}^2\sin(\theta°)\cos(\theta°)} \quad \ldots(78.4)$$

$$\text{Eqn. (C. 10)} \Longrightarrow \mathrm{TS} = \frac{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)}$$

Squaring above eqn.,

$$TS^2 = \frac{\cos^2(\theta°) \times (a^2\sin^2(\theta°) + b^2)}{\sin^2(\theta°)} \quad \ldots(78.5)$$

Multiplying eqns. (82.4) & (82.5),

$$OL' \times TS^2 = \left(\frac{b(a^2 + b^2)}{a^2\sin(\theta°)\cos(\theta°)}\right) \times \left(\frac{\cos^2(\theta°) \times (a^2\sin^2(\theta°) + b^2)}{\sin^2(\theta°)}\right)$$

$$\therefore PL \times PQ^2 = \frac{(a^2 + b^2)a\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{b} \quad \ldots(78.6)$$

Equating Eqns. (78.3) & (78.6),

$$TW \times {OF_1}^2 = PL \times PQ^2 \quad \ldots(78.7)$$

Eqn. (78.7) is mathematical expression of the theorem.

THEOREM- 79:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$

$\boldsymbol{ML' \times a^2 = QJ' \times {OF_1}^2}$

Proof of the theorem

$$\text{Eqn. (E. 7)} \Longrightarrow ML' = \frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{a^2\sin(\theta°)\cos(\theta°)}$$

Multiplying the above eqn. by a^2,

$$\therefore ML' \times a^2 = \left(\frac{\cancel{a^2}(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cancel{a^2}\sin(\theta°)\cos(\theta°)}\right) \quad \ldots(79.1)$$

$$\text{Eqn. (E. 6)} \Longrightarrow QJ' = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)\cos(\theta°)} \quad \ldots(79.2)$$

$$\text{Eqn. (A. 3)} \Longrightarrow OF_1 = \sqrt{a^2 + b^2}$$

Squaring above eqn.,

$${OF_1}^2 = a^2 + b^2 \quad \ldots(79.3)$$

Multiplying eqns. (79.2) & (79.3),

$$QJ' \times {OF_1}^2 = \frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)\cos(\theta°)} \quad \ldots(79.4)$$

Equating Eqns. (79.1) & (79.4),

$$ML' \times a^2 = QJ' \times {OF_1}^2 \quad \ldots(79.5)$$

Eqn. (79.5) is mathematical expression of the theorem.

THEOREM- 80:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$

$\boldsymbol{PM \times b^2 = TW \times QM^2}$

Proof of the theorem

$$\text{Eqn. (C.14)} \Longrightarrow \text{PM} = \frac{\text{b}\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\text{a}\cos(\theta°)}$$

Multiplying the above eqn. by b^2,

$$\text{PM} \times \text{b}^2 = \left(\frac{\text{b}\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\text{a}\cos(\theta°)}\right) \times \text{b}^2$$

$$\therefore \text{PM} \times \text{b}^2 = \frac{\text{b}^3\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\text{a}\cos(\theta°)} \qquad \ldots(80.1)$$

$$\text{Eqn. (D.3)} \Longrightarrow \text{TW} = \frac{\text{a}\cos(\theta°)\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\text{b}} \qquad \ldots(80.2)$$

$$\text{Eqn. (C.5)} \Longrightarrow \text{QM} = \frac{\text{b}^2}{\text{a}\cos(\theta°)}$$

Squaring above eqn.,

$$\text{QM}^2 = \frac{\text{b}^4}{\text{a}^2\cos^2(\theta°)} \qquad \ldots(80.3)$$

Multiplying eqns. (80.2) & (80.3),

$$\text{TW} \times \text{QM}^2 = \left(\frac{\cancel{\text{a}\cos(\theta°)}\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\text{b}}\right) \times \left(\frac{\text{b}^4}{\text{a}^2\cos^2(\theta°)}\right)$$

$$\therefore \text{TW} \times \text{QM}^2 = \frac{\text{b}^3\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\text{a}\cos(\theta°)} \qquad \ldots(80.4)$$

Equating Eqns. (80.1) & (80.4),

$$\text{PM} \times \text{b}^2 = \text{TW} \times \text{QM}^2 \qquad \ldots(80.5)$$

Eqn. (80.5) is mathematical expression of the theorem.

THEOREM- 81:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$$\boldsymbol{\frac{OQ \times TQ}{PQ^2} = \frac{a^2}{b^2}}$$

Proof of the theorem

$$\text{Eqn. (B.1)} \Longrightarrow \text{OQ} = \frac{\text{a}}{\cos(\theta°)} \qquad \ldots(81.1)$$

$$\text{Eqn. (C.4)} \Longrightarrow \text{TQ} = \frac{\text{a}\sin^2(\theta°)}{\cos(\theta°)} \qquad \ldots(81.2)$$

Multiplying eqns. (81.1), (81.2),

$$\text{OQ} \times \text{TQ} = \left(\frac{\text{a}}{\cos(\theta°)}\right) \times \left(\frac{\text{a}\sin^2(\theta°)}{\cos(\theta°)}\right)$$

$$\therefore \text{OQ} \times \text{TQ} = \frac{\text{a}^2\sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(81.3)$$

$$\text{Eqn. (B. 2)} \Longrightarrow \mathrm{PQ} = \frac{\mathrm{bsin}(\theta°)}{\cos(\theta°)}$$

Squaring above eqn.,

$$\mathrm{PQ}^2 = \left(\frac{\mathrm{bsin}(\theta°)}{\cos(\theta°)}\right)^2$$

$$\therefore \mathrm{PQ}^2 = \frac{\mathrm{b}^2\sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(81.4)$$

Dividing eqn. (81.3) by (81.4),

$$\frac{\mathrm{OQ} \times \mathrm{TQ}}{\mathrm{PQ}^2} = \left(\frac{\mathrm{a}^2\sin^2(\theta°)}{\cos^2(\theta°)}\right) \div \left(\frac{\mathrm{b}^2\sin^2(\theta°)}{\cos^2(\theta°)}\right)$$

$$\therefore \frac{\mathrm{OQ} \times \mathrm{TQ}}{\mathrm{PQ}^2} = \left(\frac{\mathrm{a}^2\cancel{\sin^2(\theta°)}}{\cancel{\cos^2(\theta°)}}\right) \times \left(\frac{\cancel{\cos^2(\theta°)}}{\mathrm{b}^2\cancel{\sin^2(\theta°)}}\right)$$

$$\therefore \frac{\mathrm{OQ} \times \mathrm{TQ}}{\mathrm{PQ}^2} = \frac{\mathrm{a}^2}{\mathrm{b}^2} \qquad \ldots(81.5)$$

Eqn. (81.5) is mathematical expression of the theorem.

THEOREM- 82:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{OM \times OY = ON \times OF_1}$

Proof of the theorem

Multiplying eqns. (91.1) & (91.2),

$$\text{Eqn. (C. 6)} \Longrightarrow \mathrm{OM} = \frac{\mathrm{a}^2 + \mathrm{b}^2}{\mathrm{acos}(\theta°)} \qquad \ldots(82.1)$$

$$\text{Eqn. (D. 11)} \Longrightarrow \mathrm{OY} = \frac{\mathrm{asin}(\theta°)\sqrt{\mathrm{a}^2 + \mathrm{b}^2}}{\mathrm{b}} \qquad \ldots(82.2)$$

Multiplying eqns. (82.1) & (82.2),

$$\mathrm{OM} \times \mathrm{OY} = \left(\frac{\mathrm{a}^2 + \mathrm{b}^2}{\cancel{\mathrm{a}}\cos(\theta°)}\right) \times \left(\frac{\cancel{\mathrm{a}}\sin(\theta°)\sqrt{\mathrm{a}^2 + \mathrm{b}^2}}{\mathrm{b}}\right)$$

$$\therefore \mathrm{OM} \times \mathrm{OY} = \frac{\sqrt{\mathrm{a}^2 + \mathrm{b}^2}(\mathrm{a}^2 + \mathrm{b}^2)\sin(\theta°)}{\mathrm{bcos}(\theta°)} \qquad \ldots(82.3)$$

$$\text{Eqn. (C. 7)} \Longrightarrow \mathrm{ON} = \frac{(\mathrm{a}^2 + \mathrm{b}^2)\sin(\theta°)}{\mathrm{bcos}(\theta°)} \qquad \ldots(82.4)$$

$$\text{Eqn. (A. 3)} \Longrightarrow \mathrm{OF}_1 = \sqrt{\mathrm{a}^2 + \mathrm{b}^2} \qquad \ldots(82.5)$$

Multiplying eqns. (82.4) & (82.5),

$$\mathrm{ON} \times \mathrm{OF}_1 = \left(\frac{(\mathrm{a}^2 + \mathrm{b}^2)\sin(\theta°)}{\mathrm{bcos}(\theta°)}\right) \times \sqrt{\mathrm{a}^2 + \mathrm{b}^2}$$

$$\therefore \mathrm{ON} \times \mathrm{OF_1} = \frac{\sqrt{a^2+b^2}(a^2+b^2)\sin(\theta°)}{b\cos(\theta°)} \quad \ldots(82.6)$$

Equating eqns. (82.3) & (82.6),

$$\mathrm{OM} \times \mathrm{OY} = \mathrm{ON} \times \mathrm{OF_1} \quad \ldots(82.7)$$

Eqn. (82.7) is mathematical expression of the theorem.

THEOREM- 83:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{PN \times TS = PS \times TW = VX \times F_1S}$

Proof of the theorem

$$\text{Eqn. (C. 19)} \Longrightarrow \mathrm{PN} = \frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)} \quad \ldots(83.1)$$

$$\text{Eqn. (C. 10)} \Longrightarrow \mathrm{TS} = \frac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)} \quad \ldots(83.2)$$

Multiplying eqns. (83.1) & (83.2),

$$\mathrm{PN} \times \mathrm{TS} = \left(\frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cancel{\cos(\theta°)}}\right) \times \left(\frac{\cancel{\cos(\theta°)}\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)}\right)$$

$$\therefore \mathrm{PN} \times \mathrm{TS} = \frac{a(a^2\sin^2(\theta°)+b^2)}{b\sin(\theta°)} \quad \ldots(83.3)$$

$$\text{Eqn. (C. 18)} \Longrightarrow \mathrm{PS} = \frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)\cos(\theta°)} \quad \ldots(83.4)$$

$$\text{Eqn. (D. 4)} \Longrightarrow \mathrm{TW} = \frac{a\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{b} \quad \ldots(83.5)$$

Multiplying eqns. (83.4) & (83.5),

$$\mathrm{PS} \times \mathrm{TW} = \left(\frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)\cancel{\cos(\theta°)}}\right) \times \left(\frac{a\cancel{\cos(\theta°)}\sqrt{a^2\sin^2(\theta°)+b^2}}{b}\right)$$

$$\therefore \mathrm{PS} \times \mathrm{TW} = \frac{a(a^2\sin^2(\theta°)+b^2)}{b\sin(\theta°)} \quad \ldots(83.6)$$

$$\text{Eqn. (D. 5)} \Longrightarrow \mathrm{VX} = \frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b} \quad \ldots(83.7)$$

$$\text{Eqn. (E. 4)} \Longrightarrow \mathrm{F_1S} = \frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)} \quad \ldots(83.8)$$

Multiplying eqns. (83.7) & (83.8),

$$\mathrm{VX} \times \mathrm{F_1S} = \left(\frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b}\right) \times \left(\frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)}\right)$$

$$\therefore \mathrm{VX} \times \mathrm{F_1S} = \frac{a(a^2\sin^2(\theta°)+b^2)}{b\sin(\theta°)} \quad \ldots(83.9)$$

Equating eqns. (83.3), (83.6), & (83.9),

$$\mathrm{PN} \times \mathrm{TS} = \mathrm{PS} \times \mathrm{TW} = \mathrm{VX} \times \mathrm{F_1S} \qquad \ldots(83.10)$$

Eqn. (83.10) is mathematical expression of the theorem.

THEOREM- 84:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{PS \times VX = PN \times F_1S}$

Proof of the theorem

$$\text{Eqn. (C. 16)} \Longrightarrow \mathrm{PS} = \frac{\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{sin(\theta°)cos(\theta°)}} \qquad \ldots(84.1)$$

$$\text{Eqn. (D. 4)} \Longrightarrow \mathrm{VX} = \frac{\mathrm{a}\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{b}} \qquad \ldots(84.2)$$

Multiplying eqns. (87.1) & (87.2),

$$\mathrm{PS} \times \mathrm{VX} = \left(\frac{\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{sin(\theta°)cos(\theta°)}}\right) \times \left(\frac{\mathrm{a}\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{b}}\right)$$

$$\therefore \mathrm{PS} \times \mathrm{VX} = \frac{\mathrm{a(a^2 sin^2(\theta°) + b^2)}}{\mathrm{bsin(\theta°)cos(\theta°)}} \qquad \ldots(84.3)$$

$$\text{Eqn. (C. 19)} \Longrightarrow \mathrm{PN} = \frac{\mathrm{a}\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{bcos(\theta°)}} \qquad \ldots(84.4)$$

$$\text{Eqn. (C. 13)} \Longrightarrow \mathrm{F_1S} = \frac{\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{sin(\theta°)}} \qquad \ldots(84.5)$$

Multiplying eqns. (84.4) & (84.5),

$$\mathrm{PN} \times \mathrm{F_1S} = \left(\frac{\mathrm{a}\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{bcos(\theta°)}}\right) \times \left(\frac{\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{sin(\theta°)}}\right)$$

$$\therefore \mathrm{PN} \times \mathrm{F_1S} = \frac{\mathrm{a(a^2 sin^2(\theta°) + b^2)}}{\mathrm{bsin(\theta°)cos^2(\theta°)}} \qquad \ldots(84.6)$$

Equating eqns. (84.3) & (84.6),

$$\mathrm{PF_1} \times \mathrm{F_2T} = \mathrm{PF_2} \times \mathrm{F_1T} \qquad \ldots(84.7)$$

Eqn. (84.7) is mathematical expression of the theorem.

THEOREM- 85:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{PL \times F_1S = PT \times VX}$

Proof of the theorem

$$\text{Eqn. (D. 2)} \Longrightarrow \mathrm{PL} = \frac{\mathrm{asin^2(\theta°)}\sqrt{\mathrm{a^2 sin^2(\theta°) + b^2}}}{\mathrm{bcos(\theta°)}} \qquad \ldots(85.1)$$

$$\text{Eqn. (C.13)} \Longrightarrow \mathrm{F_1S} = \frac{\sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{sin(\theta°)}} \qquad \ldots(85.2)$$

Multiplying eqns. (85.1) & (85.2),

$$\mathrm{PL} \times \mathrm{F_1S} = \left(\frac{\mathrm{asin^2(\theta°)}\sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{bcos(\theta°)}}\right) \times \left(\frac{\sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\cancel{\mathrm{sin(\theta°)}}}\right)$$

$$\therefore \mathrm{PL} \times \mathrm{F_1S} = \frac{\mathrm{asin(\theta°)(a^2sin^2(\theta°) + b^2)}}{\mathrm{bcos(\theta°)}} \qquad \ldots(85.3)$$

$$\text{Eqn. (C.15)} \Longrightarrow \mathrm{PT} = \frac{\mathrm{sin(\theta°)}\sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{cos(\theta°)}} \qquad \ldots(85.4)$$

$$\text{Eqn. (D.4)} \Longrightarrow \mathrm{VX} = \frac{\mathrm{a}\sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{b}} \qquad \ldots(85.5)$$

Multiplying eqns. (85.4) & (85.5),

$$\mathrm{PT} \times \mathrm{VX} = \left(\frac{\mathrm{sin(\theta°)}\sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{cos(\theta°)}}\right) \times \left(\frac{\mathrm{a}\sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{b}}\right)$$

$$\therefore \mathrm{PT} \times \mathrm{VX} = \frac{\mathrm{asin(\theta°)(a^2sin^2(\theta°) + b^2)}}{\mathrm{bcos(\theta°)}} \qquad \ldots(85.6)$$

Equating eqns. (85.3) & (85.6),

$$\mathrm{PL} \times \mathrm{F_1S} = \mathrm{PT} \times \mathrm{VX} \qquad \ldots(85.7)$$

Eqn. (85.7) is mathematical expression of the theorem.

THEOREM- 86:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{F_1N \times F_1S = PN \times F_1I' = PS \times F_1Y = SN \times OF_1}$

Proof of the theorem

$$\text{Eqn. (B.10)} \Longrightarrow \mathrm{F_1N} = \frac{\sqrt{\mathrm{a^2 + b^2}} \times \sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{bcos(\theta°)}} \qquad \ldots(86.1)$$

$$\text{Eqn. (C.13)} \Longrightarrow \mathrm{F_1S} = \frac{\sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{sin(\theta°)}} \qquad \ldots(86.2)$$

Multiplying eqns. (86.1) & (86.2),

$$\mathrm{F_1N} \times \mathrm{F_1S} = \left(\frac{\sqrt{\mathrm{a^2 + b^2}} \times \sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{bcos(\theta°)}}\right) \times \left(\frac{\sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{sin(\theta°)}}\right)$$

$$\therefore \mathrm{F_1N} \times \mathrm{F_1S} = \frac{\sqrt{\mathrm{a^2 + b^2}} \times \mathrm{(a^2sin^2(\theta°) + b^2)}}{\mathrm{bsin(\theta°)cos(\theta°)}} \qquad \ldots(86.3)$$

$$\text{Eqn. (C.19)} \Longrightarrow \mathrm{PN} = \frac{\mathrm{a}\sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{bcos(\theta°)}} \qquad \ldots(86.4)$$

$$\text{Eqn. (E.5)} \Longrightarrow F_1I' = \frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{a\sin(\theta°)} \qquad (86.5)$$

Multiplying eqns. (86.4) & (86.5),

$$PN \times F_1I' = \left(\frac{\cancel{a}\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}\right) \times \left(\frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{\cancel{a}\sin(\theta°)}\right)$$

$$\therefore PN \times F_1I' = \frac{\sqrt{a^2+b^2} \times (a^2\sin^2(\theta°)+b^2)}{b\sin(\theta°)\cos(\theta°)} \qquad \ldots(86.6)$$

$$\text{Eqn. (C.16)} \Longrightarrow PS = \frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(86.7)$$

$$\text{Eqn. (D.5)} \Longrightarrow F_1Y = \frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b} \qquad \ldots(86.8)$$

Multiplying eqns. (86.7) & (86.8),

$$PS \times F_1Y = \left(\frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)\cos(\theta°)}\right) \times \left(\frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b}\right)$$

$$\therefore PS \times F_1Y = \frac{\sqrt{a^2+b^2} \times (a^2\sin^2(\theta°)+b^2)}{b\sin(\theta°)\cos(\theta°)} \qquad \ldots(86.9)$$

$$\text{Eqn. (C.20)} \Longrightarrow SN = \frac{a^2\sin^2(\theta°)+b^2}{b\sin(\theta°)\cos(\theta°)} \qquad \ldots(86.10)$$

$$\text{Eqn. (A.3)} \Longrightarrow OF_1 = \sqrt{a^2+b^2} \qquad \ldots(86.11)$$

Multiplying eqns. (86.10) & (86.11),

$$SN \times OF_1 = \left(\frac{a^2\sin^2(\theta°)+b^2}{b\sin(\theta°)\cos(\theta°)}\right) \times \sqrt{a^2+b^2}$$

$$\therefore SN \times OF_1 = \frac{\sqrt{a^2+b^2} \times (a^2\sin^2(\theta°)+b^2)}{b\sin(\theta°)\cos(\theta°)} \qquad \ldots(86.12)$$

Equating eqns. (86.3), (86.6), (86.9) & (86.12),

$$F_1N \times F_1S = PN \times F_1I' = PS \times F_1Y = SN \times OF_1 \qquad \ldots(86.13)$$

Eqn. (86.13) is mathematical expression of the theorem.

THEOREM- 87:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{PN \times F_1Y = VX \times F_1N}$

Proof of the theorem

$$\text{Eqn. (C.19)} \Longrightarrow PN = \frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)} \qquad \ldots(87.1)$$

$$\text{Eqn. (D.5)} \Longrightarrow F_1Y = \frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b} \qquad \ldots(87.2)$$

Multiplying eqns. (87.1) & (87.2),

$$\mathrm{PN} \times \mathrm{F_1Y} = \left(\frac{\mathrm{a}\sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{bcos(\theta°)}}\right) \times \left(\frac{\sqrt{\mathrm{a^2 + b^2}} \times \sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{b}}\right)$$

$$\therefore \mathrm{PN} \times \mathrm{F_1Y} = \frac{\mathrm{a}\sqrt{\mathrm{a^2 + b^2}}(\mathrm{a^2sin^2(\theta°) + b^2})}{\mathrm{b^2cos(\theta°)}} \quad \ldots (87.3)$$

$$\mathrm{Eqn. (D.4)} \Longrightarrow \mathrm{VX} = \frac{\mathrm{a}\sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{b}} \quad \ldots (87.4)$$

$$\mathrm{Eqn. (B.10)} \Longrightarrow \mathrm{F_1N} = \frac{\sqrt{\mathrm{a^2 + b^2}} \times \sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{bcos(\theta°)}} \quad \ldots (87.5)$$

Multiplying eqns. (87.3) & (87.4),

$$\mathrm{VX} \times \mathrm{F_1N} = \left(\frac{\mathrm{a}\sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{b}}\right) \times \left(\frac{\sqrt{\mathrm{a^2 + b^2}} \times \sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{bcos(\theta°)}}\right)$$

$$\therefore \mathrm{VX} \times \mathrm{F_1N} = \frac{\mathrm{a}\sqrt{\mathrm{a^2 + b^2}}(\mathrm{a^2sin^2(\theta°) + b^2})}{\mathrm{b^2cos(\theta°)}} \quad \ldots (87.6)$$

Equating eqns. (87.3) & (87.6),

$$\mathrm{PN} \times \mathrm{F_1Y} = \mathrm{VX} \times \mathrm{F_1N} \quad \ldots (87.7)$$

Eqn. (87.7) is mathematical expression of the theorem.

THEOREM- 88:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$\boldsymbol{F_1Y \times F_1I' = MN \times TS}$

Proof of the theorem

$$\mathrm{Eqn. (D.5)} \Longrightarrow \mathrm{F_1Y} = \frac{\sqrt{\mathrm{a^2 + b^2}} \times \sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{b}} \quad \ldots (88.1)$$

$$\mathrm{Eqn. (E.5)} \Longrightarrow \mathrm{F_1I'} = \frac{\sqrt{\mathrm{a^2 + b^2}} \times \sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{asin(\theta°)}} \quad \ldots (88.2)$$

Multiplying eqns. (88.1) & (88.2),

$$\mathrm{F_1Y} \times \mathrm{F_1I'} = \left(\frac{\sqrt{\mathrm{a^2 + b^2}} \times \sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{b}}\right) \times \left(\frac{\sqrt{\mathrm{a^2 + b^2}} \times \sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{asin(\theta°)}}\right)$$

$$\therefore \mathrm{F_1Y} \times \mathrm{F_1I'} = \frac{(\mathrm{a^2 + b^2})(\mathrm{a^2sin^2(\theta°) + b^2})}{\mathrm{absin(\theta°)}} \quad \ldots (88.3)$$

$$\mathrm{Eqn. (C.21)} \Longrightarrow \mathrm{MN} = \frac{(\mathrm{a^2 + b^2})\sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{abcos(\theta°)}} \quad \ldots (88.4)$$

$$\mathrm{Eqn. (C.10)} \Longrightarrow \mathrm{TS} = \frac{\mathrm{cos(\theta°)}\sqrt{\mathrm{a^2sin^2(\theta°) + b^2}}}{\mathrm{sin(\theta°)}} \quad \ldots (88.5)$$

Multiplying eqns. (88.4) & (88.5),

$$\mathrm{MN} \times \mathrm{TS} = \left(\frac{(a^2+b^2)\sqrt{a^2\sin^2(\theta°)+b^2}}{ab\cancel{\cos(\theta°)}}\right) \times \left(\frac{\cancel{\cos(\theta°)}\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)}\right)$$

$$\therefore \mathrm{MN} \times \mathrm{TS} = \frac{(a^2+b^2)(a^2\sin^2(\theta°)+b^2)}{ab\sin(\theta°)} \quad \ldots(88.6)$$

Equating eqns. (88.5) & (88.6),

$$F_1Y \times F_1I' = \mathrm{MN} \times \mathrm{TS} \quad \ldots(88.7)$$

Eqn. (88.7) is mathematical expression of the theorem.

THEOREM- 89:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$

$\boldsymbol{RS \times OX = OV \times OQ}$

Proof of the theorem

$$\text{Eqn. (C. 18)} \Longrightarrow \mathrm{RS} = \frac{b}{\sin(\theta°)\cos(\theta°)} \quad \ldots(89.1)$$

$$\text{Eqn. (D. 10)} \Longrightarrow \mathrm{OX} = \frac{a^2\sin(\theta°)}{b} \quad \ldots(89.2)$$

Multiplying eqns. (89.1) & (89.2),

$$\mathrm{RS} \times \mathrm{OX} = \left(\frac{\cancel{b}}{\cancel{\sin(\theta°)}\cos(\theta°)}\right) \times \left(\frac{a^2\cancel{\sin(\theta°)}}{\cancel{b}}\right)$$

$$\therefore \mathrm{RS} \times \mathrm{OX} = \frac{a^2}{\cos(\theta°)} \quad \ldots(89.3)$$

$$\text{Eqn. (B. 1)} \Longrightarrow \mathrm{OQ} = \frac{a}{\cos(\theta°)}$$

Multiplying above eqn. by ‘a’,

$$\mathrm{OQ} \times a = \left(\frac{a}{\cos(\theta°)}\right) \times a$$

$$\therefore \mathrm{OQ} \times a = \frac{a^2}{\cos(\theta°)} \quad \ldots(89.4)$$

Equating eqns. (89.3) & (89.4),

$$\mathrm{RS} \times \mathrm{OX} = \mathrm{OQ} \times a \quad \ldots(89.5)$$

Eqn. (89.5) is mathematical expression of the theorem.

THEOREM- 90:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$

$\boldsymbol{OJ \times OZ = SJ \times RA' = OQ \times OS}$

Proof of the theorem

$$\text{Eqn. (D. 13)} \Longrightarrow \mathrm{OJ} = \frac{b^2\cancel{\cos(\theta°)}}{\cancel{a}\sin^{\cancel{2}}(\theta°)} \quad \ldots(90.1)$$

$$\text{Eqn. (D. 12)} \Longrightarrow \text{OZ} = \frac{\text{a}^2 \cancel{\sin(\theta°)}}{\cancel{\text{b}\cos(\theta°)}} \qquad \ldots(90.2)$$

Multiplying eqns. (90.1) & (90.2),

$$\text{OJ} \times \text{OZ} = \left(\frac{\text{b}^2 \cancel{\cos(\theta°)}}{\text{a}\sin^{\cancel{2}}(\theta°)}\right) \times \left(\frac{\text{a}^2 \cancel{\sin(\theta°)}}{\cancel{\text{b}\cos(\theta°)}}\right)$$

$$\therefore \text{OJ} \times \text{OZ} = \frac{\text{ab}}{\sin(\theta°)} \qquad \ldots(90.3)$$

$$\text{Eqn. (D. 14)} \Longrightarrow \text{SJ} = \frac{\text{b}\cos(\theta°)\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}{\text{a}\sin^2(\theta°)} \qquad \ldots(90.4)$$

$$\text{Eqn. (E. 3)} \Longrightarrow \text{RA}' = \frac{\text{a}^2\sin(\theta°)}{\cos(\theta°)\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}} \qquad \ldots(90.5)$$

Multiplying eqns. (90.4) & (90.5),

$$\text{SJ} \times \text{RA}' = \left(\frac{\text{b}\cancel{\cos(\theta°)\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}}{\cancel{\text{a}}\sin^{\cancel{2}}(\theta°)}\right) \times \left(\frac{\text{a}^{\cancel{2}} \cancel{\sin(\theta°)}}{\cancel{\cos(\theta°)\sqrt{\text{a}^2\sin^2(\theta°) + \text{b}^2}}}\right)$$

$$\therefore \text{SJ} \times \text{RA}' = \frac{\text{ab}}{\sin(\theta°)} \qquad \ldots(90.6)$$

$$\text{Eqn. (B. 1)} \Longrightarrow \text{OQ} = \frac{\text{a}}{\cos(\theta°)} \qquad \ldots(90.7)$$

$$\text{Eqn. (C. 2)} \Longrightarrow \text{OS} = \frac{\text{b}\cos(\theta°)}{\sin(\theta°)} \qquad \ldots(90.8)$$

Multiplying eqns. (90.7) & (90.8),

$$\text{OQ} \times \text{OS} = \left(\frac{\text{a}}{\cancel{\cos(\theta°)}}\right) \times \left(\frac{\text{b}\cancel{\cos(\theta°)}}{\sin(\theta°)}\right)$$

$$\therefore \text{OQ} \times \text{OS} = \frac{\text{ab}}{\sin(\theta°)} \qquad \ldots(90.9)$$

Equating eqns. (90.3), (90.6), & (90.9),

$$\text{OJ} \times \text{OZ} = \text{SJ} \times \text{RA}' = \text{OQ} \times \text{OS} \qquad \ldots(90.10)$$

Eqn. (90.10) is mathematical expression of the theorem.

THEOREM- 91:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$$\boldsymbol{PF_1 \times PF_2 = PT \times PS = PM \times PN = TM \times OQ = SN \times PQ = F_2M \times UQ = RS \times TL}$$
$$\boldsymbol{= OP^2 - (a^2 - b^2) = \left(PM \times \frac{a}{b}\right)^2 = \left(PN \times \frac{b}{a}\right)^2 = PM \times MN \times \frac{a^2}{a^2 + b^2}}$$

Proof of the theorem

$$\text{Eqn. (E. 9)} \Longrightarrow \text{PF}_1 \times \text{PF}_2 = \frac{\text{a}^2\sin^2(\theta°) + \text{b}^2}{\cos^2(\theta°)} \qquad \ldots(91.1)$$

$$\text{Eqn. (C.15)} \Longrightarrow \mathrm{PT} = \frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{\cos(\theta°)} \qquad \ldots(91.2)$$

$$\text{Eqn. (C.16)} \Longrightarrow \mathrm{PS} = \frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(91.3)$$

Multiplying eqns. (91.2) & (91.3),

$$\mathrm{PT}\times\mathrm{PS} = \left(\frac{\cancel{\sin(\theta°)}\sqrt{a^2\sin^2(\theta°)+b^2}}{\cos(\theta°)}\right)\times\left(\frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\cancel{\sin(\theta°)}\cos(\theta°)}\right)$$

$$\therefore \mathrm{PT}\times\mathrm{PS} = \frac{a^2\sin^2(\theta°)+b^2}{\cos^2(\theta°)} \qquad \ldots(91.4)$$

$$\text{Eqn. (C.14)} \Longrightarrow \mathrm{PM} = \frac{b\sqrt{a^2\sin^2(\theta°)+b^2}}{a\cos(\theta°)} \qquad \ldots(91.5)$$

$$\text{Eqn. (C.19)} \Longrightarrow \mathrm{PN} = \frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)} \qquad \ldots(91.6)$$

Multiplying eqns. (91.5) & (91.6),

$$\mathrm{PM}\times\mathrm{PN} = \left(\frac{b\sqrt{a^2\sin^2(\theta°)+b^2}}{a\cos(\theta°)}\right)\times\left(\frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}\right)$$

$$\therefore \mathrm{PM}\times\mathrm{PN} = \frac{a^2\sin^2(\theta°)+b^2}{\cos^2(\theta°)} \qquad \ldots(91.7)$$

$$\text{Eqn. (C.9)} \Longrightarrow \mathrm{TM} = \frac{a^2\sin^2(\theta°)+b^2}{a\cos(\theta°)} \qquad \ldots(91.8)$$

$$\text{Eqn. (B.1)} \Longrightarrow \mathrm{OQ} = \frac{a}{\cos(\theta°)} \qquad \ldots(91.9)$$

Multiplying eqns. (91.8) & (91.9),

$$\mathrm{TM}\times\mathrm{OQ} = \left(\frac{a^2\sin^2(\theta°)+b^2}{\cancel{a}\cos(\theta°)}\right)\times\left(\frac{\cancel{a}}{\cos(\theta°)}\right)$$

$$\therefore \mathrm{TM}\times\mathrm{OQ} = \frac{a^2\sin^2(\theta°)+b^2}{\cos^2(\theta°)} \qquad \ldots(91.10)$$

$$\text{Eqn. (C.20)} \Longrightarrow \mathrm{SN} = \frac{a^2\sin^2(\theta°)+b^2}{b\sin(\theta°)\cos(\theta°)} \qquad \ldots(91.11)$$

$$\text{Eqn. (B.2)} \Longrightarrow \mathrm{PQ} = \frac{b\sin(\theta°)}{\cos(\theta°)} \qquad \ldots(91.12)$$

Multiplying eqns. (91.11) & (91.12),

$$\mathrm{SN}\times\mathrm{PQ} = \left(\frac{a^2\sin^2(\theta°)+b^2}{\cancel{b\sin(\theta°)}\cos(\theta°)}\right)\times\left(\frac{\cancel{b\sin(\theta°)}}{\cos(\theta°)}\right)$$

$$\therefore \mathrm{SN}\times\mathrm{PQ} = \frac{a^2\sin^2(\theta°)+b^2}{\cos^2(\theta°)} \qquad \ldots(91.13)$$

$$\text{Eqn. (B.9)} \Longrightarrow F_2M = \frac{\sqrt{a^2+b^2}\left(\sqrt{a^2+b^2}+a\cos(\theta°)\right)}{a\cos(\theta°)} \qquad \ldots(91.14)$$

$$\text{Eqn. (B.3)} \Longrightarrow UQ = \frac{a\left[\sqrt{a^2+b^2}-a\cos(\theta°)\right]}{\sqrt{a^2+b^2}\times\cos(\theta°)} \qquad \ldots(91.15)$$

Multiplying eqns. (91.14) & (91.15),

$$F_2M \times UQ = \left(\frac{\cancel{\sqrt{a^2+b^2}}\left(\sqrt{a^2+b^2}+a\cos(\theta°)\right)}{\cancel{a}\cos(\theta°)}\right) \times \left(\frac{\cancel{a}\left[\sqrt{a^2+b^2}-a\cos(\theta°)\right]}{\cancel{\sqrt{a^2+b^2}}\times\cos(\theta°)}\right)$$

$$\therefore F_2M \times UQ = \left(\frac{\sqrt{a^2+b^2}+a\cos(\theta°)}{\cos(\theta°)}\right) \times \left(\frac{\sqrt{a^2+b^2}-a\cos(\theta°)}{\cos(\theta°)}\right)$$

$$\therefore F_2M \times UQ = \frac{a^2+b^2-a^2\cos^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore F_2M \times UQ = \frac{a^2\left(1-\cos^2(\theta°)\right)+b^2}{\cos^2(\theta°)}$$

$$\therefore F_2M \times UQ = \frac{a^2\sin^2(\theta°)+b^2}{\cos^2(\theta°)} \qquad \ldots(91.16)$$

$$\text{Eqn. (C.18)} \Longrightarrow RS = \frac{b}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(91.17)$$

$$\text{Eqn. (D.1)} \Longrightarrow TL = \frac{\sin(\theta°)[a^2\sin^2(\theta°)+b^2]}{b\cos(\theta°)} \qquad \ldots(91.18)$$

Multiplying eqns. (91.17) & (91.18),

$$RS \times TL = \left(\frac{\cancel{b}}{\cancel{\sin(\theta°)}\cos(\theta°)}\right) \times \left(\frac{\cancel{\sin(\theta°)}[a^2\sin^2(\theta°)+b^2]}{\cancel{b}\cos(\theta°)}\right)$$

$$\therefore RS \times TL = \frac{a^2\sin^2(\theta°)+b^2}{\cos^2(\theta°)} \qquad \ldots(91.19)$$

$$\text{Eqn. (B.5)} \Longrightarrow OP = \frac{\sqrt{a^2+b^2\sin^2(\theta°)}}{\cos(\theta°)}$$

Squaring above eqn.,

$$\therefore OP^2 = \frac{a^2+b^2\sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(91.20)$$

Subtracting $a^2 - b^2$ from eqn. (91. 20)

$$OP^2 - (a^2-b^2) = \left(\frac{a^2+b^2\sin^2(\theta°)}{\cos^2(\theta°)}\right) - (a^2-b^2)$$

$$\therefore OP^2 - (a^2-b^2) = \frac{a^2+b^2\sin^2(\theta°)-(a^2-b^2)\cos^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore OP^2 - (a^2-b^2) = \frac{a^2+b^2\sin^2(\theta°)-\left(a^2\cos^2(\theta°)-b^2\cos^2(\theta°)\right)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{OP}^2 - (a^2 - b^2) = \frac{a^2 + b^2\sin^2(\theta°) - a^2\cos^2(\theta°) + b^2\cos^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{OP}^2 - (a^2 - b^2) = \frac{a^2 - a^2\cos^2(\theta°) + b^2\sin^2(\theta°) + b^2\cos^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{OP}^2 - (a^2 - b^2) = \frac{a^2[1 - \cos^2(\theta°)] + b^2[\sin^2(\theta°) + \cos^2(\theta°)]}{\cos^2(\theta°)}$$

$$\therefore \mathrm{OP}^2 - (a^2 - b^2) = \frac{a^2\sin^2(\theta°) + b^2}{\cos^2(\theta°)} \qquad \ldots(91.21)$$

$$\text{Eqn. (C. 14)} \Longrightarrow \mathrm{PM} = \frac{b\sqrt{a^2\sin^2(\theta°) + b^2}}{a\cos(\theta°)}$$

Squaring above eqn.,

$$\mathrm{PM}^2 = \frac{b^2(a^2\sin^2(\theta°) + b^2)}{a^2\cos^2(\theta°)} \qquad \ldots(91.22)$$

Multiplying eqns. (91.22) by, $\frac{a^2}{b^2}$

$$\mathrm{PM}^2 \times \frac{a^2}{b^2} = \left(\frac{\cancel{b^2}(a^2\sin^2(\theta°) + b^2)}{\cancel{a^2}\cos^2(\theta°)}\right) \times \left(\frac{\cancel{a^2}}{\cancel{b^2}}\right)$$

$$\therefore \mathrm{PM}^2 \times \frac{a^2}{b^2} = \frac{a^2\sin^2(\theta°) + b^2}{\cos^2(\theta°)} \qquad \ldots(91.23)$$

$$\text{Eqn. (C. 19)} \Longrightarrow \mathrm{PN} = \frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)}$$

Squaring above eqn.,

$$\mathrm{PN}^2 = \frac{a^2(a^2\sin^2(\theta°) + b^2)}{b^2\cos^2(\theta°)} \qquad \ldots(91.24)$$

Multiplying eqns. (91.24) by, $\frac{b^2}{a^2}$

$$\mathrm{PN}^2 \times \frac{b^2}{a^2} = \left(\frac{\cancel{a^2}(a^2\sin^2(\theta°) + b^2)}{\cancel{b^2}\cos^2(\theta°)}\right) \times \left(\frac{\cancel{b^2}}{\cancel{a^2}}\right)$$

$$\therefore \mathrm{PN}^2 \times \frac{b^2}{a^2} = \frac{a^2\sin^2(\theta°) + b^2}{\cos^2(\theta°)} \qquad \ldots(91.25)$$

$$\text{Eqn. (C. 14)} \Longrightarrow \mathrm{PM} = \frac{b\sqrt{a^2\sin^2(\theta°) + b^2}}{a\cos(\theta°)}$$

$$\text{Eqn. (C. 21)} \Longrightarrow \mathrm{MN} = \frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{ab\cos(\theta°)}$$

Multiplying above two eqns. & $\frac{a^2}{a^2 + b^2}$

$$\mathrm{PM} \times \mathrm{MN} \times \frac{a^2}{a^2 + b^2} = \left(\frac{\cancel{b}\sqrt{a^2\sin^2(\theta°) + b^2}}{\cancel{a}\cos(\theta°)}\right) \times \left(\frac{(\cancel{a^2 + b^2})\sqrt{a^2\sin^2(\theta°) + b^2}}{\cancel{ab}\cos(\theta°)}\right) \times \left(\frac{\cancel{a^2}}{\cancel{a^2 + b^2}}\right)$$

$$\therefore \mathrm{PM} \times \mathrm{MN} \times \left(\frac{a^2}{a^2+b^2}\right) = \frac{a^2\sin^2(\theta°)+b^2}{\cos^2(\theta°)} \qquad \ldots(91.26)$$

Equating Eqns. (91.1) to (91.26),

$$\mathrm{PF_1} \times \mathrm{PF_2} = \mathrm{PT} \times \mathrm{PS} = \mathrm{PM} \times \mathrm{PN} = \mathrm{TM} \times \mathrm{OQ} = \mathrm{SN} \times \mathrm{PQ} = \mathrm{F_2M} \times \mathrm{UQ} = \mathrm{RS} \times \mathrm{TL} =$$

$$= \mathrm{OP}^2 - (a^2-b^2) = \left(\mathrm{PM} \times \frac{a}{b}\right)^2 = \left(\mathrm{PN} \times \frac{b}{a}\right)^2 = \mathrm{PM} \times \mathrm{MN} \times \frac{a^2}{a^2+b^2} \qquad \ldots(91.27)$$

Eqn. (91.27) is mathematical expression of the theorem.

THEOREM- 92:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$$\boldsymbol{\frac{PN}{F_1N} = \frac{F_1N}{MN} = \frac{VX}{F_1Y} = \frac{F_1S}{F_1I'} = \frac{PS}{ML'} = \frac{a}{\sqrt{a^2+b^2}}}$$

Proof of the theorem

$$\text{Eqn. (C.19)} \Longrightarrow \mathrm{PN} = \frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)} \qquad \ldots(92.1)$$

$$\text{Eqn. (B.10)} \Longrightarrow \mathrm{F_1N} = \frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)} \qquad \ldots(92.2)$$

Dividing eqn. (92.1) by (92.2),

$$\frac{\mathrm{PN}}{\mathrm{F_1N}} = \left(\frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}\right) \div \left(\frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}\right)$$

$$\therefore \frac{\mathrm{PN}}{\mathrm{F_1N}} = \left(\frac{a\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}{\cancel{b\cos(\theta°)}}\right) \times \left(\frac{\cancel{b\cos(\theta°)}}{\sqrt{a^2+b^2} \times \cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}\right)$$

$$\therefore \frac{\mathrm{PN}}{\mathrm{F_1N}} = \frac{a}{\sqrt{a^2+b^2}} \qquad \ldots(92.3)$$

$$\text{Eqn. (B.10)} \Longrightarrow \mathrm{F_1N} = \frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)} \qquad \ldots(92.4)$$

$$\text{Eqn. (C.21)} \Longrightarrow \mathrm{MN} = \frac{(a^2+b^2)\sqrt{a^2\sin^2(\theta°)+b^2}}{a\cos(\theta°)} \qquad \ldots(92.5)$$

Dividing eqn. (92.4) by (92.5),

$$\frac{\mathrm{F_1N}}{\mathrm{MN}} = \left(\frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}\right) \div \left(\frac{(a^2+b^2)\sqrt{a^2\sin^2(\theta°)+b^2}}{a\cos(\theta°)}\right)$$

$$\therefore \frac{\mathrm{F_1N}}{\mathrm{MN}} = \left(\frac{\sqrt{a^2+b^2} \times \cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}{\cancel{b\cos(\theta°)}}\right) \times \left(\frac{\cancel{a\cos(\theta°)}}{(a^2+b^2)\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}\right)$$

$$\therefore \frac{\mathrm{F_1N}}{\mathrm{MN}} = \frac{a}{\sqrt{a^2+b^2}} \qquad \ldots(92.6)$$

$$\text{Eqn. (D.4)} \Longrightarrow \mathrm{VX} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}} \qquad \ldots(92.7)$$

$$\text{Eqn. (D.5)} \Longrightarrow \mathrm{F_1Y} = \frac{\sqrt{\mathrm{a}^2 + \mathrm{b}^2} \times \sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}} \qquad \ldots(92.8)$$

Dividing eqn. (92.7) by (92.8),

$$\frac{\mathrm{VX}}{\mathrm{F_1Y}} = \left(\frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}}\right) \div \left(\frac{\sqrt{\mathrm{a}^2 + \mathrm{b}^2} \times \sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}}\right)$$

$$\therefore \frac{\mathrm{VX}}{\mathrm{F_1Y}} = \left(\frac{\mathrm{a}\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}{\cancel{\mathrm{b}}}\right) \times \left(\frac{\cancel{\mathrm{b}}}{\sqrt{\mathrm{a}^2 + \mathrm{b}^2} \times \sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}\right)$$

$$\therefore \frac{\mathrm{VX}}{\mathrm{F_1Y}} = \frac{\mathrm{a}}{\sqrt{\mathrm{a}^2 + \mathrm{b}^2}} \qquad \ldots(92.9)$$

$$\text{Eqn. (C.13)} \Longrightarrow \mathrm{F_1S} = \frac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)} \qquad \ldots(92.10)$$

$$\text{Eqn. (E.5)} \Longrightarrow \mathrm{F_1I'} = \frac{\sqrt{\mathrm{a}^2 + \mathrm{b}^2} \times \sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\sin(\theta°)} \qquad \ldots(92.11)$$

Dividing eqn. (92.10) by (92.11),

$$\frac{\mathrm{F_1S}}{\mathrm{F_1I'}} = \left(\frac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)}\right) \div \left(\frac{\sqrt{\mathrm{a}^2 + \mathrm{b}^2} \times \sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\sin(\theta°)}\right)$$

$$\therefore \frac{\mathrm{F_1S}}{\mathrm{F_1I'}} = \left(\frac{\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}{\cancel{\sin(\theta°)}}\right) \times \left(\frac{\cancel{\mathrm{a}\sin(\theta°)}}{\sqrt{\mathrm{a}^2 + \mathrm{b}^2} \times \cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}\right)$$

$$\therefore \frac{\mathrm{F_1S}}{\mathrm{F_1I'}} = \frac{\mathrm{a}}{\sqrt{\mathrm{a}^2 + \mathrm{b}^2}} \qquad \ldots(92.12)$$

$$\text{Eqn. (C.16)} \Longrightarrow \mathrm{PS} = \frac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(92.13)$$

$$\text{Eqn. (E.7)} \Longrightarrow \mathrm{ML'} = \frac{(\mathrm{a}^2 + \mathrm{b}^2)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}^2\sin(\theta°)\cos(\theta°)} \qquad \ldots(92.14)$$

Dividing eqn. (92.13) by (92.14),

$$\frac{\mathrm{PS}}{\mathrm{ML'}} = \left(\frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)}\right) \div \left(\frac{(\mathrm{a}^2 + \mathrm{b}^2)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}^2\sin(\theta°)\cos(\theta°)}\right)$$

$$\therefore \frac{\mathrm{PS}}{\mathrm{ML'}} = \left(\frac{\mathrm{a}\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}{\cancel{\mathrm{b}\cos(\theta°)}}\right) \times \left(\frac{\cancel{\mathrm{b}\cos(\theta°)}}{\sqrt{\mathrm{a}^2 + \mathrm{b}^2} \times \cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}\right)$$

$$\therefore \frac{\mathrm{PS}}{\mathrm{ML'}} = \frac{\mathrm{a}}{\sqrt{\mathrm{a}^2 + \mathrm{b}^2}} \qquad \ldots(92.15)$$

Equating eqns. (92.3), (92.6), (92.9), (92.12) & (92.15),

$$\frac{\mathrm{PN}}{\mathrm{F_1N}} = \frac{\mathrm{F_1N}}{\mathrm{MN}} = \frac{\mathrm{VX}}{\mathrm{F_1Y}} = \frac{\mathrm{F_1S}}{\mathrm{F_1I'}} = \frac{\mathrm{PS}}{\mathrm{ML'}} = \frac{\mathrm{a}}{\sqrt{\mathrm{a}^2 + \mathrm{b}^2}} \qquad \ldots(92.16)$$

Eqn. (92.16) is mathematical expression of the theorem.

THEOREM- 93:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$$\boldsymbol{\frac{PM}{MN} = \frac{b^2}{a^2 + b^2}}$$

Proof of the theorem

$$\text{Eqn. (C.14)} \Longrightarrow \mathrm{PM} = \frac{b\sqrt{a^2\sin^2(\theta°) + b^2}}{a\cos(\theta°)} \qquad \ldots(93.1)$$

$$\text{Eqn. (C.21)} \Longrightarrow \mathrm{MN} = \frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{ab\cos(\theta°)} \qquad \ldots(93.2)$$

Dividing eqns. (93.1) & (93.2),

$$\frac{\mathrm{PM}}{\mathrm{MN}} = \left(\frac{b\sqrt{a^2\sin^2(\theta°) + b^2}}{a\cos(\theta°)}\right) \div \left(\frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{ab\cos(\theta°)}\right)$$

$$\frac{\mathrm{PM}}{\mathrm{MN}} = \left(\frac{b\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{\cancel{a\cos(\theta°)}}\right) \times \left(\frac{\cancel{ab\cos(\theta°)}}{(a^2 + b^2)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \frac{\mathrm{PM}}{\mathrm{MN}} = \frac{b^2}{a^2 + b^2} \qquad \ldots(93.3)$$

Eqn. (93.3) is mathematical expression of the theorem.

THEOREM- 94:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$$\boldsymbol{\frac{PM}{F_1N} = \frac{b^2}{a\sqrt{a^2 + b^2}}}$$

Proof of the theorem

$$\text{Eqn. (C.14)} \Longrightarrow \mathrm{PM} = \frac{b\sqrt{a^2\sin^2(\theta°) + b^2}}{a\cos(\theta°)} \qquad \ldots(94.1)$$

$$\text{Eqn. (B.10)} \Longrightarrow \mathrm{F_1N} = \frac{\sqrt{a^2 + b^2} \times \sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)} \qquad \ldots(94.2)$$

Dividing eqns. (94.1) & (94.2),

$$\frac{\mathrm{PM}}{\mathrm{F_1N}} = \left(\frac{b\sqrt{a^2\sin^2(\theta°) + b^2}}{a\cos(\theta°)}\right) \div \left(\frac{\sqrt{a^2 + b^2} \times \sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)}\right)$$

$$\frac{\mathrm{PM}}{\mathrm{F_1N}} = \left(\frac{b\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{a\cancel{\cos(\theta°)}}\right) \times \left(\frac{b\cancel{\cos(\theta°)}}{\sqrt{a^2 + b^2} \times \cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \frac{\mathrm{PM}}{\mathrm{F_1N}} = \frac{b^2}{a\sqrt{a^2 + b^2}} \qquad \ldots(94.3)$$

Eqn. (94.3) is mathematical expression of the theorem.

THEOREM- 95:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$$\boldsymbol{\frac{UF_1}{UF_2} = \frac{b^2}{2a^2 + b^2}}$$

Proof of the theorem

$$\text{Eqn. (A.9)} \Longrightarrow \mathrm{UF_1} = \frac{b^2}{\sqrt{a^2 + b^2}} \qquad \dots (95.1)$$

$$\text{Eqn. (A.10)} \Longrightarrow \mathrm{UF_2} = \frac{2a^2 + b^2}{\sqrt{a^2 + b^2}} \qquad \dots (95.2)$$

Dividing eqn. (95.1) from (95.2),

$$\frac{\mathrm{UF_1}}{\mathrm{UF_2}} = \left(\frac{b^2}{\sqrt{a^2 + b^2}}\right) \div \left(\frac{2a^2 + b^2}{\sqrt{a^2 + b^2}}\right)$$

$$\therefore \frac{\mathrm{UF_1}}{\mathrm{UF_2}} = \left(\frac{b^2}{\cancel{\sqrt{a^2 + b^2}}}\right) \times \left(\frac{\cancel{\sqrt{a^2 + b^2}}}{2a^2 + b^2}\right)$$

$$\therefore \frac{\mathrm{UF_1}}{\mathrm{UF_2}} = \frac{b^2}{2a^2 + b^2} \qquad \dots (95.3)$$

Eqn. (95.3) is mathematical expression of the theorem.

THEOREM- 96:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$$\boldsymbol{\frac{PN}{PM} = \frac{OQ}{QM} = \frac{RN}{PQ} = \frac{OU}{OF_1} = \left(\frac{OV}{OB}\right)^2 = \frac{a^2}{b^2}}$$

Proof of the theorem

$$\text{Eqn. (C.19)} \Longrightarrow \mathrm{PN} = \frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)} \qquad \dots (96.1)$$

$$\text{Eqn. (C.14)} \Longrightarrow \mathrm{PM} = \frac{b\sqrt{a^2\sin^2(\theta°) + b^2}}{a\cos(\theta°)} \qquad \dots (96.2)$$

Dividing eqn. (96.1) by (96.2),

$$\frac{\mathrm{PN}}{\mathrm{PM}} = \left(\frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)}\right) \div \left(\frac{b\sqrt{a^2\sin^2(\theta°) + b^2}}{a\cos(\theta°)}\right)$$

$$\therefore \frac{\mathrm{PN}}{\mathrm{PM}} = \left(\frac{a\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{b\cancel{\cos(\theta°)}}\right) \times \left(\frac{a\cancel{\cos(\theta°)}}{b\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \frac{\mathrm{PN}}{\mathrm{PM}} = \frac{a^2}{b^2} \qquad \dots (96.3)$$

$$\text{Eqn. (B.1)} \Longrightarrow \mathrm{OQ} = \frac{a}{\cos(\theta°)} \quad \ldots(96.4)$$

$$\text{Eqn. (C.5)} \Longrightarrow \mathrm{QM} = \frac{b^2}{a\cos(\theta°)} \quad \ldots(96.5)$$

Dividing eqn. (96.4) by (96.5),

$$\frac{\mathrm{OQ}}{\mathrm{QM}} = \left(\frac{a}{\cos(\theta°)}\right) \div \left(\frac{b^2}{a\cos(\theta°)}\right)$$

$$\therefore \frac{\mathrm{OQ}}{\mathrm{QM}} = \left(\frac{a}{\cancel{\cos(\theta°)}}\right) \times \left(\frac{a\cancel{\cos(\theta°)}}{b^2}\right)$$

$$\therefore \frac{\mathrm{OQ}}{\mathrm{QM}} = \frac{a^2}{b^2} \quad \ldots(96.6)$$

$$\text{Eqn. (C.17)} \Longrightarrow \mathrm{RN} = \frac{a^2\sin(\theta°)}{b\cos(\theta°)} \quad \ldots(96.7)$$

$$\text{Eqn. (B.2)} \Longrightarrow \mathrm{PQ} = \frac{b\sin(\theta°)}{\cos(\theta°)} \quad \ldots(96.8)$$

Dividing eqn. (96.7) by (96.8),

$$\frac{\mathrm{RN}}{\mathrm{PQ}} = \left(\frac{a^2\sin(\theta°)}{b\cos(\theta°)}\right) \div \left(\frac{b\sin(\theta°)}{\cos(\theta°)}\right)$$

$$\therefore \frac{\mathrm{RN}}{\mathrm{PQ}} = \left(\frac{a^2\cancel{\sin(\theta°)}}{b\cancel{\cos(\theta°)}}\right) \times \left(\frac{\cancel{\cos(\theta°)}}{b\cancel{\sin(\theta°)}}\right)$$

$$\therefore \frac{\mathrm{RN}}{\mathrm{PQ}} = \frac{a^2}{b^2} \quad \ldots(96.9)$$

$$\text{Eqn. (C.19)} \Longrightarrow \mathrm{OV} = a \quad \ldots(96.10)$$

$$\text{Eqn. (B.2)} \Longrightarrow \mathrm{OB} = b \quad \ldots(96.11)$$

Dividing squares of eqn. (96.10) by (96.11),

$$\therefore \frac{\mathrm{OV}^2}{\mathrm{OB}^2} = \frac{a^2}{b^2} \quad \ldots(96.12)$$

$$\text{Eqn. (A.7)} \Longrightarrow \mathrm{OU} = \frac{a^2}{\sqrt{a^2+b^2}} \quad \ldots(96.13)$$

$$\text{Eqn. (A.9)} \Longrightarrow \mathrm{UF_1} = \frac{b^2}{\sqrt{a^2+b^2}} \quad \ldots(96.14)$$

Dividing eqn. (96.13) by (96.14),

$$\frac{\mathrm{OU}}{\mathrm{OF_1}} = \left(\frac{a^2}{\sqrt{a^2+b^2}}\right) \div \left(\frac{b^2}{\sqrt{a^2+b^2}}\right)$$

$$\therefore \frac{\mathrm{OU}}{\mathrm{OF_1}} = \frac{a^2}{\cancel{\sqrt{a^2+b^2}}} \times \frac{\cancel{\sqrt{a^2+b^2}}}{b^2}$$

$$\therefore \frac{\mathrm{OU}}{\mathrm{OF_1}} = \frac{a^2}{b^2} \quad \ldots(96.15)$$

$$\frac{\mathrm{PN}}{\mathrm{PM}}=\frac{\mathrm{OQ}}{\mathrm{QM}}=\frac{\mathrm{RN}}{\mathrm{PQ}}=\frac{\mathrm{OU}}{\mathrm{OF_1}}=\left(\frac{\mathrm{OV}}{\mathrm{OB}}\right)^2=\frac{a^2}{b^2} \qquad \ldots(96.16)$$

Eqn. (96.16) is mathematical expression of the theorem.

THEOREM- 97:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1}$,

$$\boldsymbol{\frac{PM}{PS}=\frac{PT}{PN}=\frac{TM}{SN}=\frac{QM}{RS}=\frac{QE'}{RA'}}$$

Proof of the theorem

$$\text{Eqn. (C.14)} \Longrightarrow \mathrm{PM}=\frac{b\sqrt{a^2\sin^2(\theta°)+b^2}}{a\cos(\theta°)} \qquad \ldots(97.1)$$

$$\text{Eqn. (C.16)} \Longrightarrow \mathrm{PS}=\frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(97.2)$$

Dividing eqns. (97.1) by (97.2),

$$\therefore \frac{\mathrm{PM}}{\mathrm{PS}}=\left(\frac{b\sqrt{a^2\sin^2(\theta°)+b^2}}{a\cos(\theta°)}\right)\div\left(\frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)\cos(\theta°)}\right)$$

$$\frac{\mathrm{PM}}{\mathrm{PS}}=\left(\frac{b\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}{a\cancel{\cos(\theta°)}}\right)\times\left(\frac{\sin(\theta°)\cancel{\cos(\theta°)}}{\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}\right)$$

$$\therefore \frac{\mathrm{PM}}{\mathrm{PS}}=\frac{b\sin(\theta°)}{a} \qquad \ldots(97.3)$$

$$\text{Eqn. (C.15)} \Longrightarrow \mathrm{PT}=\frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{\cos(\theta°)} \qquad \ldots(97.4)$$

$$\text{Eqn. (C.19)} \Longrightarrow \mathrm{PN}=\frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)} \qquad \ldots(97.5)$$

Dividing eqns. (97.4) by (97.5),

$$\frac{\mathrm{PT}}{\mathrm{PN}}=\left(\frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{\cos(\theta°)}\right)\div\left(\frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}\right)$$

$$\therefore \frac{\mathrm{PT}}{\mathrm{PN}}=\left(\frac{\sin(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}{\cancel{\cos(\theta°)}}\right)\times\left(\frac{b\cancel{\cos(\theta°)}}{a\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}\right)$$

$$\therefore \frac{\mathrm{PT}}{\mathrm{PN}}=\frac{b\sin(\theta°)}{a} \qquad \ldots(97.6)$$

$$\text{Eqn. (C.9)} \Longrightarrow \mathrm{TM}=\frac{a^2\sin^2(\theta°)+b^2}{a\cos(\theta°)} \qquad \ldots(97.7)$$

$$\text{Eqn. (C.20)} \Longrightarrow \mathrm{SN}=\frac{a^2\sin^2(\theta°)+b^2}{b\sin(\theta°)\cos(\theta°)} \qquad \ldots(97.8)$$

Dividing eqns. (97.7) by (97.8),

$$\frac{TM}{SN} = \left(\frac{a^2\sin^2(\theta°) + b^2}{a\cos(\theta°)}\right) \div \left(\frac{a^2\sin^2(\theta°) + b^2}{b\sin(\theta°)\cos(\theta°)}\right)$$

$$\therefore \frac{TM}{SN} = \left(\frac{\cancel{a^2\sin^2(\theta°) + b^2}}{a\cancel{\cos(\theta°)}}\right) \times \left(\frac{b\sin(\theta°)\cancel{\cos(\theta°)}}{\cancel{a^2\sin^2(\theta°) + b^2}}\right)$$

$$\therefore \frac{TM}{SN} = \frac{b\sin(\theta°)}{a} \qquad \ldots(97.9)$$

$$\text{Eqn. (C. 5)} \Longrightarrow QM = \frac{b^2}{a\cos(\theta°)} \qquad \ldots(97.10)$$

$$\text{Eqn. (C. 18)} \Longrightarrow RS = \frac{b}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(97.11)$$

Dividing eqns. (97.10) by (97.11),

$$\frac{QM}{RS} = \left(\frac{b^2}{a\cos(\theta°)}\right) \div \left(\frac{b}{\sin(\theta°)\cos(\theta°)}\right)$$

$$\therefore \frac{QM}{RS} = \left(\frac{b^{\cancel{2}}}{a\cancel{\cos(\theta°)}}\right) \times \left(\frac{\sin(\theta°)\cancel{\cos(\theta°)}}{\cancel{b}}\right)$$

$$\therefore \frac{QM}{RS} = \frac{b\sin(\theta°)}{a} \qquad \ldots(97.12)$$

$$\text{Eqn. (E. 2)} \Longrightarrow QE' = \frac{ab\sin^2(\theta°)}{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}} \qquad \ldots(97.13)$$

$$\text{Eqn. (E. 3)} \Longrightarrow RA' = \frac{a^2\sin(\theta°)}{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}} \qquad \ldots(97.14)$$

Dividing eqns. (97.13) by (97.14),

$$\frac{QE'}{RA'} = \left(\frac{ab\sin^2(\theta°)}{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}\right) \div \left(\frac{a^2\sin(\theta°)}{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}\right)$$

$$\therefore \frac{QE'}{RA'} = \left(\frac{\cancel{a}b\sin^{\cancel{2}}(\theta°)}{\cancel{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}}\right) \times \left(\frac{\cancel{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}}{a^{\cancel{2}}\cancel{\sin(\theta°)}}\right)$$

$$\therefore \frac{QE'}{RA'} = \frac{b\sin(\theta°)}{a} \qquad \ldots(97.15)$$

Equating eqns. (97.3), (97.6), (97.9), (97.12) & (97.15),

$$\frac{PM}{PS} = \frac{PT}{PN} = \frac{TM}{SN} = \frac{QM}{RS} = \frac{QE'}{RA'} \qquad \ldots(97.16)$$

Eqn. (97.16) is mathematical expression of the theorem.

THEOREM- 98:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$$\boldsymbol{\frac{F_1Y}{F_1N} = \frac{OV}{OQ} = \frac{VX}{PN} = \frac{OX}{OZ} = \frac{TS}{F_1S} = \frac{F_2S}{PS} = \frac{TW}{VX} = \sqrt{\frac{TK}{RA'}}}$$

Proof of the theorem

$$\text{Eqn. (D.5)} \Longrightarrow F_1Y = \frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b} \qquad \ldots(98.1)$$

$$\text{Eqn. (B.10)} \Longrightarrow F_1N = \frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)} \qquad \ldots(98.2)$$

Dividing eqns. (98.1) by (98.2),

$$\frac{F_1Y}{F_1N} = \left(\frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b}\right) \div \left(\frac{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}\right)$$

$$\therefore \frac{F_1Y}{F_1N} = \left(\frac{\cancel{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}}{\cancel{b}}\right) \times \left(\frac{\cancel{b}\cos(\theta°)}{\cancel{\sqrt{a^2+b^2} \times \sqrt{a^2\sin^2(\theta°)+b^2}}}\right)$$

$$\therefore \frac{F_1Y}{F_1N} = \cos(\theta°) \qquad \ldots(98.3)$$

$$\text{Eqn. (A.1)} \Longrightarrow OV = a \qquad \ldots(98.4)$$

$$\text{Eqn. (B.1)} \Longrightarrow OQ = \frac{a}{\cos(\theta°)} \qquad \ldots(98.5)$$

Dividing eqns. (98.4) by (98.5),

$$\frac{OV}{OQ} = a \div \left(\frac{a}{\cos(\theta°)}\right)$$

$$\therefore \frac{OV}{OQ} = a \times \left(\frac{\cos(\theta°)}{a}\right)$$

$$\therefore \frac{OV}{OQ} = \cos(\theta°) \qquad \ldots(98.6)$$

$$\text{Eqn. (D.4)} \Longrightarrow VX = \frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b} \qquad \ldots(98.7)$$

$$\text{Eqn. (C.19)} \Longrightarrow PN = \frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)} \qquad \ldots(98.8)$$

Dividing eqns. (98.7) by (98.8),

$$\frac{VX}{PN} = \left(\frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b}\right) \div \left(\frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}\right)$$

$$\therefore \frac{VX}{PN} = \left(\frac{\cancel{a\sqrt{a^2\sin^2(\theta°)+b^2}}}{\cancel{b}}\right) \times \left(\frac{\cancel{b}\cos(\theta°)}{\cancel{a\sqrt{a^2\sin^2(\theta°)+b^2}}}\right)$$

$$\therefore \frac{VX}{PN} = \cos(\theta°) \qquad \ldots(98.9)$$

$$\text{Eqn. (D.10)} \Longrightarrow OX = \frac{a^2\sin(\theta°)}{b} \qquad \ldots(98.10)$$

$$\text{Eqn. (D.12)} \Longrightarrow OZ = \frac{a^2\sin(\theta°)}{b\cos(\theta°)} \qquad \ldots(98.11)$$

Dividing eqns. (98.10) by (98.11),

$$\therefore \frac{OX}{OZ} = \left(\frac{a^2\sin(\theta°)}{b}\right) \div \left(\frac{a^2\sin(\theta°)}{b\cos(\theta°)}\right)$$

$$\therefore \frac{OX}{OZ} = \left(\frac{\cancel{a^2\sin(\theta°)}}{\cancel{b}}\right) \times \left(\frac{\cancel{b}\cos(\theta°)}{\cancel{a^2\sin(\theta°)}}\right)$$

$$\therefore \frac{OX}{OZ} = \cos(\theta°) \qquad \ldots(98.12)$$

$$\text{Eqn. (C.10)} \Longrightarrow TS = \frac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)} \qquad \ldots(98.13)$$

$$\text{Eqn. (C.13)} \Longrightarrow F_1S = \frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)} \qquad \ldots(98.14)$$

Dividing eqns. (98.13) by (98.14),

$$\therefore \frac{TS}{F_1S} = \left(\frac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)}\right) \div \left(\frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)}\right)$$

$$\therefore \frac{TS}{F_1S} = \left(\frac{\cos(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}{\cancel{\sin(\theta°)}}\right) \times \left(\frac{\cancel{\sin(\theta°)}}{\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}\right)$$

$$\therefore \frac{TS}{F_1S} = \cos(\theta°) \qquad \ldots(98.15)$$

$$\text{Eqn. (C.13)} \Longrightarrow F_1S = \frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)} \qquad \ldots(98.16)$$

$$\text{Eqn. (C.16)} \Longrightarrow PS = \frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(98.17)$$

Dividing eqns. (98.16) by (98.17),

$$\frac{F_1S}{PS} = \left(\frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)}\right) \div \left(\frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)\cos(\theta°)}\right)$$

$$\therefore \frac{F_1S}{PS} = \left(\frac{\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}{\cancel{\sin(\theta°)}}\right) \times \left(\frac{\cancel{\sin(\theta°)}\cos(\theta°)}{\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}\right)$$

$$\therefore \frac{F_1S}{PS} = \cos(\theta°) \qquad \ldots(98.18)$$

$$\text{Eqn. (D.3)} \Longrightarrow TW = \frac{a\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{b} \qquad \ldots(98.19)$$

$$\text{Eqn. (D.4)} \Longrightarrow VX = \frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b} \qquad \ldots(98.20)$$

Dividing eqns. (98.19) by (98.20),

$$\frac{TW}{VX} = \left(\frac{a\cos(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{b}\right) \div \left(\frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b}\right)$$

$$\therefore \frac{TW}{VX} = \left(\frac{a\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cancel{b}}\right) \times \left(\frac{\cancel{b}}{a\sqrt{a^2\sin^2(\theta°) + b^2}}\right)$$

$$\therefore \frac{TW}{VX} = \cos(\theta°) \quad \ldots (98.21)$$

$$\text{Eqn. (D.16)} \Longrightarrow TK = \frac{a^2\sin(\theta°)\cos(\theta°)}{\sqrt{a^2\sin^2(\theta°) + b^2}} \quad \ldots (98.22)$$

$$\text{Eqn. (E.3)} \Longrightarrow RA' = \frac{a^2\sin(\theta°)}{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}} \quad \ldots (98.23)$$

Dividing eqns. (98.22) by (98.23),

$$\frac{TK}{RA'} = \left(\frac{a^2\sin(\theta°)\cos(\theta°)}{\sqrt{a^2\sin^2(\theta°) + b^2}}\right) \div \left(\frac{a^2\sin(\theta°)}{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}\right)$$

$$\therefore \frac{TK}{RA'} = \left(\frac{\cancel{a^2\sin(\theta°)}\cos(\theta°)}{\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right) \times \left(\frac{\cos(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{\cancel{a^2\sin(\theta°)}}\right)$$

$$\therefore \frac{TK}{RA'} = \cos^2(\theta°)$$

$$\therefore \sqrt{\frac{TK}{RA'}} = \cos(\theta°) \quad \ldots (98.24)$$

Equating eqns. (98.3), (98.6), (98.9), (98.12), (98.15), (98.18), (98.21) & (98.24)

$$\frac{F_1Y}{F_1N} = \frac{OV}{OQ} = \frac{VX}{PN} = \frac{OX}{OZ} = \frac{TS}{F_1S} = \frac{F_1S}{PS} = \frac{TW}{VX} = \sqrt{\frac{TK}{RA'}} \quad \ldots (98.25)$$

Eqn. (98.25) is mathematical expression of the theorem.

THEOREM- 99:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$$\boldsymbol{\frac{PT}{PS} = \frac{PL}{PN} = \frac{TL}{SN} = \frac{PQ}{RS}}$$

Proof of the theorem

$$\text{Eqn. (C.15)} \Longrightarrow PT = \frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)} \quad \ldots (99.1)$$

$$\text{Eqn. (C.16)} \Longrightarrow PS = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)\cos(\theta°)} \quad \ldots (99.2)$$

Dividing eqns. (99.1) by (99.2),

$$\therefore \frac{PT}{PS} = \left(\frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)}\right) \div \left(\frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)\cos(\theta°)}\right)$$

$$\frac{PT}{PS} = \left(\frac{\sin(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{\cancel{\cos(\theta°)}}\right) \times \left(\frac{\sin(\theta°)\cancel{\cos(\theta°)}}{\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \frac{PT}{PS} = \sin^2(\theta°) \quad \ldots(99.3)$$

$$\text{Eqn. (D. 2)} \Longrightarrow PL = \frac{a\sin^2(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)} \quad \ldots(99.4)$$

$$\text{Eqn. (C. 19)} \Longrightarrow PN = \frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)} \quad \ldots(99.5)$$

Dividing eqns. (99.4) by (99.5),

$$\frac{PL}{PN} = \left(\frac{a\sin^2(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)}\right) \div \left(\frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)}\right)$$

$$\therefore \frac{PL}{PN} = \left(\frac{\cancel{a}\sin^2(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{\cancel{b\cos(\theta°)}}\right) \times \left(\frac{\cancel{b\cos(\theta°)}}{\cancel{a\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \frac{PL}{PN} = \sin^2(\theta°) \quad \ldots(99.6)$$

$$\text{Eqn. (D. 1)} \Longrightarrow TL = \frac{\sin(\theta°)[a^2\sin^2(\theta°) + b^2]}{b\cos(\theta°)} \quad \ldots(99.7)$$

$$\text{Eqn. (C. 20)} \Longrightarrow SN = \frac{a^2\sin^2(\theta°) + b^2}{b\sin(\theta°)\cos(\theta°)} \quad \ldots(99.8)$$

Dividing eqns. (99.7) by (99.8),

$$\frac{TL}{SN} = \left(\frac{\sin(\theta°)[a^2\sin^2(\theta°) + b^2]}{b\cos(\theta°)}\right) \div \left(\frac{a^2\sin^2(\theta°) + b^2}{b\sin(\theta°)\cos(\theta°)}\right)$$

$$\therefore \frac{TL}{SN} = \left(\frac{\sin(\theta°)\cancel{[a^2\sin^2(\theta°) + b^2]}}{\cancel{b\cos(\theta°)}}\right) \times \left(\frac{\cancel{b}\sin(\theta°)\cancel{\cos(\theta°)}}{\cancel{a^2\sin^2(\theta°) + b^2}}\right)$$

$$\therefore \frac{TL}{SN} = \sin^2(\theta°) \quad \ldots(99.9)$$

$$\text{Eqn. (B. 2)} \Longrightarrow PQ = \frac{b\sin(\theta°)}{\cos(\theta°)} \quad \ldots(99.10)$$

$$\text{Eqn. (C. 18)} \Longrightarrow RS = \frac{b}{\sin(\theta°)\cos(\theta°)} \quad \ldots(99.11)$$

Dividing eqns. (99.10) by (99.11),

$$\frac{PQ}{RS} = \left(\frac{b\sin(\theta°)}{\cos(\theta°)}\right) \div \left(\frac{b}{\sin(\theta°)\cos(\theta°)}\right)$$

$$\therefore \frac{PQ}{RS} = \left(\frac{\cancel{b}\sin(\theta°)}{\cancel{\cos(\theta°)}}\right) \times \left(\frac{\sin(\theta°)\cancel{\cos(\theta°)}}{\cancel{b}}\right)$$

$$\therefore \frac{PQ}{RS} = \sin^2(\theta°) \quad \ldots(99.12)$$

Equating eqns. (99.3), (99.6), (99.9) & (99.12),

$$\frac{PT}{PS} = \frac{PL}{PN} = \frac{TL}{SN} = \frac{PQ}{RS} \quad \ldots(99.13)$$

Eqn. (99.13) is mathematical expression of the theorem.

THEOREM- 100:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$$\boldsymbol{\frac{PN}{F_1S} = \frac{VX}{TS} = \frac{OV}{OS} = \sqrt{\frac{PN}{SJ}}}$$

Proof of the theorem

$$\text{Eqn. (C.19)} \Longrightarrow \mathrm{PN} = \frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)} \qquad \ldots(100.1)$$

$$\text{Eqn. (C.13)} \Longrightarrow \mathrm{F_1S} = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)} \qquad (100.2)$$

Dividing eqns. (100.1) by (100.2),

$$\therefore \frac{\mathrm{PN}}{\mathrm{F_1S}} = \left(\frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)}\right) \div \left(\frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)}\right)$$

$$\frac{\mathrm{PN}}{\mathrm{F_1S}} = \left(\frac{a\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{b\cos(\theta°)}\right) \times \left(\frac{\sin(\theta°)}{\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \frac{\mathrm{PN}}{\mathrm{F_1S}} = \frac{a\sin(\theta°)}{b\cos(\theta°)} \qquad \ldots(100.3)$$

$$\text{Eqn. (D.4)} \Longrightarrow \mathrm{VX} = \frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b} \qquad \ldots(100.4)$$

$$\text{Eqn. (C.10)} \Longrightarrow \mathrm{TS} = \frac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)} \qquad \ldots(100.5)$$

Dividing eqns. (100.4) by (100.5),

$$\frac{\mathrm{VX}}{\mathrm{TS}} = \left(\frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b}\right) \div \left(\frac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)}\right)$$

$$\therefore \frac{\mathrm{VX}}{\mathrm{TS}} = \left(\frac{a\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{b}\right) \times \left(\frac{\sin(\theta°)}{\cos(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \frac{\mathrm{VX}}{\mathrm{TS}} = \frac{a\sin(\theta°)}{b\cos(\theta°)} \qquad \ldots(100.6)$$

$$\text{Eqn. (A.1)} \Longrightarrow \mathrm{OV} = a \qquad \ldots(100.7)$$

$$\text{Eqn. (C.2)} \Longrightarrow \mathrm{OS} = \frac{b\cos(\theta°)}{\sin(\theta°)} \qquad \ldots(100.8)$$

Dividing eqns. (100.7) by (100.8),

$$\frac{\mathrm{OV}}{\mathrm{OS}} = a \div \left(\frac{b\cos(\theta°)}{\sin(\theta°)}\right)$$

$$\therefore \frac{\mathrm{OV}}{\mathrm{OS}} = \mathrm{a} \times \left(\frac{\sin(\theta°)}{\mathrm{b}\cos(\theta°)}\right)$$

$$\therefore \frac{\mathrm{OV}}{\mathrm{OS}} = \frac{\mathrm{a}\sin(\theta°)}{\mathrm{b}\cos(\theta°)} \qquad \ldots(100.9)$$

$$\text{Eqn. (C.19)} \Longrightarrow \mathrm{PN} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \qquad \ldots(100.10)$$

$$\text{Eqn. (D.14)} \Longrightarrow \mathrm{SJ} = \frac{\mathrm{b}\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\sin^2(\theta°)} \qquad \ldots(100.11)$$

Dividing eqn. (100.10) by (100.11),

$$\frac{\mathrm{PN}}{\mathrm{SJ}} = \left(\frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)}\right) \div \left(\frac{\mathrm{b}\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\sin^2(\theta°)}\right)$$

$$\therefore \frac{\mathrm{PN}}{\mathrm{SJ}} = \left(\frac{\mathrm{a}\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}{\mathrm{b}\cos(\theta°)}\right) \times \left(\frac{\mathrm{a}\sin^2(\theta°)}{\mathrm{b}\cos(\theta°)\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}\right)$$

$$\therefore \frac{\mathrm{PN}}{\mathrm{SJ}} = \frac{\mathrm{a}^2\sin^2(\theta°)}{\mathrm{b}^2\cos^2(\theta°)}$$

$$\therefore \sqrt{\frac{\mathrm{PN}}{\mathrm{SJ}}} = \frac{\mathrm{a}\sin(\theta°)}{\mathrm{b}\cos(\theta°)} \qquad \ldots(100.12)$$

Equating eqns. (100.3), (100.6), (100.9) & (100.12),

$$\frac{\mathrm{PN}}{\mathrm{F_1S}} = \frac{\mathrm{VX}}{\mathrm{TS}} = \frac{\mathrm{OV}}{\mathrm{OS}} = \sqrt{\frac{\mathrm{PN}}{\mathrm{SJ}}} \qquad \ldots(100.13)$$

Eqn. (100.13) is mathematical expression of the theorem.

THEOREM- 101:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,}$

$$\boldsymbol{\frac{PS}{ML'} = \frac{PN}{MN} = \frac{a^2}{a^2 + b^2}}$$

Proof of the theorem

$$\text{Eqn. (C.16)} \Longrightarrow \mathrm{PS} = \frac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(101.1)$$

$$\text{Eqn. (E.7)} \Longrightarrow \mathrm{ML'} = \frac{(\mathrm{a}^2 + \mathrm{b}^2)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}^2\sin(\theta°)\cos(\theta°)} \qquad \ldots(101.2)$$

Dividing eqns. (101.1) & (101.2),

$$\frac{\mathrm{PS}}{\mathrm{ML'}} = \left(\frac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)\cos(\theta°)}\right) \div \left(\frac{(\mathrm{a}^2 + \mathrm{b}^2)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}^2\sin(\theta°)\cos(\theta°)}\right)$$

$$\frac{\mathrm{PS}}{\mathrm{ML'}} = \left(\frac{\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{\cancel{\sin(\theta°)\cos(\theta°)}}\right) \times \left(\frac{a^2\cancel{\sin(\theta°)\cos(\theta°)}}{(a^2 + b^2)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \frac{\mathrm{PS}}{\mathrm{ML'}} = \frac{a^2}{a^2 + b^2} \qquad \ldots(101.3)$$

$$\text{Eqn. (C. 19)} \Longrightarrow \mathrm{PN} = \frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)} \qquad \ldots(101.4)$$

$$\text{Eqn. (C. 21)} \Longrightarrow \mathrm{MN} = \frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{ab\cos(\theta°)} \qquad \ldots(101.5)$$

Dividing eqns. (101.4) & (101.5),

$$\frac{\mathrm{PN}}{\mathrm{MN}} = \left(\frac{a\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)}\right) \div \left(\frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{ab\cos(\theta°)}\right)$$

$$\frac{\mathrm{PN}}{\mathrm{MN}} = \left(\frac{a\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{\cancel{b\cos(\theta°)}}\right) \times \left(\frac{a\cancel{b\cos(\theta°)}}{(a^2 + b^2)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \frac{\mathrm{PN}}{\mathrm{MN}} = \frac{a^2}{a^2 + b^2} \qquad \ldots(101.6)$$

Equating eqns. (101.3) & (101.6),

$$\frac{\mathrm{PS}}{\mathrm{ML'}} = \frac{\mathrm{PN}}{\mathrm{MN}} = \frac{a^2}{a^2 + b^2} \qquad \ldots(101.7)$$

Eqn. (101.7) is mathematical expression of the theorem.

THEOREM- 102:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$$\boldsymbol{\frac{TM}{PT} = \frac{SN}{PN} = \frac{TL}{PL}}$$

Proof of the theorem

$$\text{Eqn. (C. 9)} \Longrightarrow \mathrm{TM} = \frac{a^2\sin^2(\theta°) + b^2}{a\cos(\theta°)} \qquad \ldots(102.1)$$

$$\text{Eqn. (C. 15)} \Longrightarrow \mathrm{PT} = \frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)} \qquad \ldots(102.2)$$

Dividing eqns. (102.1) & (102.2),

$$\frac{\mathrm{TM}}{\mathrm{PT}} = \left(\frac{a^2\sin^2(\theta°) + b^2}{a\cos(\theta°)}\right) \div \left(\frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)}\right)$$

$$\therefore \frac{\mathrm{TM}}{\mathrm{PT}} = \left(\frac{a^2\sin^2(\theta°) + b^2}{a\cancel{\cos(\theta°)}}\right) \times \left(\frac{\cancel{\cos(\theta°)}}{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}\right)$$

$$\therefore \frac{\mathrm{TM}}{\mathrm{PT}} = \frac{\sqrt{a^2\sin^2(\theta°) + b^2}}{a\sin(\theta°)} \qquad \ldots(102.3)$$

$$\text{Eqn. (C. 20)} \Longrightarrow \mathrm{SN} = \frac{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}{\mathrm{b}\sin(\theta°)\cos(\theta°)} \qquad \ldots(102.4)$$

$$\text{Eqn. (C. 19)} \Longrightarrow \mathrm{PN} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \qquad \ldots(102.5)$$

Dividing eqns. (102.4) & (102.5),

$$\frac{\mathrm{SN}}{\mathrm{PN}} = \left(\frac{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}{\mathrm{b}\sin(\theta°)\cos(\theta°)}\right) \div \left(\frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)}\right)$$

$$\therefore \frac{\mathrm{SN}}{\mathrm{PN}} = \left(\frac{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}{\cancel{\mathrm{b}}\sin(\theta°)\cancel{\cos(\theta°)}}\right) \times \left(\frac{\cancel{\mathrm{b}\cos(\theta°)}}{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}\right)$$

$$\therefore \frac{\mathrm{SN}}{\mathrm{PN}} = \frac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\sin(\theta°)} \qquad \ldots(102.6)$$

$$\text{Eqn. (D. 1)} \Longrightarrow \mathrm{TL} = \frac{\sin(\theta°)[\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2]}{\mathrm{b}\cos(\theta°)} \qquad \ldots(102.7)$$

$$\text{Eqn. (D. 2)} \Longrightarrow \mathrm{PL} = \frac{\mathrm{a}\sin^2(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)} \qquad \ldots(102.8)$$

Dividing eqns. (102.7) & (102.8),

$$\frac{\mathrm{TL}}{\mathrm{PL}} = \left(\frac{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}{\mathrm{a}\cos(\theta°)}\right) \div \left(\frac{\mathrm{a}\sin^2(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}\cos(\theta°)}\right)$$

$$\therefore \frac{\mathrm{TL}}{\mathrm{PL}} = \left(\frac{\cancel{\sin(\theta°)}[\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2]}{\cancel{\mathrm{b}\cos(\theta°)}}\right) \times \left(\frac{\cancel{\mathrm{b}\cos(\theta°)}}{\mathrm{a}\sin^{\cancel{2}}(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}\right)$$

$$\therefore \frac{\mathrm{TL}}{\mathrm{PL}} = \frac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{a}\sin(\theta°)} \qquad \ldots(102.9)$$

Equating Eqns. (102.3), (102.6) & (102.9),

$$\frac{\mathrm{TM}}{\mathrm{PT}} = \frac{\mathrm{SN}}{\mathrm{PN}} = \frac{\mathrm{TL}}{\mathrm{PL}} \qquad \ldots(102.10)$$

Eqn. (102.10) is mathematical expression of the theorem.

THEOREM- 103:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$$\boldsymbol{\frac{PT}{TS} = \frac{PL}{TW} = \frac{QM}{OJ} = \frac{PM}{SJ} = \frac{QE'}{OK}}$$

Proof of the theorem

$$\text{Eqn. (C. 15)} \Longrightarrow \mathrm{PT} = \frac{\sin(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\cos(\theta°)} \qquad \ldots(103.1)$$

$$\text{Eqn. (C. 10)} \Longrightarrow \mathrm{TS} = \frac{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)} \qquad \ldots(103.2)$$

Dividing eqns. (103.1) & (103.2),

$$\frac{PT}{TS} = \left(\frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)}\right) \div \left(\frac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)}\right)$$

$$\therefore \frac{PT}{TS} = \left(\frac{\sin(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{\cos(\theta°)}\right) \times \left(\frac{\sin(\theta°)}{\cos(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \frac{PT}{TS} = \frac{\sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(103.3)$$

$$\text{Eqn. (D.2)} \Longrightarrow PL = \frac{a\sin^2(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)} \qquad \ldots(103.4)$$

$$\text{Eqn. (D.3)} \Longrightarrow TW = \frac{a\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{b} \qquad \ldots(103.5)$$

Dividing eqns. (103.4) & (103.5),

$$\frac{PL}{TW} = \left(\frac{a\sin^2(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)}\right) \div \left(\frac{a\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{b}\right)$$

$$\therefore \frac{PL}{TW} = \left(\frac{\cancel{a}\sin^2(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{\cancel{b}\cos(\theta°)}\right) \times \left(\frac{\cancel{b}}{\cancel{a}\cos(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \frac{PL}{TW} = \frac{\sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(103.6)$$

$$\text{Eqn. (C.5)} \Longrightarrow QM = \frac{b^2}{a\cos(\theta°)} \qquad \ldots(103.7)$$

$$\text{Eqn. (D.13)} \Longrightarrow OJ = \frac{b^2\cos(\theta°)}{a\sin^2(\theta°)} \qquad \ldots(103.8)$$

Dividing eqns. (103.7) & (103.8),

$$\therefore \frac{QM}{OJ} = \left(\frac{b^2}{a\cos(\theta°)}\right) \div \left(\frac{b^2\cos(\theta°)}{a\sin^2(\theta°)}\right)$$

$$\therefore \frac{QM}{OJ} = \left(\frac{\cancel{b^2}}{\cancel{a}\cos(\theta°)}\right) \times \left(\frac{\cancel{a}\sin^2(\theta°)}{\cancel{b^2}\cos(\theta°)}\right)$$

$$\therefore \frac{QM}{OJ} = \frac{\sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(103.9)$$

$$\text{Eqn. (C.14)} \Longrightarrow PM = \frac{b\sqrt{a^2\sin^2(\theta°) + b^2}}{a\cos(\theta°)} \qquad \ldots(103.10)$$

$$\text{Eqn. (D.14)} \Longrightarrow SJ = \frac{b\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{a\sin^2(\theta°)} \qquad \ldots(103.11)$$

Dividing eqns. (103.10) & (103.11),

$$\frac{PM}{SJ} = \left(\frac{b\sqrt{a^2\sin^2(\theta°) + b^2}}{a\cos(\theta°)}\right) \div \left(\frac{b\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{a\sin^2(\theta°)}\right)$$

$$\therefore \frac{PM}{SJ} = \left(\frac{\cancel{b\sqrt{a^2\sin^2(\theta°) + b^2}}}{\cancel{a}\cos(\theta°)}\right) \times \left(\frac{\cancel{a}\sin^2(\theta°)}{\cancel{b}\cos(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \frac{PM}{SJ} = \frac{\sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(103.12)$$

$$\text{Eqn. (E.2)} \Rightarrow QE' = \frac{ab\sin^2(\theta°)}{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}} \qquad \ldots(103.13)$$

$$\text{Eqn. (D.15)} \Rightarrow OK = \frac{ab\cos(\theta°)}{\sqrt{a^2\sin^2(\theta°) + b^2}} \qquad \ldots(103.14)$$

Dividing eqns. (103.13) & (103.14),

$$\frac{QE'}{OK} = \left(\frac{ab\sin^2(\theta°)}{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}\right) \div \left(\frac{ab\cos(\theta°)}{\sqrt{a^2\sin^2(\theta°) + b^2}}\right)$$

$$\therefore \frac{QE'}{OK} = \left(\frac{\cancel{ab}\sin^2(\theta°)}{\cos(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right) \times \left(\frac{\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{\cancel{ab}\cos(\theta°)}\right)$$

$$\therefore \frac{QE'}{OK} = \frac{\sin^2(\theta°)}{\cos^2(\theta°)} \qquad \ldots(103.15)$$

Equating Eqns. (103.3), (103.6), (103.9), (103.12) & (103.15),

$$\frac{PT}{TS} = \frac{PL}{TW} = \frac{QM}{OJ} = \frac{PM}{SJ} = \frac{QE'}{OK} \qquad \ldots(103.16)$$

Eqn. (103.16) is mathematical expression of the theorem.

THEOREM- 104:

In a Hyperbola $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$,

$$\boldsymbol{\frac{ON}{OM} = \frac{OS}{OJ} = \frac{VX}{F_1S} = \frac{TS}{SJ} = \frac{PL}{PT} = \frac{F_1Y}{F_1I'} = \frac{MN}{ML'} = \frac{TN}{SM} = \frac{PT}{PM} = \frac{PN}{PS} = \frac{TL}{TM} = \frac{TW}{TS} = \frac{OQ}{RS} = \sqrt{\frac{RN}{RS}} = \sqrt{\frac{TW}{SJ}}}$$

Proof of the theorem

$$\text{Eqn. (C.7)} \Rightarrow ON = \frac{(a^2 + b^2)\sin(\theta°)}{b\cos(\theta°)} \qquad \ldots(104.1)$$

$$\text{Eqn. (C.6)} \Rightarrow OM = \frac{a^2 + b^2}{a\cos(\theta°)} \qquad \ldots(104.2)$$

Dividing eqn. (104.1) by (104.2),

$$\frac{ON}{OM} = \left(\frac{(a^2 + b^2)\sin(\theta°)}{b\cos(\theta°)}\right) \div \left(\frac{a^2 + b^2}{a\cos(\theta°)}\right)$$

$$\therefore \frac{ON}{OM} = \left(\frac{\cancel{(a^2 + b^2)}\sin(\theta°)}{b\cancel{\cos(\theta°)}}\right) \times \left(\frac{a\cancel{\cos(\theta°)}}{\cancel{(a^2 + b^2)}}\right)$$

$$\therefore \frac{\mathrm{ON}}{\mathrm{OM}} = \frac{\mathrm{asin}(\theta°)}{\mathrm{b}} \qquad \ldots(104.3)$$

$$\text{Eqn. (C. 2)} \Longrightarrow \mathrm{OS} = \frac{\mathrm{bcos}(\theta°)}{\sin(\theta°)} \qquad \ldots(104.4)$$

$$\text{Eqn. (D. 13)} \Longrightarrow \mathrm{OJ} = \frac{\mathrm{b}^2\cos(\theta°)}{\mathrm{asin}^2(\theta°)} \qquad \ldots(104.5)$$

Dividing eqn. (104.4) by (104.5),

$$\frac{\mathrm{OS}}{\mathrm{OJ}} = \left(\frac{\mathrm{bcos}(\theta°)}{\sin(\theta°)}\right) \div \left(\frac{\mathrm{b}^2\cos(\theta°)}{\mathrm{asin}^2(\theta°)}\right)$$

$$\therefore \frac{\mathrm{OS}}{\mathrm{OJ}} = \frac{\cancel{\mathrm{bcos}(\theta°)}}{\cancel{\sin(\theta°)}} \times \left(\frac{\mathrm{asin}^{\cancel{2}}(\theta°)}{\mathrm{b}^{\cancel{2}}\cancel{\cos(\theta°)}}\right)$$

$$\therefore \frac{\mathrm{OS}}{\mathrm{OJ}} = \frac{\mathrm{asin}(\theta°)}{\mathrm{b}} \qquad \ldots(104.6)$$

$$\text{Eqn. (D. 4)} \Longrightarrow \mathrm{VX} = \frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}} \qquad \ldots(104.7)$$

$$\text{Eqn. (C. 13)} \Longrightarrow \mathrm{F_1S} = \frac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)} \qquad \ldots(104.8)$$

Dividing eqn. (104.7) by (104.8),

$$\frac{\mathrm{VX}}{\mathrm{F_1S}} = \left(\frac{\mathrm{a}\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{b}}\right) \div \left(\frac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)}\right)$$

$$\therefore \frac{\mathrm{VX}}{\mathrm{F_1S}} = \frac{\mathrm{a}\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}{\mathrm{b}} \times \left(\frac{\sin(\theta°)}{\cancel{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}\right)$$

$$\therefore \frac{\mathrm{VX}}{\mathrm{F_1S}} = \frac{\mathrm{asin}(\theta°)}{\mathrm{b}} \qquad \ldots(104.9)$$

$$\text{Eqn. (C. 10)} \Longrightarrow \mathrm{TS} = \frac{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)} \qquad \ldots(104.10)$$

$$\text{Eqn. (D. 14)} \Longrightarrow \mathrm{SJ} = \frac{\mathrm{bcos}(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{asin}^2(\theta°)} \qquad \ldots(104.11)$$

Dividing eqn. (104.10) by (104.11),

$$\frac{\mathrm{TS}}{\mathrm{SJ}} = \left(\frac{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\sin(\theta°)}\right) \div \left(\frac{\mathrm{bcos}(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{asin}^2(\theta°)}\right)$$

$$\therefore \frac{\mathrm{TS}}{\mathrm{SJ}} = \frac{\cancel{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}{\cancel{\sin(\theta°)}} \times \left(\frac{\mathrm{asin}^{\cancel{2}}(\theta°)}{\mathrm{b}\cancel{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}}\right)$$

$$\therefore \frac{\mathrm{TS}}{\mathrm{SJ}} = \frac{\mathrm{asin}(\theta°)}{\mathrm{b}} \qquad \ldots(104.12)$$

$$\text{Eqn. (D. 2)} \Longrightarrow \mathrm{PL} = \frac{\mathrm{asin}^2(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2}}{\mathrm{bcos}(\theta°)} \qquad \ldots(104.13)$$

$$\text{Eqn. (C.15)} \Longrightarrow \mathrm{PT} = \frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)} \quad \ldots(104.14)$$

Dividing eqns. (104.13) & (104.14),

$$\frac{\mathrm{PL}}{\mathrm{PT}} = \left(\frac{a\sin^2(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{b\cos(\theta°)}\right) \div \left(\frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\cos(\theta°)}\right)$$

$$\frac{\mathrm{PL}}{\mathrm{PT}} = \left(\frac{a\sin^{\cancel{2}}(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{b\cancel{\cos(\theta°)}}\right) \times \left(\frac{\cancel{\cos(\theta°)}}{\cancel{\sin(\theta°)}\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \frac{\mathrm{PL}}{\mathrm{PT}} = \frac{a\sin(\theta°)}{b} \quad \ldots(104.15)$$

$$\text{Eqn. (D.5)} \Longrightarrow \mathrm{F_1Y} = \frac{\sqrt{a^2 + b^2} \times \sqrt{a^2\sin^2(\theta°) + b^2}}{b} \quad \ldots(104.16)$$

$$\text{Eqn. (E.5)} \Longrightarrow \mathrm{F_1I'} = \frac{\sqrt{a^2 + b^2} \times \sqrt{a^2\sin^2(\theta°) + b^2}}{a\sin(\theta°)} \quad \ldots(104.17)$$

Dividing eqns. (104.16) & (104.17),

$$\frac{\mathrm{F_1Y}}{\mathrm{F_1I'}} = \left(\frac{\sqrt{a^2 + b^2} \times \sqrt{a^2\sin^2(\theta°) + b^2}}{b}\right) \div \left(\frac{\sqrt{a^2 + b^2} \times \sqrt{a^2\sin^2(\theta°) + b^2}}{a\sin(\theta°)}\right)$$

$$\frac{\mathrm{F_1Y}}{\mathrm{F_1I'}} = \left(\frac{\cancel{\sqrt{a^2 + b^2}} \times \cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{b}\right) \times \left(\frac{a\sin(\theta°)}{\cancel{\sqrt{a^2 + b^2}} \times \cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \frac{\mathrm{F_1Y}}{\mathrm{F_1I'}} = \frac{a\sin(\theta°)}{b} \quad \ldots(104.18)$$

$$\text{Eqn. (C.21)} \Longrightarrow \mathrm{MN} = \frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{ab\cos(\theta°)} \quad \ldots(104.19)$$

$$\text{Eqn. (E.7)} \Longrightarrow \mathrm{ML'} = \frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{a^2\sin(\theta°)\cos(\theta°)} \quad \ldots(104.20)$$

Dividing eqns. (104.19) & (104.20),

$$\frac{\mathrm{MN}}{\mathrm{ML'}} = \left(\frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{ab\cos(\theta°)}\right) \div \left(\frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta°) + b^2}}{a^2\sin(\theta°)\cos(\theta°)}\right)$$

$$\frac{\mathrm{MN}}{\mathrm{ML'}} = \left(\frac{\cancel{(a^2 + b^2)}\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}{ab\cancel{\cos(\theta°)}}\right) \times \left(\frac{a^2\sin(\theta°)\cancel{\cos(\theta°)}}{\cancel{(a^2 + b^2)}\cancel{\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \frac{\mathrm{MN}}{\mathrm{ML'}} = \frac{a\sin(\theta°)}{b} \quad \ldots(104.21)$$

$$\text{Eqn. (C.23)} \Longrightarrow \mathrm{TN} = \frac{\sqrt{(a^2 + b^2)^2\sin^2(\theta°) + a^2b^2\cos^4(\theta°)}}{b\cos(\theta°)} \quad \ldots(104.22)$$

$$\text{Eqn. (C.22)} \Longrightarrow \mathrm{SM} = \frac{\sqrt{(a^2 + b^2)^2\sin^2(\theta°) + a^2b^2\cos^4(\theta°)}}{a\sin(\theta°)\cos(\theta°)} \quad \ldots(104.23)$$

Dividing eqn. (104.22) by (104.23),

$$\frac{TN}{SM}=\left(\frac{\sqrt{(a^2+b^2)^2\sin^2(\theta°)+a^2b^2\cos^4(\theta°)}}{b\cos(\theta°)}\right)\div\left(\frac{\sqrt{(a^2+b^2)^2\sin^2(\theta°)+a^2b^2\cos^4(\theta°)}}{a\sin(\theta°)\cos(\theta°)}\right)$$

$$\therefore\frac{TN}{SM}=\left(\frac{\cancel{\sqrt{(a^2+b^2)^2\sin^2(\theta°)+a^2b^2\cos^4(\theta°)}}}{b\cancel{\cos(\theta°)}}\right)\times\left(\frac{a\sin(\theta°)\cancel{\cos(\theta°)}}{\cancel{\sqrt{(a^2+b^2)^2\sin^2(\theta°)+a^2b^2\cos^4(\theta°)}}}\right)$$

$$\therefore\frac{TN}{SM}=\frac{a\sin(\theta°)}{b}\qquad\ldots(104.24)$$

$$\text{Eqn. (C.15)}\Longrightarrow PT=\frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{\cos(\theta°)}\qquad\ldots(104.25)$$

$$\text{Eqn. (C.14)}\Longrightarrow PM=\frac{b\sqrt{a^2\sin^2(\theta°)+b^2}}{a\cos(\theta°)}\qquad\ldots(104.26)$$

Dividing eqn. (104.25) by (104.26),

$$\frac{PT}{PM}=\left(\frac{\sin(\theta°)\sqrt{a^2\sin^2(\theta°)+b^2}}{\cos(\theta°)}\right)\div\left(\frac{b\sqrt{a^2\sin^2(\theta°)+b^2}}{a\cos(\theta°)}\right)$$

$$\therefore\frac{PT}{PM}=\left(\frac{\sin(\theta°)\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}{\cancel{\cos(\theta°)}}\right)\times\left(\frac{a\cancel{\cos(\theta°)}}{b\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}\right)$$

$$\therefore\frac{PT}{PM}=\frac{a\sin(\theta°)}{b}\qquad\ldots(104.27)$$

$$\text{Eqn. (C.19)}\Longrightarrow PN=\frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}\qquad\ldots(104.28)$$

$$\text{Eqn. (C.16)}\Longrightarrow PS=\frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)\cos(\theta°)}\qquad\ldots(104.29)$$

Dividing eqn. (104.28) by (104.29),

$$\frac{PN}{PS}=\left(\frac{a\sqrt{a^2\sin^2(\theta°)+b^2}}{b\cos(\theta°)}\right)\div\left(\frac{\sqrt{a^2\sin^2(\theta°)+b^2}}{\sin(\theta°)\cos(\theta°)}\right)$$

$$\therefore\frac{PN}{PS}=\left(\frac{a\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}{b\cancel{\cos(\theta°)}}\right)\times\left(\frac{\sin(\theta°)\cancel{\cos(\theta°)}}{\cancel{\sqrt{a^2\sin^2(\theta°)+b^2}}}\right)$$

$$\therefore\frac{PN}{PS}=\frac{a\sin(\theta°)}{b}\qquad\ldots(104.30)$$

$$\text{Eqn. (D.1)}\Longrightarrow TL=\frac{\sin(\theta°)[a^2\sin^2(\theta°)+b^2]}{b\cos(\theta°)}\qquad\ldots(104.31)$$

$$\text{Eqn. (C.9)}\Longrightarrow TM=\frac{a^2\sin^2(\theta°)+b^2}{a\cos(\theta°)}\qquad\ldots(104.32)$$

Dividing eqn. (104.31) by (104.32),

$$\frac{TL}{TM}=\left(\frac{\sin(\theta°)[a^2\sin^2(\theta°)+b^2]}{b\cos(\theta°)}\right)\div\left(\frac{a^2\sin^2(\theta°)+b^2}{a\cos(\theta°)}\right)$$

$$\therefore \frac{TL}{TM} = \left(\frac{\sin(\theta°)[\cancel{a^2\sin^2(\theta°) + b^2}]}{b\cancel{\cos(\theta°)}}\right) \times \left(\frac{a\cancel{\cos(\theta°)}}{\cancel{a^2\sin^2(\theta°) + b^2}}\right)$$

$$\therefore \frac{TL}{TM} = \frac{a\sin(\theta°)}{b} \qquad \ldots(104.33)$$

$$\text{Eqn. (D.3)} \Longrightarrow TW = \frac{a\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{b} \qquad \ldots(104.34)$$

$$\text{Eqn. (C.10)} \Longrightarrow TS = \frac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)} \qquad \ldots(104.35)$$

Dividing eqn. (104.34) by (104.35),

$$\frac{TW}{TS} = \left(\frac{a\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{b}\right) \div \left(\frac{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}{\sin(\theta°)}\right)$$

$$\therefore \frac{TW}{TS} = \left(\frac{a\cancel{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}}{b}\right) \times \left(\frac{\sin(\theta°)}{\cancel{\cos(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2}}}\right)$$

$$\therefore \frac{TW}{TS} = \frac{a\sin(\theta°)}{b} \qquad \ldots(104.36)$$

$$\text{Eqn. (B.1)} \Longrightarrow OQ = \frac{a}{\cos(\theta°)} \qquad \ldots(104.37)$$

$$\text{Eqn. (C.18)} \Longrightarrow RS = \frac{b}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(104.38)$$

Dividing eqn. (104.37) by (104.38),

$$\frac{OQ}{RS} = \left(\frac{a}{\cos(\theta°)}\right) \div \left(\frac{b}{\sin(\theta°)\cos(\theta°)}\right)$$

$$\therefore \frac{OQ}{RS} = \left(\frac{a}{\cancel{\cos(\theta°)}}\right) \times \left(\frac{\sin(\theta°)\cancel{\cos(\theta°)}}{b}\right)$$

$$\therefore \frac{OQ}{RS} = \frac{a\sin(\theta°)}{b} \qquad \ldots(104.39)$$

$$\text{Eqn. (C.17)} \Longrightarrow RN = \frac{a^2\sin(\theta°)}{b\cos(\theta°)} \qquad \ldots(104.40)$$

$$\text{Eqn. (C.18)} \Longrightarrow RS = \frac{b}{\sin(\theta°)\cos(\theta°)} \qquad \ldots(104.41)$$

Dividing eqn. (104.40) by (104.41),

$$\frac{RN}{RS} = \left(\frac{a^2\sin(\theta°)}{b\cos(\theta°)}\right) \div \left(\frac{b}{\sin(\theta°)\cos(\theta°)}\right)$$

$$\therefore \frac{RN}{RS} = \left(\frac{a^2\sin(\theta°)}{b\cancel{\cos(\theta°)}}\right) \times \left(\frac{\sin(\theta°)\cancel{\cos(\theta°)}}{b}\right)$$

$$\therefore \frac{RN}{RS} = \frac{a^2\sin^2(\theta°)}{b^2}$$

Taking root of the above,

$$\therefore \sqrt{\frac{\mathrm{RN}}{\mathrm{RS}}} = \frac{\mathrm{a}\sin(\theta°)}{\mathrm{b}} \qquad \ldots(104.42)$$

$$\text{Eqn. (D.3)} \Longrightarrow \mathrm{TW} = \frac{\mathrm{a}\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{b}} \qquad \ldots(104.43)$$

$$\text{Eqn. (D.14)} \Longrightarrow \mathrm{SJ} = \frac{\mathrm{b}\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{a}\sin^2(\theta°)} \qquad \ldots(104.44)$$

Dividing eqn. (104.43) by (104.44),

$$\frac{\mathrm{TW}}{\mathrm{SJ}} = \left(\frac{\mathrm{a}\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{b}}\right) \div \left(\frac{\mathrm{b}\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}{\mathrm{a}\sin^2(\theta°)}\right)$$

$$\therefore \frac{\mathrm{TW}}{\mathrm{SJ}} = \frac{\mathrm{a}\cancel{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}}{\mathrm{b}} \times \left(\frac{\mathrm{a}\sin^2(\theta°)}{\mathrm{b}\cancel{\cos(\theta°)\sqrt{\mathrm{a}^2\sin^2(\theta°)+\mathrm{b}^2}}}\right)$$

$$\therefore \frac{\mathrm{TW}}{\mathrm{SJ}} = \frac{\mathrm{a}^2\sin^2(\theta°)}{\mathrm{b}^2}$$

$$\therefore \sqrt{\frac{\mathrm{TW}}{\mathrm{SJ}}} = \frac{\mathrm{a}\sin(\theta°)}{\mathrm{b}} \qquad \ldots(104.45)$$

Equating Eqns. (104.3), (104.6), (104.9), (104.12), (104.15), (104.18), (104.21), (104.24), (104.27), (104.30), (104.33), (104.36), (104.39), (104.42) & (104.45),

$$\frac{\mathrm{ON}}{\mathrm{OM}} = \frac{\mathrm{OS}}{\mathrm{OJ}} = \frac{\mathrm{VX}}{\mathrm{F_1S}} = \frac{\mathrm{TS}}{\mathrm{SJ}} = \frac{\mathrm{PL}}{\mathrm{PT}} = \frac{\mathrm{F_1Y}}{\mathrm{F_1I'}} = \frac{\mathrm{MN}}{\mathrm{ML'}} = \frac{\mathrm{TN}}{\mathrm{SM}} = \frac{\mathrm{PT}}{\mathrm{PM}} = \frac{\mathrm{PN}}{\mathrm{PS}} = \frac{\mathrm{TL}}{\mathrm{TM}} = \frac{\mathrm{TW}}{\mathrm{TS}} = \frac{\mathrm{OQ}}{\mathrm{RS}} = \sqrt{\frac{\mathrm{RN}}{\mathrm{RS}}} = \sqrt{\frac{\mathrm{TW}}{\mathrm{SJ}}} \qquad \ldots(104.46)$$

Eqn. (104.46) is mathematical expression of the theorem.

CONCLUSION

This research has presented a comprehensive mathematical investigation into the properties of the hyperbola, with particular emphasis on its tangent, normal, pair focal distances and vertex. Through the application of parametric equations, 104 new theorems have been systematically derived and rigorously proven, supported by illustrative diagrams to enhance clarity and comprehension. These results extend the classical understanding of the parabola, offering deeper insights into its geometric structure. Beyond pure mathematics, the findings carry significant implications for applied fields such as radar systems, antenna design, and optical devices like torch reflectors. The integration of parametric methods with detailed visualizations strengthens both the precision and accessibility of the work. Overall, this study provides a valuable reference for scholars and researchers engaged in advanced studies of conic sections, celestial mechanics, and geometric theory, marking a meaningful contribution to both mathematical knowledge and its interdisciplinary applications.

REFERENCES

Borowski E.J & Borwein J.M (1991). Collins Dictionary of Mathematics, Harper Collins publishers, Glasgow, p.261, 158, 316, 174, 556, 390, 216, 415, 34, 108, 598, 2, 405, 545.

Christopher Clapham & James Nicolson (2009). The concise Oxford Dictionary of Mathematics, (Oxford, UK, Oxford University Press) p.380, 321.

Mark D Licker (2003). Dictionary of Mathematics, Second edition, Mc Graw Hill Companies, Inc., New York, USA, p.239.