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Research Article

ANALYSIS TO DEVELOP NEW PROPERTIES ON VARIOUS ELEMENTS OF AN ELLIPSE

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ABSTRACT

Ellipse is one of the conic sections. It is an elongated circle. It is the locus of a point that moves in such a way that the ratio of its distance from a fixed point (called Focus) to its distance from a fixed line (called Directrix) equals to constant 'e' which is less than or equal to unity. According to the Keplar's law of Planetary Motion, the ellipse has very important role in geometry and the field of Astronomy, since in universe every planet is orbiting its star in an elliptical path and its star is as one of the foci. The objective of this research article is to establish some new theorems for mathematical properties related to various parameters of ellipse. These 14 properties have been defined with necessary drawings and derivations wherever necessary. The mathematical expressions of each property have also been given. These properties will be very useful for reference to the research scholars to higher level research works in Geometry.

***Keywords**: Astronomy, Planetary motion, Ellipse, Conic sections, Focal distance, Eccentricity, Eccentric angle of ellipse, Conjugate axis, Transverse axis, Auxiliary circle, Focus, Parametric equation of ellipse*

INTRODUTION

An *ellipse* is the set of all points in a plane such that the sum of the distances from two fixed points called *foci* is a given constant. Things that are in the shape of an ellipse are said to be elliptical. In the 17th century, a mathematician Mr. Johannes Kepler discovered that the orbits along which the planets travel around the Sun are ellipses with the Sun at one of the foci, in his First law of planetary motion. Later, Isaac Newton explained that this as a corollary of his law of universal gravitation. One of the physical properties of ellipse is that sound or light rays emanating from one focus will reflect back to the other focus. The longest and shortest diameters of an ellipse is called *Major axis* and *Minor axis* respectively. The *eccentricity*, of an ellipse, usually denoted by ε or e, is the ratio of the distance between the two foci to the length of the major axis. A straight line passing an ellipse and touching it at just one point is called *tangent*. A straight line which passing through the centre of ellipse is called *diameter*. A straight line that passing through the centre of the parallel lines to the diameter ellipse is called *conjugate diameter*. The distances between foci and a point of contact by tangent with ellipse is called *focal distances* at that point. There are some existing properties of ellipse such as "sum of focal distances is a constant" (called focal constant), reflection property of focus of the ellipse, etc.
Now an attempt has been made by the Author to develop mathematical properties regarding various parameters such as tangent, normal, focal distance, semi-conjugate diameters, semi-major axis, semi-minor axis, linear eccentricity, eccentric angle of ellipse, eccentric angle, and intercepts of the tangent mostly using Parametric equation of ellipse. The new properties have been derived mathematically. In this article, step by step derivations have been presented wherever necessary. The geometrical properties, which have been defined in this research article is very useful for those doing research works or further study in the field of Astronomy, Conics and Euclidean geometry, since this is also one of the important properties of an ellipse. This may also be very important to scientists who work in the field of Optics.

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DERIVATIONS AND DRAWINGS

Property-1: α° in terms of θ°

A tangent is drawn at point 'P' on ellipse. x-axis and y-axis are called as transverse axis *(Christopher Clapham & James Nicolson)* and conjugate axis *(Christopher Clapham & James Nicolson)* respectively, points T & S are x & y intercepts *(Borowski E.J & Borwein J.M)* respectively, points Q and R are the projection of point P on transverse axis and conjugate axis respectively. In other words, OQ & OR are the abscissa *(Borowski E.J & Borwein J.M)* and ordinate *(Borowski E.J & Borwein J.M)* of point P respectively.

Let, $\angle UOQ = \theta°$ is eccentric angle *(Henry Burchard Fine & Henry Dallas Thompson)*, $\angle PTQ = \alpha°$, F_1 & F_2 are the foci *(Borowski E.J & Borwein J.M)*, OA is semi-major axis *(Borowski E.J & Borwein J.M)*, OB is semi-minor axis *(Borowski E.J & Borwein J.M)*, O is centre of the ellipse, PF_1 & PF_2 are pair focal distances *(Borowski E.J & Borwein J.M)* at point P.

Referring fig.1, in right-triangle TOS, $\angle OTS = 90°$. We know already that $OQ \times OT = a^2$ *(Bali N.P)*

OQ is abscissa of point P in parametric *(Borowski E.J & Borwein J.M)* form.

Substituting $OQ = a \times \cos(\theta°)$ in above eqn.,

$$OT \times a \times \cos(\theta°) = a^2$$

$$\therefore OT = \frac{a^2}{a \times \cos(\theta°)}$$

$$\therefore OT = \frac{a}{\cos(\theta°)} \quad ------------------------------------(1.1)$$

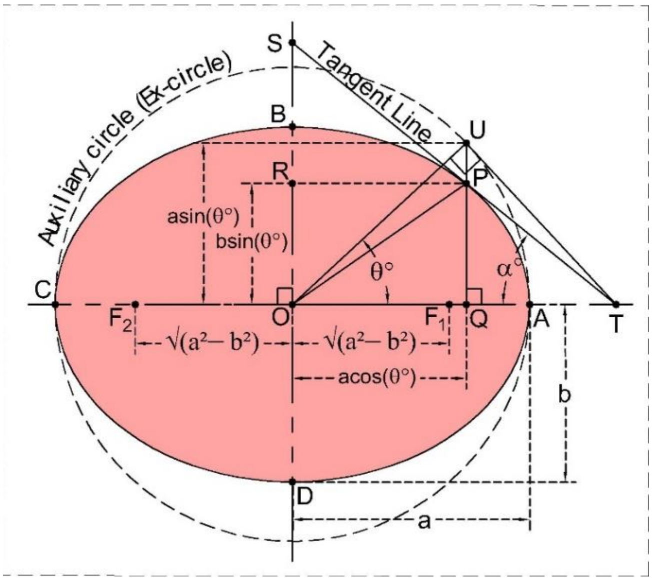

Fig: 1

OR is ordinate of point P in parametric form. $OR \times OS = b^2$ *(Bali N.P)*

Substituting $OR = b \times \sin(\theta°)$ in above eqn.,

$$OS \times b \times \sin(\theta°) = b^2$$

$$\therefore OS = \frac{b^2}{b \times \sin(\theta°)}$$

$$\therefore OS = \frac{b}{\sin(\theta°)} \quad ------------------------------------(1.2)$$

With reference to the fig.1,

$$\tan(\alpha°) = \frac{OS}{OT}$$

Substituting eqn. (1.1) and (1.2) in above eqn.,

$$\tan(\alpha°) = \frac{OS}{OT} = \left(\frac{b}{\sin(\theta°)}\right) \div \left(\frac{a}{\cos(\theta°)}\right)$$

$$\therefore \tan(\alpha°) = \left(\frac{b}{\sin(\theta°)}\right) \times \left(\frac{\cos(\theta°)}{a}\right)$$

$$\therefore \tan(\alpha°) = \frac{b}{a \times \tan(\theta°)} \qquad ------------------------------ (1.1)$$

Eqn. (1.1) is the mathematical expression of the property.

Property-2: β° in terms of θ° (Fig: 2)

A tangent is drawn at point 'P' on ellipse. x-axis and y-axis are called as transverse axis and conjugate axis respectively, points T & 'S' are x & y intercepts respectively, points Q & R are the projection of point P on transverse axis & conjugate axis respectively. In other words, OQ & OR are the abscissa and ordinate of point P respectively.

Let $\angle UOQ = \theta°$ is eccentric angle, $\angle PF_1Q = \beta°$. F_1 & F_2 are foci, OA is semi-major axis. OB is semi-minor axis. O is centre of the ellipse. PF_1 & PF_2 are pair focal distances at point P.

In right-triangle F_1RP, $\angle F_1QP = 90°$.

In right-triangle PQF_1

$$\tan(\beta°) = \frac{PQ}{F_1Q} = \frac{OR}{F_1Q}$$

$$\tan(\beta°) = \frac{PQ}{F_1Q} = \frac{b\sin(\theta°)}{OQ - OF_1}$$

OF_1 is called linear eccentricity *(Borowski E.J & Borwein J.M)* and its value is equal to $\sqrt{a^2 - b^2}$.

Substituting $OQ = a\cos(\theta°)$, $OR = b \times \sin(\theta°)$ and $OF_1 = \sqrt{a^2 - b^2}$ in above eqn.,

$$\therefore \tan(\beta°) = \frac{b\sin(\theta°)}{a\cos(\theta°) - \sqrt{a^2 - b^2}} \qquad ---------------------------(2.1)$$

Eqn. (2.1) is the mathematical expression of the property.

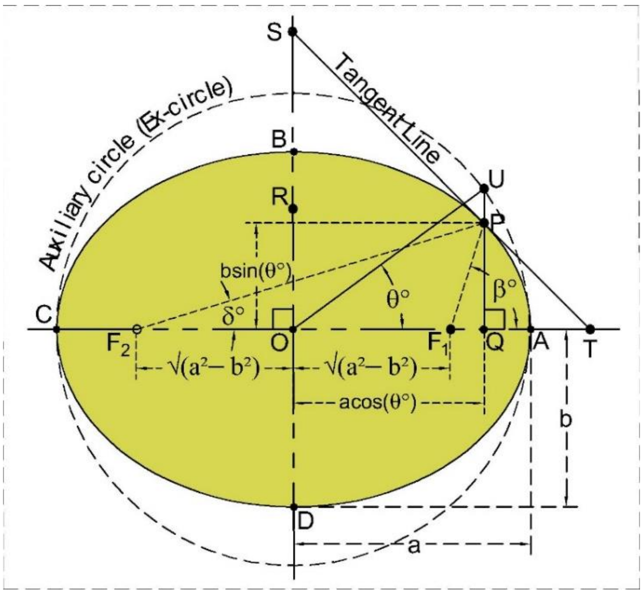

Fig: 2

Property-3: φ° in terms of θ°

A tangent is drawn at point 'P' on ellipse. x-axis and y-axis are called as transverse axis and conjugate axis respectively, points T & S are x & y intercepts respectively, points Q & R are the projection of point P on transverse axis & conjugate axis respectively. In other words, OQ & OR are the abscissa and ordinate of point P respectively.

Let $\angle UOQ = \theta°$ is eccentric angle, $\angle POQ = \varphi°$. F_1 & F_2 are foci, OA is semi-major axis. OB is semi-minor axis. O is centre of the ellipse. PF_1 & PF_2 are pair focal distances at point P.

In fig. 15, $\angle OQP = 90°$.

$$\therefore \tan(\varphi°) = \frac{b\sin(\theta°)}{a\cos(\theta°)}$$

$$\therefore \tan(\varphi°) = \frac{b\tan(\theta°)}{a} \quad ---------------------------------- (3.1)$$

Eqn. (3.1) is the mathematical expression of the property.

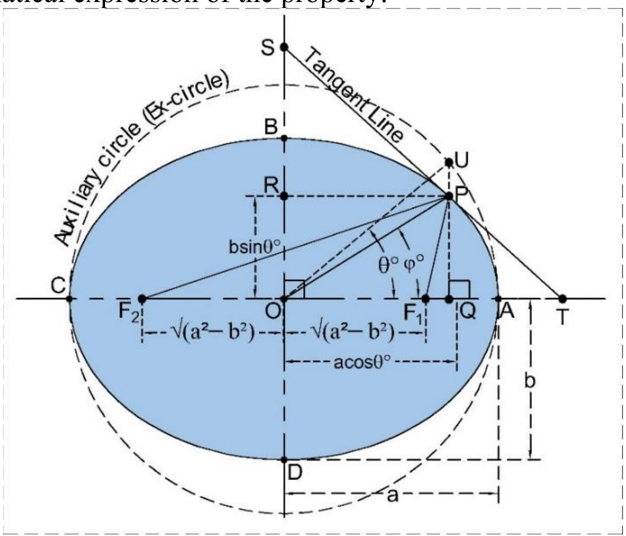

Fig: 3

Property-4: δ° in terms of θ°

A tangent is drawn at point 'P' on ellipse. x-axis and y-axis are called as transverse axis and conjugate axis respectively, points T & S are x & y intercepts respectively, points Q & R are the projection of point P on transverse axis & conjugate axis respectively. In other words, OQ & OR are the abscissa and ordinate of point P respectively.

Let $\angle UOQ = \theta°$ is eccentric angle, $\angle PF_2Q = \delta°$. F_1 & F_2 are foci, OA is semi-major axis. OB is semi-minor axis. O is centre of the ellipse. PF_1 & PF_2 are pair focal distances at point P.

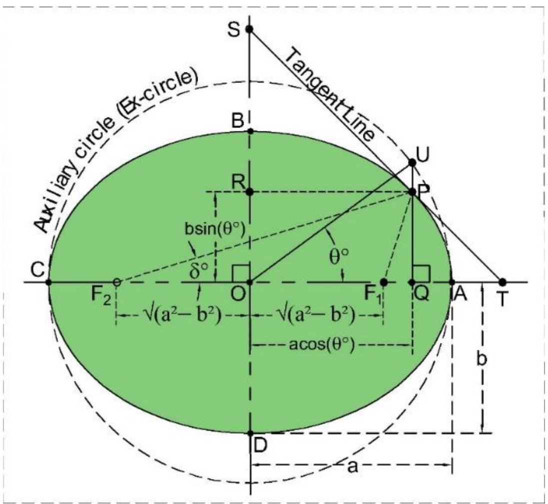

Fig: 4

In fig.4, $\angle F_2QP = 90°$.

$$\therefore \tan(\delta°) = \frac{PQ}{F_2Q}$$

$$\therefore \tan(\delta°) = \frac{PQ}{F_2O + OQ} = \frac{OR}{F_2O + OQ}$$

Substituting $OQ = a \times \cos(\theta°), OR = b \times \sin(\theta°)$ and $F_2O = \sqrt{a^2 - b^2}$ in above eqn.,

$$\therefore \tan(\delta°) = \frac{b\sin(\theta°)}{\sqrt{a^2 - b^2} + a\cos(\theta°)} \qquad (4.1)$$

Eqn. (4.1) is the mathematical expression of the property.

Property-5: Mathematical relation between α° and β°

A tangent is drawn at point 'P' on ellipse. x-axis and y-axis are called as transverse axis and conjugate axis respectively, points T & S are x & y intercepts respectively, points Q & R are the projection of point P on transverse axis & conjugate axis respectively. In other words, OQ & OR are the abscissa and ordinate of point P respectively.
Let $\angle UOQ = \theta°$ is eccentric angle, $\angle PTQ = \alpha°$, $\angle PF_1T = \beta°$. F_1 & F_2 are foci, OA is semi-major axis. OB is semi-minor axis. O is centre of the ellipse. PF_1 & PF_2 are pair focal distances at point P.

In fig. 5, $\angle F_2QP = 90°$.

$$\text{Eqn. (1.1)} \Rightarrow \tan(\alpha°) = \frac{b}{a\tan(\theta°)} \qquad (5.1)$$

$$\therefore \tan(\alpha°) = \frac{b\cos(\theta°)}{a\sin(\theta°)} \qquad (5.2)$$

$$\text{From eqn. (5.1)}, \tan(\theta°) = \frac{b}{a\tan(\alpha°)} = \frac{b\cos(\alpha°)}{a\sin(\alpha°)}$$

$$\tan(\theta°) = \frac{b\cos(\alpha°)}{a\sin(\alpha°)} \qquad (5.3)$$

$$\text{If } \tan(\theta°) = \frac{b\cos(\alpha°)}{a\sin(\alpha°)},$$

$$\sin(\theta°) = \frac{b\cos(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)} \qquad (5.4)$$

$$\cos(\theta°) = \frac{a\sin(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)} \qquad (5.5)$$

$$\text{Eqn. (2.1)} \Rightarrow \tan(\beta°) = \frac{b\sin(\theta°)}{a\cos(\theta°) - \sqrt{a^2 - b^2}}$$

Substituting eqns. (5.4) & (5.5) in above, we get

$$\tan(\beta°) = b\left[\frac{b\cos(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}\right] \div \left(a\left[\frac{a\sin(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}\right] - \sqrt{a^2 - b^2}\right)$$

$$\therefore \tan(\beta°) = \left[\frac{b^2\cos(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}\right] \div \left(\left[\frac{a^2\sin(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}\right] - \sqrt{a^2 - b^2}\right)$$

$$\therefore \tan(\beta°) = \left[\frac{b^2\cos(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}\right] \div \left[\left(\frac{a^2\sin(\alpha°) - \sqrt{a^2 - b^2}[a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)]}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}\right)\right]$$

$$\therefore \tan(\beta°) = \left[\frac{b^2\cos(\alpha°)}{\cancel{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}}\right] \times \left[\left(\frac{\cancel{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}}{a^2\sin(\alpha°) - \sqrt{a^2 - b^2}[a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)]}\right)\right]$$

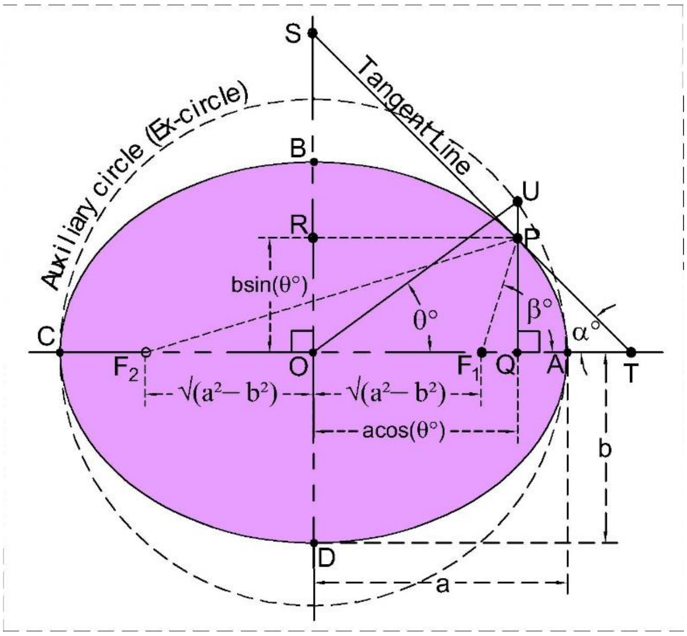

Fig: 5

$$\therefore \tan(\beta°) = \frac{b^2\cos(\alpha°)}{a^2\sin(\alpha°) - \sqrt{a^2 - b^2}[a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)]} \qquad ------------------(5.6)$$

Eqn. (5.6) is the mathematical expression of the property.

Property-6: Mathematical relation between α° and φ°

A tangent is drawn at point 'P' on ellipse. x-axis and y-axis are called as transverse axis and conjugate axis respectively, points T & S are x & y intercepts respectively, points Q & R are the projection of point P on transverse axis & conjugate axis respectively. In other words, OQ & OR are the abscissa and ordinate of point P respectively.

Let $\angle UOQ = \theta°$ is eccentric angle, $\angle PTQ = \alpha°$, $\angle TOP = \varphi°$. F_1 & F_2 are foci. OA is semi-major axis. OB is semi-minor axis. O is centre of the ellipse. PF_1 & PF_2 are pair focal distances at point P.

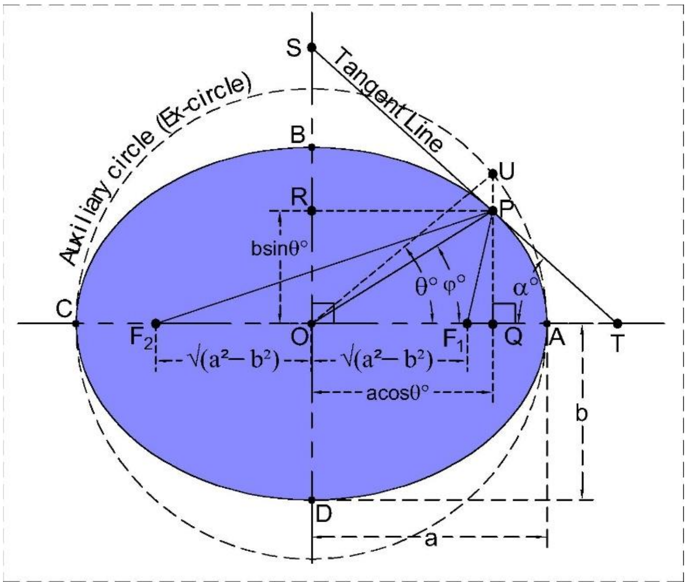

Fig: 6

$$\text{Eqn. (3.1)} \Rightarrow \tan(\varphi°) = \frac{b \times \tan(\theta°)}{a}$$

$$\text{Eqn. (1.1)} \Rightarrow \tan(\alpha°) = \frac{b}{a \times \tan(\theta°)}$$

$$\tan(\varphi°) \times \tan(\alpha°) = \left(\frac{b \times \tan(\theta°)}{a}\right) \times \left(\frac{b}{a \times \tan(\theta°)}\right)$$

$$\therefore \tan(\varphi°) \times \tan(\alpha°) = \frac{b^2}{a^2} \qquad \text{------------------------} (6.1)$$

Where, $\frac{b^2}{a^2}$ is an arbitrary constant.

Eqn. (6.1) is the mathematical expression of the property.

Property-7: Mathematical relation between α° and δ°

A tangent is drawn at point 'P' on ellipse. x-axis and y-axis are called as transverse axis and conjugate axis respectively, points T & S are x & y intercepts respectively, points Q & R are the projection of point P on transverse axis & conjugate axis respectively. In other words, OQ & OR are the abscissa and ordinate of point P respectively.

Let $\angle UOQ = \theta°$ is eccentric angle, $\angle PTQ = \alpha°$ and $\angle PF_2Q = \delta°$. F_1 & F_2 are foci, OA is semi-major axis. OB is semi-minor axis. O is centre of the ellipse. PF_1 & PF_2 are pair focal distances at point P.

$$\text{Eqn. (4.1)} \Rightarrow \tan(\delta°) = \frac{b\sin(\theta°)}{\sqrt{a^2 - b^2} + a\cos(\theta°)} \qquad \text{---------------------} (7.1)$$

$$\text{Eqn. (5.4)} \Rightarrow \frac{b\cos(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}$$

$$\text{Eqn. (5.5)} \Rightarrow \frac{a\sin(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}$$

Substituting eqn. (5.4) & (5.5) in eqn. (7.1), we get

$$\tan(\delta°) = b\left[\frac{b\cos(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}\right] \div \left[\sqrt{a^2 - b^2} + a\left(\frac{a\sin(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}\right)\right]$$

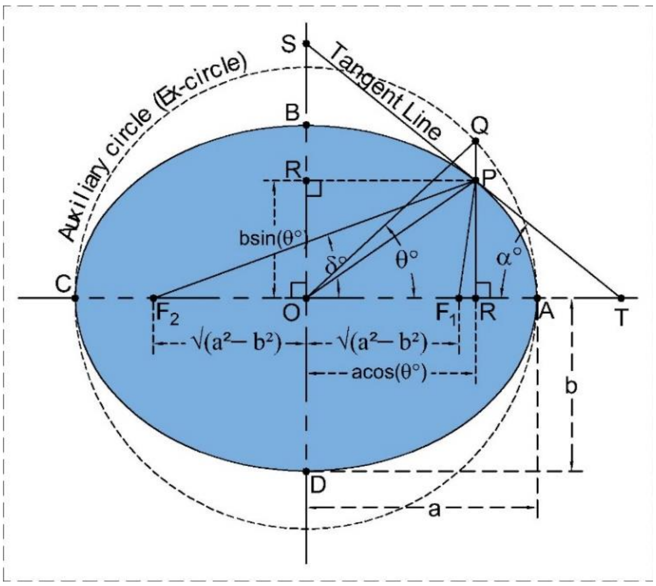

Fig: 7

$$\therefore \tan(\delta°) = \left[\frac{b^2\cos(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}\right] \div \left[\sqrt{a^2-b^2} + \left(\frac{a^2\sin(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}\right)\right]$$

$$\therefore \tan(\delta°) = \left[\frac{b^2\cos(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}\right] \div \left[\left(\frac{\sqrt{a^2-b^2}[a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)] + a^2\sin(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}\right)\right]$$

$$\therefore \tan(\delta°) = \left[\frac{b^2\cos(\alpha°)}{\cancel{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}}\right] \times \left[\left(\frac{\cancel{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}}{\sqrt{a^2-b^2}[a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)] + a^2\sin(\alpha°)}\right)\right]$$

$$\therefore \tan(\delta°) = \frac{b^2\cos(\alpha°)}{a^2\sin(\alpha°) + \sqrt{a^2-b^2}[a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)]} \quad ------------(7.2)$$

Eqn. (7.2) is the mathematical expression of the property.

Property-8: Mathematical relation between β° and φ° (Fig: 14)

A tangent is drawn at point 'P' on ellipse. x-axis and y-axis are called as transverse axis and conjugate axis respectively, points T & S are x & y intercepts respectively, points Q & R are the projection of point P on transverse axis & conjugate axis respectively. In other words, OQ & OR are the abscissa and ordinate of point P respectively.

Let $\angle UOQ = \theta°$ is eccentric angle, $\angle PTQ = \alpha°$, $\angle PF_1T = \beta°$, $\angle TOP = \varphi°$. F_1 & F_2 are foci. OB is semi major axis. OA is semi-major axis. OB is semi-minor axis. O is centre of the ellipse. PF_1 & PF_2 are pair focal distances at point P. We know already that, $OF_1 = \sqrt{a^2-b^2}$.

Let, (x, y) is the coordinate of point P with respect to O. Therefore, $OQ = x$, $PQ = y$,

F_1P is the focal distance. $\angle F_1QP = 90°$ and $\angle QF_1P = \beta°$.

Therefore,

$$F_1Q = PF_1\cos\beta° \quad ------------------------------(8.1)$$

$$PQ = PF_1\sin\beta° \quad ------------------------------(8.2)$$

In fig. 1, $OQ = x = OF_1 + F_1Q$.

$$\therefore\ x = \sqrt{a^2-b^2} + PF_1\cos\beta° \quad --------------------------(8.3)$$

$$\therefore y = PF_1\sin\beta° \quad ------------------------------(8.4)$$

We know that, the equation of ellipse with respect to centre O (0,0) is

$$\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1$$

Substituting (14.3) and (14.4) in above eqn., we get

$$\frac{\left(\sqrt{a^2-b^2} + PF_1\cos\beta°\right)^2}{a^2} + \frac{(PF_1\sin\beta°)^2}{b^2} = 1$$

$$\Rightarrow \frac{a^2 - b^2 + {PF_1}^2\cos^2\beta° + 2PF_1\sqrt{a^2-b^2}\cos\beta°}{a^2} + \frac{{PF_1}^2\sin^2\beta°}{b^2} = 1$$

$$\Rightarrow b^2\left(a^2 - b^2 + {PF_1}^2\cos^2\beta° + 2PF_1\sqrt{a^2-b^2}\cos\beta°\right) + a^2\left({PF_1}^2\sin^2\beta°\right) = a^2b^2$$

$$\Rightarrow a^2b^2 - b^4 + b^2{PF_1}^2\cos^2\beta° + 2b^2PF_1\sqrt{a^2-b^2}\cos\beta° + a^2{PF_1}^2\sin^2\beta° = a^2b^2$$

$$\Rightarrow -b^4 + b^2{PF_1}^2\cos^2\beta° + 2b^2PF_1\sqrt{a^2-b^2}\cos\beta° + a^2{PF_1}^2\sin^2\beta° = 0$$

$$\Rightarrow PF_1{}^2(a^2\sin^2\beta° + b^2\cos^2\beta°) + PF_1\left(2b^2\sqrt{a^2-b^2}\cos\beta°\right) - b^4 = 0$$

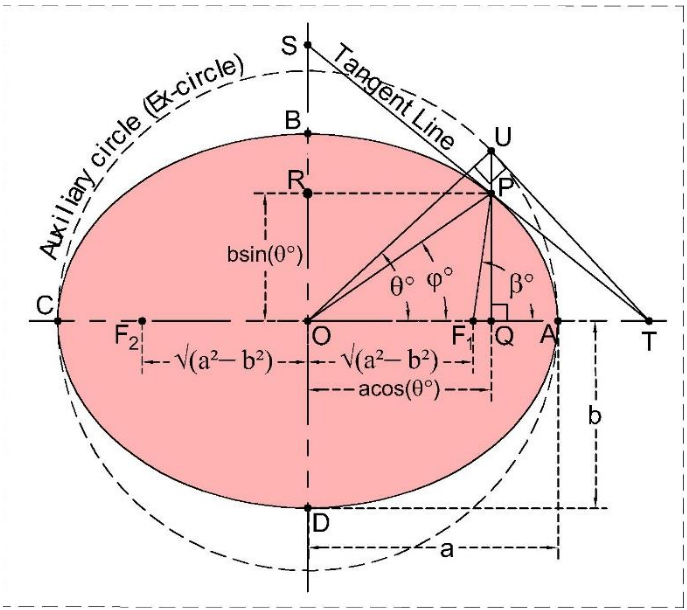

Fig: 8

Solving the above quadratic equation (Christopher Clapham & James Nicolson), we get

$$\therefore PF_1 = \frac{b^2}{a + \left(\sqrt{a^2-b^2}\cos\beta°\right)} \quad ------------------------(8.5)$$

$$\therefore PF_2 = 2a - \left(\frac{b^2}{a + \left(\sqrt{a^2-b^2}\cos\beta°\right)}\right) \quad ----------------------(8.6)$$

$$\text{Eqn. }(3.1) \Rightarrow \tan(\varphi°) = \frac{b\tan(\theta°)}{a} = \frac{a\sin(\varphi°)}{b\cos(\varphi°)}$$

$$\text{If } \tan(\theta°) = \frac{a\sin(\varphi°)}{b\cos(\varphi°)},$$

$$\cos(\theta°) = \frac{b\cos(\varphi°)}{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)} \quad ------------------------(8.7)$$

Referring fig:1,

$$\tan(\beta°) = \frac{PQ}{F_1Q} = \frac{PQ}{OQ - OF_1}$$

$$\therefore \tan(\beta°) = \frac{a\cos(\theta°)}{a\cos(\theta°) - \sqrt{a^2-b^2}}$$

Substituting eqn. (8.7) in above eqn., we get

$$\therefore \tan(\beta°) = \frac{a\left[\frac{b\cos(\varphi°)}{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}\right]}{a\left[\frac{b\cos(\varphi°)}{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}\right] - \sqrt{a^2-b^2}}$$

$$\therefore \tan(\beta°) = \frac{\left[\dfrac{ab\cos(\varphi°)}{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}\right]}{\left[\dfrac{ab\cos(\varphi°)}{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}\right] - \sqrt{a^2 - b^2}}$$

$$\therefore \tan(\beta°) = \frac{\left[\dfrac{ab\cos(\varphi°)}{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}\right]}{\left[\dfrac{ab\cos(\varphi°) - \sqrt{a^2 - b^2}[a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)]}{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}\right]}$$

$$\therefore \tan(\beta°) = \left[\frac{ab\cos(\varphi°)}{\cancel{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}}\right] \times \left[\frac{\cancel{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}}{ab\cos(\varphi°) - \sqrt{a^2 - b^2}[a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)]}\right]$$

$$\therefore \tan(\beta°) = \frac{ab\cos(\varphi°)}{ab\cos(\varphi°) - \sqrt{a^2 - b^2}[a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)]} \quad ------------(8.8)$$

Eqn. (8.8) is the mathematical expression of the property.

Property-9: Mathematical relation between and β° and δ°

A tangent is drawn at point 'P' on ellipse. x-axis and y-axis are called as transverse axis and conjugate axis respectively, points T & S are x & y intercepts respectively, points Q & R are the projection of point P on transverse axis & conjugate axis respectively. In other words, OQ & OR are the abscissa and ordinate of point P respectively.

Let $\angle UOQ = \theta°$ is eccentric angle, $\angle PTQ = \alpha°$, $\angle PF_1T = \beta°$, $\angle TOP = \varphi°$, $\angle PF_2Q = \delta°$. F_1 & F_2 are foci. OA is semi-major axis. OB is semi-minor axis. O is centre of the ellipse. PF_1 & PF_2 are pair focal distances at point P.

In fig. 1, $OQ = x = OF_1 + F_1Q$.

$$\text{Eqn. (2.2)} \Rightarrow \tan(\beta°) = \frac{b\sin(\theta°)}{a\cos(\theta°) - \sqrt{a^2 - b^2}}$$

$$\text{Eqn. (4.1)} \Rightarrow \tan(\delta°) = \frac{b\sin(\theta°)}{\sqrt{a^2 - b^2} + a\cos(\theta°)}$$

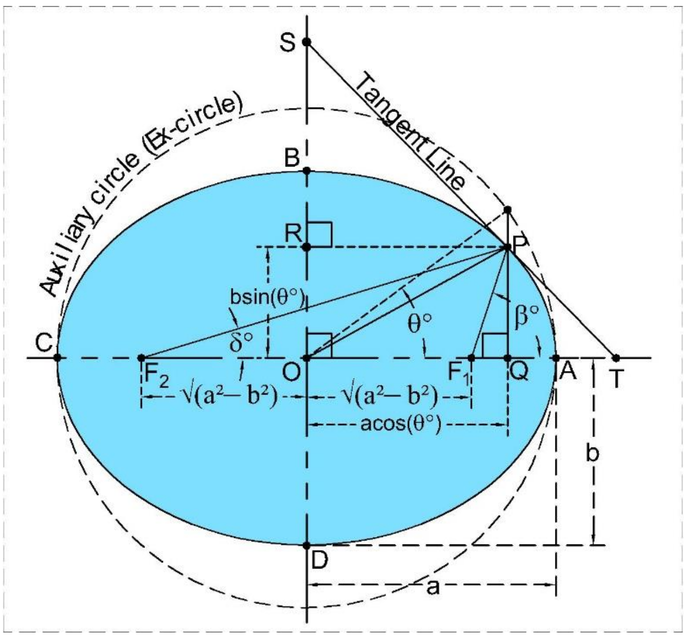

Fig: 9

$$\tan(\beta°) \times \tan(\delta°) = \frac{b\sin(\theta°)}{a\cos(\theta°) - \sqrt{a^2 - b^2}} \times \frac{b\sin(\theta°)}{\sqrt{a^2 - b^2} + a\cos(\theta°)}$$

$$\tan(\beta°) \times \tan(\delta°) = \frac{b^2\sin^2(\theta°)}{a^2\cos^2(\theta°) - (a^2 - b^2)}$$

$$\tan(\beta°) \times \tan(\delta°) = \frac{b^2\sin^2(\theta°)}{a^2\cos^2(\theta°) - a^2 + b^2}$$

$$\tan(\beta°) \times \tan(\delta°) = \frac{b^2\sin^2(\theta°)}{a^2[\cos^2(\theta°) - 1] + b^2}$$

$$\tan(\beta°) \times \tan(\delta°) = \frac{b^2\sin^2(\theta°)}{a^2\sin^2(\theta°) + b^2} \qquad ------------------------(9.1)$$

Eqn. (9.1) is the mathematical expression of the property.

Property-10: Mathematical relation between α°, β° and δ°

A tangent is drawn at point ‘P’ on ellipse. x-axis and y-axis are called as transverse axis and conjugate axis respectively, points T & S are x & y intercepts respectively, points Q & R are the projection of point P on transverse axis & conjugate axis respectively. In other words, OQ & OR are the abscissa and ordinate of point P respectively.

Let $\angle UOQ = \theta°$ is eccentric angle, $\angle PTQ = \alpha°$, $\angle PF_1T = \beta°$, $\angle TOP = \varphi°$, $\angle PF_2Q = \delta°$. F_1 & F_2 are foci. OA is semi-major axis. OB is semi-minor axis. O is centre of the ellipse. PF_1 & PF_2 are pair focal distances at point P.

In fig. 10, $OQ = x = OF_1 + F_1Q$.

$$\text{Eqn. (2.2)} \Rightarrow \tan(\beta°) = \frac{b\sin(\theta°)}{a\cos(\theta°) - \sqrt{a^2 - b^2}}$$

$$\text{Eqn. (4.1)} \Rightarrow \tan(\delta°) = \frac{b\sin(\theta°)}{\sqrt{a^2 - b^2} + a\cos(\theta°)}$$

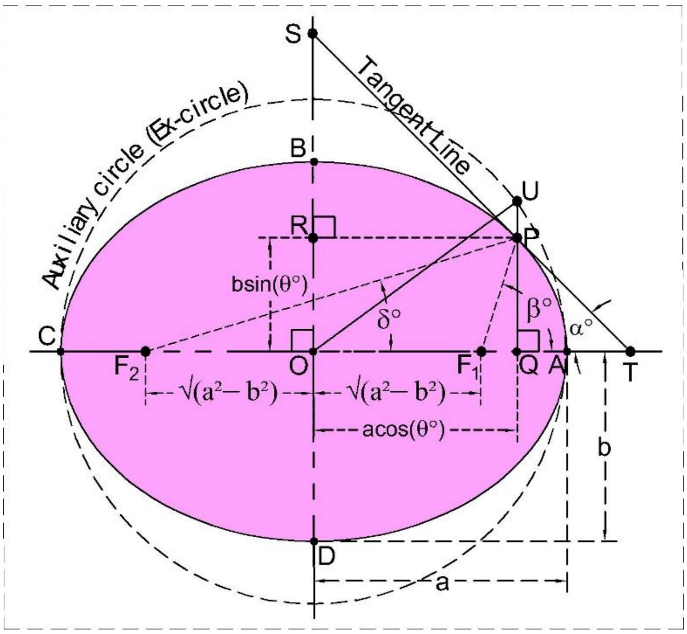

Fig: 10

$$\therefore \frac{1}{\tan(\beta°)} + \frac{1}{\tan(\delta°)} = \left(\frac{a\cos(\theta°) - \sqrt{a^2 - b^2}}{b\sin(\theta°)}\right) + \left(\frac{\sqrt{a^2 - b^2} + a\cos(\theta°)}{b\sin(\theta°)}\right)$$

$$\therefore \frac{1}{\tan(\beta°)} + \frac{1}{\tan(\delta°)} = \frac{a\cos(\theta°) \cancel{- \sqrt{a^2 - b^2} + \sqrt{a^2 - b^2}} + a\cos(\theta°)}{b\sin(\theta°)}$$

$$\therefore \frac{1}{\tan(\beta°)} + \frac{1}{\tan(\delta°)} = \frac{2a}{b\tan(\theta°)} \quad ------------------------(10.1)$$

$$\text{Eqn. (5.1)} \Rightarrow \tan(\alpha°) = \frac{b}{a \times \tan(\theta°)}$$

Substituting the above in eqn. (7.1), we get

$$\left(\frac{1}{\tan(\beta°)} + \frac{1}{\tan(\delta°)}\right)\frac{1}{\tan(\alpha°)} = \left(\frac{2a}{b\tan(\theta°)}\right) \times \left(\frac{a \times \tan(\theta°)}{b}\right)$$

$$\therefore \left(\frac{1}{\tan(\beta°)} + \frac{1}{\tan(\delta°)}\right)\frac{1}{\tan(\alpha°)} = \left(\frac{2a}{b \times \cancel{\tan(\theta°)}}\right) \times \left(\frac{a \times \cancel{\tan(\theta°)}}{b}\right)$$

$$\therefore \left(\frac{1}{\tan(\beta°)} + \frac{1}{\tan(\delta°)}\right) \times \frac{1}{\tan(\alpha°)} = \left(\frac{2a^2}{b^2}\right) \text{ where, } \frac{2a^2}{b^2} \text{ is an arbitrary constant} \quad ----(10.2)$$

Eqns. (10.1) is the mathematical expression of the property.

Property-11: Mathematical relation between φ° and δ°

A tangent is drawn at point 'P' on ellipse. x-axis and y-axis are called as transverse axis and conjugate axis respectively, points T & S are x & y intercepts respectively, points Q & R are the projection of point P on transverse axis & conjugate axis respectively. In other words, OQ & OR are the abscissa and ordinate of point P respectively.
Let $\angle UOQ = \theta°$ is eccentric angle, $\angle TOP = \varphi°$, $\angle PF_2Q = \delta°$. F_1 & F_2 are foci. OA is semi-major axis. OB is semi-minor axis. O is centre of the ellipse. PF_1 & PF_2 are pair focal distances at point P.

Referring fig. 1, In right-triangle PQO, $\angle POQ = \varphi°$

$$PQ = OP \times \sin(\varphi°) \quad ------------------------(11.1)$$

$$OQ = OP \times \cos(\varphi°) \quad ------------------------(11.2)$$

$$OP = \frac{ab}{\sqrt{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}}$$

Referring fig. 1, In right-triangle PQO, $\angle PF_2Q = \delta°$

$$\tan(\delta°) = \frac{PQ}{F_2Q} = \frac{PQ}{F_2F_1 + F_1Q}$$

$$\tan(\delta°) = \frac{PQ}{F_2F_1 + (OQ - OF_1)}$$

$$\tan(\delta°) = \frac{PQ}{F_2F_1 + (OQ - OF_1)}$$

Substituting eqns. (11.1), (11.2), $F_1F_2 = 2\sqrt{a^2 - b^2}$ and $OF_1 = \sqrt{a^2 - b^2}$ in above eqn.,

$$\tan(\delta°) = \frac{OP \times \sin(\varphi°)}{2\sqrt{a^2 - b^2} + \left(OP \times \cos(\varphi°) - \sqrt{a^2 - b^2}\right)}$$

$$\therefore \tan(\delta°) = \frac{ab \times \sin(\varphi°)}{\sqrt{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}} \div \left(\sqrt{a^2 - b^2} + \frac{ab \times \cos(\varphi°)}{\sqrt{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}}\right)$$

$$\therefore \tan(\delta°) = \frac{ab \times \sin(\varphi°)}{\sqrt{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}} \div \left(\frac{\left(\sqrt{a^2 - b^2}\sqrt{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}\right) + ab\cos(\varphi°)}{\sqrt{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}}\right)$$

$$\therefore \tan(\delta°) = \frac{ab \times \sin(\varphi°)}{\cancel{\sqrt{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}}} \times \left(\frac{\cancel{\sqrt{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}}}{\left(\sqrt{a^2 - b^2}\sqrt{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}\right) + ab\cos(\varphi°)}\right)$$

Simplifying the above eqn., we get

$$\tan(\delta°) = \frac{ab\sin(\varphi°)}{ab\cos(\varphi°) + \sqrt{a^2(a^2 - b^2)\sin^2(\varphi°) + b^2(a^2 - b^2)\cos^2(\varphi°)}} \quad ---------(11.3)$$

Eqn. (11.3) is the mathematical expression of the property.

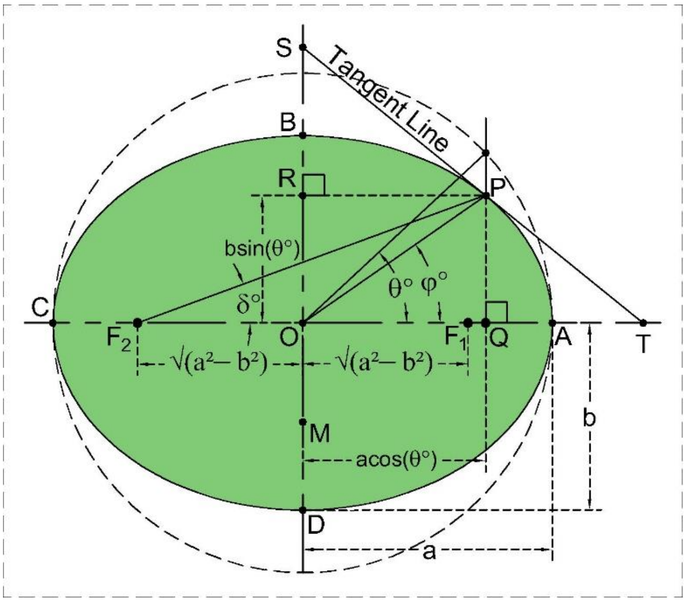

Fig: 11

Property-12: Miscellaneous properties with respect to PT, PS, a, b & θ°

A tangent is drawn at point ‘P’ on ellipse. x-axis and y-axis are called as transverse axis and conjugate axis respectively, points T & S are x & y intercepts respectively, points Q & R are the projection of point P on transverse axis & conjugate axis respectively. In other words, OQ & OR are the abscissa and ordinate of point P respectively.

Let $\angle UOQ = \theta°$ is eccentric angle, F_1 & F_2 are foci. OA is semi-major axis. OB is semi-minor axis. O is centre of the ellipse. PF_1 & PF_2 are pair focal distances at point P.

Referring fig. 12.

$$OQ = a \times \cos(\theta°)$$

$$QT = OT - OQ$$

Substituting eqns. (1.1) and $OQ = a\cos(\theta°)$ in above, we have

$$QT = \left[\frac{a}{\cos(\theta°)}\right] - a\cos(\theta°)$$

$$\therefore QT = \frac{a - a\cos^2(\theta°)}{\cos(\theta°)}$$

$$\therefore QT = \frac{a[1-\cos^2(\theta°)]}{\cos(\theta°)}$$

$$\therefore QT = \frac{a\sin^2(\theta°)}{\cos(\theta°)} \quad ----------------------------(12.1)$$

In right-triangle PQT, $\angle PQT = 90°$

$$\therefore PT^2 = QT^2 + PQ^2$$

Substituting eqns. (1.2) in above, we have

$$\therefore PT^2 = \left[\frac{a\sin^2(\theta°)}{\cos(\theta°)}\right]^2 + [b\sin(\theta°)]^2$$

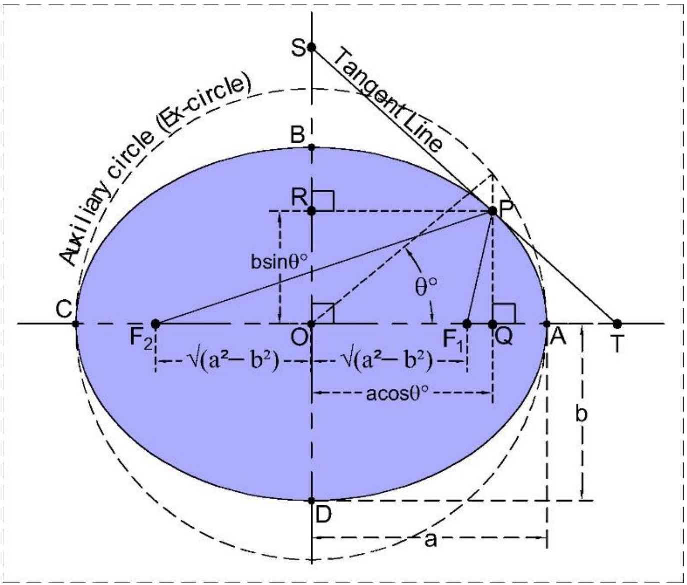

Fig: 12

$$\therefore PT^2 = \frac{a^2\sin^4(\theta°)}{\cos^2(\theta°)} + b^2\sin^2(\theta°)$$

$$\therefore PT^2 = \frac{\sin^2(\theta°)\left(a^2\sin^2(\theta°) + b^2\cos^2(\theta°)\right)}{\cos^2(\theta°)}$$

$$\therefore PT^2 = \tan^2(\theta°)\left(a^2\sin^2(\theta°) + b^2\cos^2(\theta°)\right) \quad ----------------------(12.2)$$

Referring fig. 12.

$OR = b \times \sin(\theta°)$

$RS = OS - OR$

$$OS = \frac{b}{\sin(\theta°)}$$

Substituting eqns. (1.2) and $OR = b \times \sin(\theta°)$in above, we have

$$RS = \left[\frac{b}{\sin(\theta°)}\right] - b\sin(\theta°)$$

$$\therefore RS = \frac{b - b\sin^2(\theta°)}{\sin(\theta°)}$$

$$\therefore \mathrm{RS} = \frac{\mathrm{b}[1 - \sin^2(\theta°)]}{\sin(\theta°)}$$

$$\therefore \mathrm{RS} = \frac{\mathrm{b}\cos^2(\theta°)}{\sin(\theta°)} \quad ----------------------------(12.3)$$

In right-triangle PQT, $\angle \mathrm{PQT} = 90°$

$$\therefore \mathrm{PS}^2 = \mathrm{RS}^2 + \mathrm{PR}^2$$

$$\therefore \mathrm{PS}^2 = \mathrm{RS}^2 + \mathrm{OQ}^2$$

Substituting eqns. (12.2) in above, we have

$$\therefore \mathrm{PS}^2 = \left[\frac{\mathrm{b}\cos^2(\theta°)}{\sin(\theta°)}\right]^2 + [\mathrm{a}\cos(\theta°)]^2$$

$$\therefore \mathrm{PS}^2 = \frac{\mathrm{b}^2\cos^4(\theta°)}{\sin^2(\theta°)} + \mathrm{a}^2\cos^2(\theta°)$$

$$\therefore \mathrm{PS}^2 = \frac{\cos^2(\theta°)\left(\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2\cos^2(\theta°)\right)}{\sin^2(\theta°)}$$

$$\therefore \mathrm{PS}^2 = \cot^2(\theta°)\left(\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2\cos^2(\theta°)\right) \quad ---------------------(12.4)$$

Dividing eqn. (12.2) by eqn. (12.4)

$$\frac{\mathrm{PT}^2}{\mathrm{PS}^2} = \frac{\tan^2(\theta°)\left(\cancel{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2\cos^2(\theta°)}\right)}{\cot^2(\theta°)\left(\cancel{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2\cos^2(\theta°)}\right)} = \tan^4(\theta°)$$

$$\therefore \frac{\mathrm{PT}}{\mathrm{PS}} = \tan^2(\theta°) \quad ----------------------------------(12.5)$$

Eqn. (12.5) is the mathematical expression of the property.

Property-13: Properties with respect to TS, a, b & θ°

A tangent is drawn at point 'P' on ellipse. x-axis and y-axis are called as transverse axis and conjugate axis respectively, points T & S are x & y intercepts respectively, points Q & R are the projection of point P on transverse axis & conjugate axis respectively. In other words, OQ & OR are the abscissa and ordinate of point P respectively.

Let $\angle \mathrm{UOQ} = \theta°$ is eccentric angle, F_1 & F_2 are foci. OA is semi-major axis. OB is semi-minor axis. O is centre of the ellipse. PF_1 & PF_2 are pair focal distances at point P.

Referring fig:12, In right triangle TOS, $\angle \mathrm{TOS} = 90°$.

As per Pythagoras theorem, $\mathrm{TS}^2 = \mathrm{OT}^2 + \mathrm{OS}^2$

Substituting eqns. (1.1) & (1.2) in above, we get

$$\mathrm{TS}^2 = \left[\frac{\mathrm{a}}{\cos(\theta°)}\right]^2 + \left[\frac{\mathrm{b}}{\sin(\theta°)}\right]^2$$

$$\therefore \mathrm{TS}^2 = \frac{\mathrm{a}^2}{\cos^2(\theta°)} + \frac{\mathrm{b}^2}{\sin^2(\theta°)}$$

$$\therefore \mathrm{TS}^2 = \frac{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2\cos^2(\theta°)}{\sin^2(\theta°)\cos^2(\theta°)}$$

$$\therefore \mathrm{TS} = \frac{\sqrt{\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2\cos^2(\theta°)}}{\sin(\theta°)\cos(\theta°)} \quad ---------------------------(13.1)$$

Eqn. (13.1) is the mathematical expression of the property.

Property-14: Properties with respect to $\overline{PF_1}$, $\overline{PF_2}$, $\overline{F_1V}$, $\overline{F_2W}$ & θ°

A tangent is drawn at point 'P' on ellipse. x-axis and y-axis are called as transverse axis and conjugate axis respectively, points T & S are x & y intercepts respectively, points Q & R are the projection of point P on transverse axis & conjugate axis respectively. In other words, OQ & OR are the abscissa and ordinate of point P respectively.
Let $\angle UOQ = \theta°$ is eccentric angle, F_1& F_2 are foci. OA is semi-major axis. OB is semi-minor axis. O is centre of the ellipse. PF_1 & PF_2 are pair focal distances at point P. Points V & W are intersection of chord passing through the focus and point P.

We already known that the parametric equation of ellipse is $P\,(x, y) = (a\,\cos\theta°, b\sin\theta°)$. θ° is eccentric angle of the ellipse.

Referring fig. 8. In right-triangle UQO, $\angle UQO = 90°$ and $\angle UOQ = \theta°$.

PF_1 & PF_2 are pair focal distances *(Borowski EJ and Borwein JM)* at a point P

$PF_1{}^2 = F_1Q^2 + F_1Q^2$

$\therefore PF_1{}^2 = (OQ - OF_1)^2 + F_1Q^2$

$\therefore PF_1{}^2 = \left(a\cos\theta° - \sqrt{a^2 - b^2}\right)^2 + (b\sin\theta°)^2$ [where $OF_1 = \sqrt{a^2 - b^2}$ is called linear eccentricity

Simplifying the above, we get

$PF_1{}^2 = \left(a - \sqrt{a^2 - b^2}\cos\theta°\right)^2$

$\therefore PF_1 = a - \left(\sqrt{a^2 - b^2} \times \cos\theta°\right)$ ---------------------------(14.1)

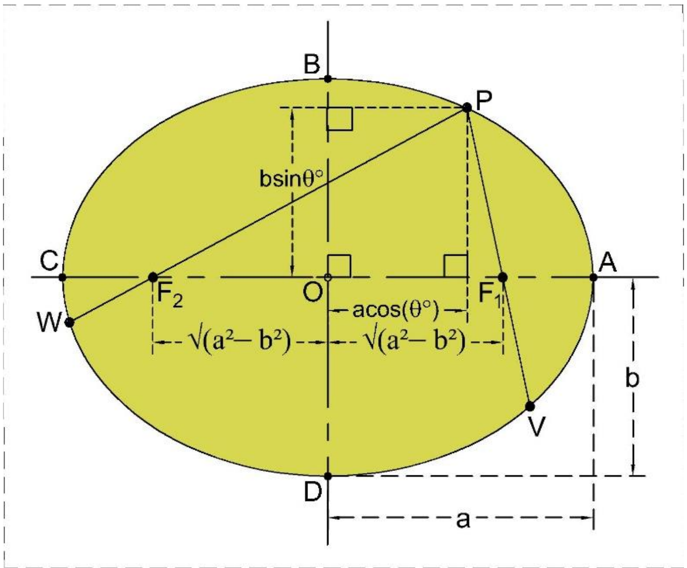

Fig: 13

Referring fig. 8. In right-triangle PQO, $\angle PQO = 90°$ and $\angle POQ = \theta°$

We know already that $PF_1 + PF_2 = 2a$

$\therefore\ PF_2 = 2a - PF_1$

Substituting eqn. (11.1) in above

$\therefore\ PF_2 = 2a - \left(a - \sqrt{a^2 - b^2}\cos\theta°\right)$

Simplifying the above, we get

$$PF_2 = a + \left(\sqrt{a^2 - b^2} \times \cos\theta°\right) \quad ------------------------(14.2)$$

Subtracting eqn. (14.1) from (14.2)

$$PF_2 - PF_1 = a + \left(\sqrt{a^2 - b^2} \times \cos\theta°\right) - \left[a - \left(\sqrt{a^2 - b^2} \times \cos\theta°\right)\right]$$

$$\therefore PF_2 - PF_1 = 2\sqrt{a^2 - b^2} \times \cos\theta° \quad ------------------------(14.3)$$

Multiplying eqn. (14.1) and (14.2)

$$PF_2 \times PF_1 = a + \left(\sqrt{a^2 - b^2} \times \cos\theta°\right) \times \left[a - \left(\sqrt{a^2 - b^2} \times \cos\theta°\right)\right]$$

$$PF_2 \times PF_1 = a^2 - (a^2 - b^2)\cos^2\theta°$$

Simplifying the above eqn. we get

$$PF_2 \times PF_1 = a^2\sin^2\theta° + b^2\cos^2\theta° \quad ------------------------(14.4)$$

We already know that $\frac{1}{PF_1} + \frac{1}{F_1V} = \frac{2a}{b^2}$ $\quad$ ------------------------(14.5)

Similarly, We already know that $\frac{1}{PF_2} + \frac{1}{F_2W} = \frac{2a}{b^2}$ $\quad$ ----------------(14.6)

Equating eqns. (14.5) and (14.6), we have

$$\frac{1}{PF_1} + \frac{1}{F_1V} = \frac{1}{PF_2} + \frac{1}{F_2W}$$

$$\therefore \frac{1}{PF_1} - \frac{1}{PF_2} = \frac{1}{F_2W} - \frac{1}{F_1V}$$

$$\therefore \frac{PF_2 - PF_1}{PF_1 \times PF_2} = \frac{F_1V - F_2W}{F_1V \times F_2W} \quad ----------------------------(14.7)$$

From eqns. (14.3) & (14.4), we have

$$\therefore \frac{PF_2 - PF_1}{PF_1 \times PF_2} = \frac{2\sqrt{a^2 - b^2}\cos\theta°}{a^2\sin^2\theta° + b^2\cos^2\theta°} \quad ------------------------(14.8)$$

Substituting eqn. (14.8) in (14.7),

$$\frac{PF_2 - PF_1}{PF_1 \times PF_2} = \frac{F_1V - F_2W}{F_1V \times F_2W} = \frac{2\sqrt{a^2 - b^2}\cos\theta°}{a^2\sin^2\theta° + b^2\cos^2\theta°} \quad ----------------(14.9)$$

Eqn. (14.9) is the mathematical expression of the property.

CONCLUSION

The author has developed and established 14 geometrical properties among various parameters of ellipse with necessary derivations and equations including appropriate drawings in detail. These properties which have been defined in this article will be very useful for those do research or further study in the field of conics & Euclidean geometry, also very useful as reference to the research scholars for higher-level research works, since these are also one of the important properties of an ellipse.

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