International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Online, International Journal, Available at http://www.cibtech.org/jpms.htm
2023 Vol. 13, pp. 59-65/Jhankal
Research Article (Open Access)

HOMOTOPY PERTURBATION METHOD FOR UNSTEADY MHD BOUNDARY LAYER FLOW OVER A STRETCHING PLATE

***A K Jhankal**
Department of Mathematics, Army Cadet College, Indian Military Academy, Dehradun-248007 (INDIA)
**Author for Correspondence: anujjhankal@yahoo.com*

ABSTRACT
The present study deals with the unsteady MHD boundary layer flow over a stretching plate. The governing nonlinear partial differential equations of the physical problem are presented and then converted into nonlinear ordinary differential equations by using similarity transformations. The resulting ordinary differential equations are successfully solved using He's homotopy perturbation method (HPM). The main advantage of HPM is that it does not require the small parameters in the equations, and hence the limitations of traditional perturbation can be eliminated. The results reveal that the proposed method is very effective and simple and can be applied to other nonlinear problems. The influence of various relevant physical characteristics is presented and discussed.

Keywords: *Homotopy Perturbation Method (HPM), MHD, Boundary Layer Flow, Stretching Plate*

NOMENCLATURE

A	Unsteady parameter $(= b/a)$, $[-]$
a, b	Constant, $[-]$
B_0	Constant applied magnetic field, $[\mathrm{Wb\ m^{-2}}]$
f	Dimensionless stream function, $[-]$
M	Magnetic parameter $(= \sigma_e B_0^2/a\rho)$, $[-]$
t	Dimensionless time, $[s]$
U	Free stream velocity, $[\mathrm{m\ s^{-1}}]$
u, v	Velocity component of the fluid along the x and y directions, respectively, $[\mathrm{m\ s^{-1}}]$
x, y	Cartesian coordinates along the surface and normal to it, respectively, $[\mathrm{m}]$
Greek symbols	
ρ	Density of the fluid, $[\mathrm{Kg\ m^{-3}}]$
υ	Kinematic viscosity, $[\mathrm{m^2\ s^{-1}}]$
σ_e	Electrical conductivity, $[\mathrm{m^2 s^{-1}}]$
η	Dimensionless similarity variable, $[= (U_w/\nu x)^{1/2}]$
Ψ	Stream function, $[= (U_w \nu x)^{\frac{1}{2}} f(\eta)]$
Superscript	
$'$	Derivative with respect to η
Subscripts:	
w	Properties at the plate
∞	Free stream condition

INTRODUCTION
The phenomenon of magnetohydrodynamic (MHD) boundary layer flow over a stretching sheet involving incompressible fluids is a common occurrence in various engineering and industrial applications. Over the past few decades, this field has garnered significant attention from researchers due to its relevance in numerous industries, such as aerodynamic extrusion of polymer sheets, hot rolling processes, cooling of metallic plates in

cooling baths, liquid film boundary layers in condensation, and the production of glass fibers Tadmor and Klein (1970), Fisher (1976), Alten and Gegrl (1979). The cooling of continuous filaments in metallurgical processes by drawing them through a quiescent fluid plays a pivotal role in determining product mechanical properties. Notably, Sakiadis (1961) laid the foundation for the study of stretching sheets and boundary layer flows. Indeed, MHD laminar boundary layer behaviour over a stretching surface holds great practical significance in fields like chemical engineering, electrochemistry, and polymer processing. Gupta and Gupta (1977) delved into the magnetic field's influence on boundary layer flow over a stretching sheet, while Anderson (1992) explored the effects of porosity in the presence of conducting fluid particles on the stretching sheet. Ariel (1994) investigated the combined impacts of viscoelasticity and magnetic fields on the Cranes problem. Since stretching can occur in various ways, the flow across a stretched sheet need not always have two dimensions; if the extension is radial, it can be three. Wang (1984) examined a flat surface with three stretches of the same width, and Brady and Acrivos (1981) monitored flow both inside and outside the channel or tube, as well as the Wang (1988) flow beyond the tube's performance region. Wang (1990) and Usha and Sidharan (1995) experimented with the unstable stretching sheet, and Ariel et al. (2006) employed HPM and extended HPM methods to derive solutions for analyzing axisymmetric flow across the flat layer of the boundary layer. Ariel (2004) provided a noniterative solution for MHD boundary layer flow over a stretching sheet. Magnetohydrodynamics (MHD) investigates the interaction of electromagnetic conditions with the transfer of heat in liquids, particularly conducting fluids, and has crucial applications in fields such as MHD power generation, MHD generators, and MHD pumps. In recent times, many researchers have focused on different aspects of boundary layer flow Sheikholeslami and Ellahi (2015), Javed and Nadeem (2019), Siavashi et al. (2019), Subhani and Nadeem (2019), Sadiq et al. (2019), Jhankal (2022), considering the MHD effects in various scenarios with stretching sheets. Furthermore, numerous scientists have explored the influence of MHD and heat transfer on a range of boundary layer flows for different parameter sets Ahmad et al. (2017), Hosseinzadeh et al. (2021, 2020). Jhankal and Kumar (2013, 2015) examined the impact of magnetic fields in conjunction with stretching surfaces.

Nonlinear phenomena play a prominent role within the boundary layer flow regime. Addressing these nonlinear equations necessitates alternative methods due to the scarcity of precise analytical solutions, except for a limited subset with established accuracy. It is widely acknowledged among the scientific community that a combination of numerical and semi-exact analytical techniques can yield valuable insights. In this research, we introduce and employ a semi-exact methodology known as the Homotopy Perturbation Method (HPM) to investigate the boundary layer flow over a stretched vertical surface with heat flux. The foundation of HPM was laid by J. H. He (1998, 2000, 2001, 2005, 2009), and subsequent studies by researchers such as Ganji and Rajabi (2006), Ganji and Sadighi (2006), Ariel et al. (2006), Zhang and He (2006), Ganji and Ganjin (2008), Beléndez et al. (2008), Ma et al. (2008), Siddiqui et al. (2008), Zhang et al. (2008), and Jhankal (2014, 2015) have been inspired to utilize this method in addressing nonlinear equations.

In the context of the current investigation, we explore the problem of unsteady boundary layer flow over a stretching plate under the influence of a transverse magnetic field. We employ the Homotopy Perturbation Method (HPM) to delve into this matter. The study examines the influence of various relevant parameters and provides insights and discussions on their effects.

1.1 Basic idea of homotopy perturbation method (HPM):

To illustrate the basic ideas of the HPM, we consider the following nonlinear differential equation.

$$A(u) - f(r) = 0, \; r \in \Omega \qquad \ldots(1)$$

Subject to the boundary conditions

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, \; r \in \Gamma \qquad \ldots(2)$$

Where A is a general differential operator, B is a boundary operator, f(r) is a known analytical function and Γ is the boundary of the domain Ω. A can be divided into two parts which are L and N, where L is linear and N is nonlinear. Therefore equation (1) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \ r \in \Omega \qquad \ldots(3)$$

By the homotopy perturbation technique, we construct a homotopy

$v(r, p): \Omega \times [0,1] \rightarrow R$, which satisfies:

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \ \ p \in [0, 1], r \in \Omega \qquad \ldots(4)$$

Where $p \in [0, 1]$ is an embedding parameter and u_0 is an initial approximation that satisfies the boundary condition. Obviously, from these definitions we will have:

$$H(v, 0) = L(v) - L(u_0) = 0$$
$$H(v, 1) = A(v) - f(r) = 0$$

The changing process of p from zero to one is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. According to the HPM, we can first use the embedding parameter p as a "small parameter" and assuming that the solution of (4) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 \ldots \qquad \ldots(5)$$

Setting $p = 1$, results in the approximate solution of (1):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \qquad \ldots(6)$$

The convergence and stability of this method was shown in Hosein Nia et al. (2008).

1. MATHEMATICAL FORMULATION OF THE PROBLEM

Consider an unsteady; two dimensional boundary layer flows over a stretching plate of a viscous incompressible electrically conducting fluid, when t = 0, the plate is impulsively stretched with the velocity U_w, where the x-axis is along the stretching plate and y-axis perpendicular to it, the applied magnetic field B_0 is transversely to x-axis. The magnetic Reynolds number for this flow has been deliberately kept at a low value to enable us to disregard the influence of any induced magnetic field. Within the framework of the boundary layer approximation, the governing equations for both fluid continuity and momentum under the influence of externally applied transverse magnetic field are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \ldots(7)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2} - \frac{\sigma_e B^2}{\rho}u \qquad \ldots(8)$$

Along with the boundary conditions for the problem are given by:

$$y = 0: u = U_w, \ v = 0$$
$$y \rightarrow \infty: u = 0 \qquad \ldots(9)$$

Here, we assume the stretching velocity $U_w(x, t)$ is of the form: $U_w(x, t) = \frac{ax}{1-bt}$, where a and b are constants with $a > 0$ and $b \geq 0$ (with $bt < 1$).

The continuity equation (1) is satisfied by introducing a stream function Ψ such that $u = \frac{\partial \Psi}{\partial y}$ and $v = -\frac{\partial \Psi}{\partial x}$. The momentum equation can be transformed into corresponding nonlinear ordinary differential equation by using the following transformations: $\eta = \left(\frac{U_w}{\nu x}\right)^{1/2} y$, $f(\eta) = \frac{\Psi}{(U_w \nu x)^{1/2}}$, and $B = \frac{B_0}{\sqrt{1-bt}}$ $\qquad \ldots(10)$

Substituting in (8), then the transformed non-linear momentum equations is:

$$f''' + ff'' - f'^2 - A\left(f' + \frac{1}{2}\eta f''\right) - Mf' = 0 \qquad \ldots(11)$$

The transformed boundary conditions are:

$$\eta = 0: f = 0, f' = 1$$
$$\eta \rightarrow \infty: f' = 0 \qquad \ldots(12)$$

Where prime denotes differentiation with respect to η, $M = \frac{\sigma_e B_0^2}{a\rho}$ is the magnetic parameter, and $A = \frac{b}{a}$ is the unsteadiness parameter.

SOLUTION WITH HOMOTOPY PERTURBATION METHOD:

According to the HPM, the homotopy form of equation (11) is constructed as follows:

$$(1-p)\left[f''' - Af' - Mf'\right] + p\left[f''' + ff'' - f'^2 - A(f' + \tfrac{1}{2}\eta f'') - Mf'\right] = 0 \qquad \ldots(13)$$

We consider f as the following:

$$f = f_0 + pf_1 + p^2 f_2 \ldots \qquad \ldots(14)$$

By substituting equation (14) into (13), and then

(I) Terms independent of p give

$$f_0''' - (A+M)f_0' = 0 \qquad \ldots(15)$$

The boundary conditions are

$$f_0(0) = 0,\ f_0'(0) = 1,\ f_0'(\infty) = 0. \qquad \ldots(16)$$

(II) Terms containing only p give

$$f_1''' - f_0'^2 + f_0 f_0'' - \frac{A}{2}\eta f_0'' - (A+M)f_1' = 0 \qquad \ldots(17)$$

The boundary conditions are

$$f_1(0) = 0,\ f_1'(0) = 0,\ f_1'(\infty) = 0. \qquad \ldots(18)$$

(III) Terms containing only p^2 give

$$f_2''' - 2f_0' f_1' + f_0 f_1'' + f_1 f_0'' - \frac{A}{2}\eta f_1'' - (A+M)f_2' = 0 \qquad \ldots(19)$$

The boundary conditions are

$$f_2(0) = 0,\ f_2'(0) = 0,\ f_2'(\infty) = 0. \qquad \ldots(20)$$

(III) Terms containing only p^3 give

$$f_3''' - f_1'^2 - 2f_0' f_2' + f_2 f_0'' + f_0 f_2'' + f_1 f_1'' - \frac{A}{2}\eta f_2'' - (A+M)f_3' = 0 \qquad \ldots(21)$$

The boundary conditions are

$$f_3(0) = 0,\ f_3'(0) = 0,\ f_3'(\infty) = 0. \qquad \ldots(22)$$

The equations (15), (17), (19), and (21) are solved with boundary conditions (16), (18), (20), and (22) respectively, the boundary condition $\eta = \infty$ were replaced by those at $\eta = 4$ in accordance with standard practice in the boundary layer analysis. If $p \to 1$, we can find the approximate solution of equations (11).

RESULTS AND DISCUSSION

To gain a deeper understanding of the problem, we have represented our numerical calculations through non-dimensional velocity profiles. These calculations have been executed across various settings, encompassing different values of two key parameters: the magnetic field parameter (M) and the unsteady parameter (A).

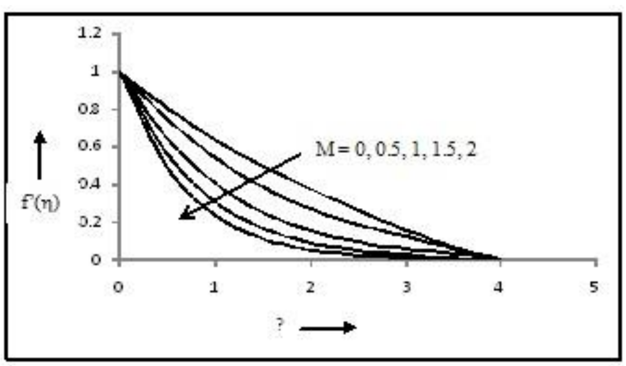

Figure 1: Velocity profile for various values of M, when A=1.5.

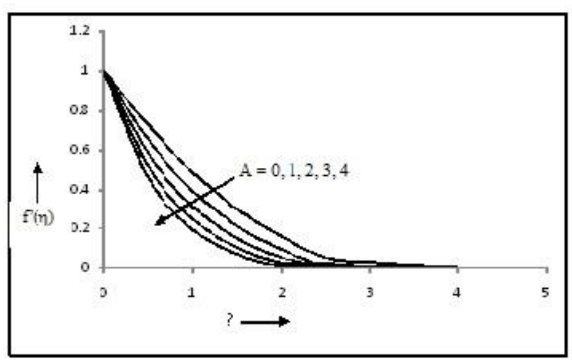

Figure 2: Velocity profile for various values of A, when M = 1.

In Figure 1, we illustrate how the magnetic parameter (M) influences the velocity profile for a fixed value of A, which is set at 1.5. Upon closer examination of this figure, it becomes evident that the dimensionless velocity f'(η) experiences a decline as M increases. This behaviour is attributed to the Lorentz force, an outcome of the interaction between the magnetic and electric fields that transpires during the motion of electrically conducting fluid. This force exerts a decelerating effect on the fluid's motion within the boundary layer.
Moving on to Figure 2, we shift our focus to the unsteady parameter (A) and its impact on the velocity profile, with M fixed at 1. Our analysis of this graph reveals that as A increases, the velocity experiences a reduction.

CONCLUSIONS

In the present study, we entail the formulation of a mathematical model to analyze the unsteady boundary layer flow over a stretching plate, subject to the effects of a transverse magnetic field. To tackle this problem, the governing momentum equation is converted into a nonlinear ordinary differential equation using a similarity transformation. This transformed equation is subsequently addressed through the application of He's Homotopy Perturbation Method (HPM). The results are then visualized in graphical representations, which shed light on the impact of magnetic and unsteady parameters on the velocity profile. The key takeaway from this investigation is that an increase in magnetic field strength (M) and unsteady parameter (A) leads to a reduction in the velocity gradient at the plate's surface.

REFERENCES

Ahmad I, Muhammad AZR, Muhammad B and Farooq A (2017). Neural network methods to solve the Lane–Emden type equations arising in thermodynamic studies of the spherical gas cloud model. *Neural Computing & Applications* **28(1)** 929-944.

Alten T, Oh S and Gegrl H (1979). Metal Forming Fundamentals and Applications. *American Society for Metals, Metal Park, OH, USA*.

Anderson HI (1992). MHD flow of a viscoelastic fluid past a stretching surface. *Acta Mechanica.* **95** 227-230.

Ariel PD (1994). MHD flow of a viscoelastic fluid past a stretching sheet with suction. *Acta Mechanica* **105(1-4)** 49-56.

Ariel PD (2004). Computation of MHD flow due to moving boundary. *Tech. Rep., Department of Mathematical Sciences: Trinity Western University, Canada (Technical report MCS-2004-01)*.

Ariel PD, Hayat T and Asghar S (2006). Homotopy perturbation method and axisymmetric flow over a stretching sheet. *International Journal of Nonlinear Sciences and Numerical Stimulation* **7(4)** 399-406.

Beléndez A, Beléndez T, Márquez A and Neipp C (2008). Application of He's homotopy perturbation method to conservative truly nonlinear oscillators. *Chaos, Solitons & Fractals*. **37**(3), 770-780.

Brady JF and Acrivos A (1981). Steady flow in a channel or tube with an accelerating surface velocity: An exact solution to the Navier-Stokes equations with reverse flow. *Journal of Fluid Mechanics* **112(1)** 127-150.

Fisher EG (1976). *Extrusion of Plastics, Wiley, New York, NY, USA.*

Ganji DD and Rajabi A (2006). Assessment of homotopy-perturbation and perturbation methods in heat radiation equations. *Internat. Comm. Heat Mass Transfer*. *33*, 391-400.

Ganji DD and Sadighi A (2006). Application of He's Homotopy–perturbation Method to Nonlinear Coupled Systems of Reaction–diffusion Equations. *Int. J. Nonl. Sci. and Num. Simu*. **7**(4), 411-418.

Ganji ZZ and Ganji DD (2008). Approximate Solutions of Thermal Boundary-layer Problems in a Semi-infinite Flat Plate by using He's Homotopy Perturbation Method. *International Journal of Nonlinear Sciences and Numerical Simulation*. **9**(4), 415-422.

Gupta PS and Gupta AS (1977). Heat and mass transfer on a stretching sheet with suction or blowing. *Canadian Journal of Chemical Engineering* **55(6)** 744-746.

He JH (1998). Approximate analytical solution for seepage flow with fractional derivatives in porous media. *J. Comput. Math. Appl. Mech. Eng*. **167**, 57-68.
He JH (2000). A review on some new recently developed nonlinear analytical techniques. *Int J Non-linear Sci Numer Simul*. **1**, 51-70.
He JH (2001). Modified Lindstedt–Poincare methods for some non-linear oscillations. Part III: double series expansion. *Int. J. Nonlinear Sci Numer Simul*. **2**:317-20.
He JH (2005). Homotopy perturbation method for bifurcation on nonlinear problems. *Int. J. Non-linear Sci. Numer. Simul*. **6**:207-8.
He JH (2009). An elementary introduction to the homotopy perturbation method. *Computers & Mathematics with Applications*. **57**(3), 410-412.
Hosein Nia SH, Ranjbar AN, Ganji DD, Soltani H and Ghasemi J (2008). Maintaining the stability of nonlinear differential equations by the enhancement of HPM. *Physics Letters A*, **372**(16), 2855-2861.
Hosseinzadeh K, Mardani MR, Sajad S, Salehi S, Waqas M and Ganji DD (2021). Entropy generation of three-dimensional Bödewadt flow of water and hexanol base fluid suspended by Fe_3O_4 and MoS_2 hybrid nanoparticles. *Pramana* **95(2)** 57.
Hosseinzadeh Kh, Roghani S, Mogharrebi AR, Waqas M and Ganji DD (2020). Investigation of cross-fluid flow containing motile gyrotactic microorganisms and nanoparticles over a three-dimensional cylinder. *Alexandria Engineering Journal* **59(5)** 3297-3307.
Javed F and Nadeem S (2019). Numerical solution of a casson nanofluid flow and heat transfer analysis between concentric cylinders. *Journal of Power Technologies* **99(1)** 25-30.
Jhankal AK and Kumar M (2013). MHD Boundary Layer Flow Past a Stretching Plate with Heat Transfer. *International Journal of Engineering and Science* **2**(3) 9-13.
Jhankal AK and Kumar M (2015). Heat and Mass Transfer Effects on MHD Boundary Layer Stagnation-Point Flow over a Nonlinear Stretching/Shrinking Sheet In Porous Media. *International Journal of Mathematics Archive* **6**(9), 1-8.
Jhankal AK (2014). Homotopy Perturbation Method for MHD Boundary Layer Flow With Low Pressure Gradient Over a Flat Plate. *Journal of Applied Fluid Mechanics* **7**, No. 1, 177-185.
Jhankal AK (2015). Application of Homotopy Perturbation Method for MHD Boundary Layer Flow of an Upper-Convected Maxwell Fluid in a Porous Medium. *Chemical Engineering Research Bulletin* **18**, 12-17.
Jhankal AK (2022). A Reliable method for boundary layer flow past a porous stretching plane with heat transfer in presence of a magnetic field. *International Journal of Physics and Mathematical Sciences* **12** 35-41.
Ma X, Wei L and Guo Z (2008). He's homotopy perturbation method to periodic solutions of nonlinear Jerk equations. *Journal of Sound and Vibration* **314**, 217-227.
Sadiq MA, Khan AU, Saleem S and Nadeem S (2019). Numerical simulation of oscillatory oblique stagnation point flow of a magneto micropolar nanofluid. *RSC Advances* **9(9)** 4751-4764.
Sakiadis BC (1961). Boundary-layer behaviour on continuous solid surfaces: I. Boundary-layer equations for two-dimensional and axisymmetric flow. *AIChE Journal* **7(1)** 26-28.
Sakiadis BC (1961). Boundary-layer behaviour on continuous solid surfaces: II. The boundary layer on a continuous flat surface. *AIChE Journal* **7(2)** 221–225.
Siavashi M, Rasam H and Izadi A (2019). Similarity solution of air and nanofluid impingement cooling of a cylindrical porous heat sink. *Journal of Thermal Analysis and Calorimetry* **135(2)** 1399-1415.
Siddiqui AM, Zeb A, Ghori QK and Benharbit AM (2008). Homotopy perturbation method for heat transfer flow of a third grade fluid between parallel plates. *Chaos, Solitons & Fractals* **36**(1), 182-192.
Sheikholeslami M and Ellahi R (2015). Three dimensional macroscopic simulation of magnetic field effect on natural convection of nanofluid. *International Journal of Heat and Mass Transfer* **89** 799-808.
Subhani M and Nadeem S (2019). Numerical analysis of micropolar hybrid nanofluid. *Applied Nanoscience* **9(4)** 447-459.

Tadmor Z and Klein I (1970). Engineering Principles of Plasticating Extrusion. *Polymer Science and Engineering Series. Van Nostrand Reinhold, New York, NY, USA.*
Usha R and Sridharan R (1995). The axisymmetric motion of a liquid film on an unsteady stretching surface. *Journal of Fluids Engineering* **117(1)** 81-85.
Wang CY (1984). The three-dimensional flow due to a stretching flat surface. *Physics of Fluids* **27(8)** 1915-1917.
Wang CY (1988). Fluid flow due to a stretching cylinder. *Physics of Fluids* **31(3)** 466-468.
Wang CY (1990). Liquid film on an unsteady stretching surface. *Quarterly of Applied Mathematics* **48(4)** 601-610.
Zhang LN and He JH (2006). Homotopy perturbation method for the solution of the electrostatic potential differential equation. *Mathematical Problems in Engineering*, Art. No. 83878.
Zhang BG, Li SY and Liu ZR (2008). Homotopy perturbation method for modified Camassa-Holm and Degasperis-Procesi. *Physics Letters A* **372**(11), 1867-1872.