International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Online, International Journal, Available at http://www.cibtech.org/jpms.htm
2022 Vol. 12, pp. 1-5/Sudin
Research Article (Open Access)

A THEORETICAL STUDY ON THE CONFLUENT HYPER-GEOMETRIC TYPE ANALYTICAL SOLUTION OF THE YUKAWA POTENTIAL

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ABSTRACT
In this paper, a confluent hyper-geometric type analytical solution of the Yukawa potential is obtained with the help of the Schrödinger equation. The solution gives an approximate value for the eigenfunction. The energy eigenvalue obtained from the solution of this eigenfunction is found directly proportional to the fine-structure constant (α) and inversely proportional to the square of the diminished screening parameter 'n'; where $n = (a - 1)$ and 'a' being the screening parameter.

Keywords: *Analytical Solution, Confluent Hyper-geometric Function, Screening Parameter, Yukawa Potential*

INTRODUCTION
Hydrogen is the first member in the periodic table. In quantum mechanics the eigenvalue problem for this atom can be solved analytically. However, as we move to the next atom (Helium) in the periodic table, the eigenvalue problem cannot be solved by analytical methods. Solutions to the Schrödinger equation are obtained for some potential (Singh, 2009; Miranda, 2010; Arda, 2012; Ikot, 2012). However, some authors have been able to find the analytical solutions for several other potentials (Rajabi, 2007; Deta, 2013; Rajabi, 2013) using different approaches. Many of which are remarkably used in the field of physics and chemistry.
The pioneering work of Yukawa potential or Static Screened Coulomb Potential (*SSCP*) has been illustrated here in the literature. The problem of finding an accurate solution for the screened Coulomb potential (spherically symmetric in general) has drawn a lot of attention in the literature. The screened potential finds its relevant role in the study of hydrogen under pressure (Ferraz, 1984; Ferraz, 1987; Mao, 1989). The binding energy of a hydrogen atom can be estimated using this potential. The Yukawa potential which is a crude model for the binding force in an atomic nucleus has the form in one dimension

$$V(x) = \alpha \frac{e^{-\lambda x}}{x} \qquad (1)$$

where, the length, $\frac{1}{\lambda} = a$ may be considered as the 'range of the potential' and $\alpha = a\,Z$ is the fine-structure constant, 'Z' being the atomic number. Here 'a' is the atomic radius which is termed as the screening parameter with the value $a = \frac{1}{137.037}$.
Thus, the expression for the Yukawa potential in equation (1) can also be written as follows:

$$V(x) = \alpha \frac{e^{-(x/a)}}{x} \qquad (2)$$

The potential expressed in equation (2) is known as the Debye-Hückel potential and the Thomas-Fermi potential respectively in plasma and solid-state physics. In both the cases, the potential arises from a screened Coulomb potential.
The aim of the present work is to obtain an analytical solution of the Yukawa potential first by solving the Schrödinger equation in the form of a confluent hyper-geometric function for this type of potential and then the solution is used to find the expression for the energy eigenvalue (E). Recently, attempts were

made by Adeyemi *et al.* to infuse the Yukawa's potential into the Schrodinger equation and they obtained an approximate value of the eigenfunction and the eigenvalue (Adeyemi, 2017). Adeyemi *et al.* had obtained their results, for a large value of 'a' and found the solution, $\psi(x)$ as a '*cosine function*' with a normalization constant (Adeyemi, 2017). In comparison with the work done by Adeyemi *et al.* here in the present work, the approximate solution has been found to be expressed in terms of a '*sine function*' and with a different normalization constant. This is the essence of the present work. The paper is organized as follows:

Section-2 is focused on obtaining an analytical solution of the Yukawa potential by solving the Schrödinger equation in the form of a confluent hyper-geometric function. The solution gives an approximate value for the eigenfunction.

Section-3 contains the result and discussion where the final expression of the analytical solution is depicted. The energy eigenvalue obtained here, shows that it is directly proportional to the square of the fine-structure constant (α) and inversely proportional to the square of the diminished screening parameter 'n'; where $n = (a-1)$ and 'a' being the screening parameter.

Finally, Section-4 has been devoted for summarizing the present work and the inferences drawn from this work have also been embedded in this section.

MATERIALS AND METHODS

The Yukawa potential in the chosen work is expressed as follows: $V(x) = \alpha \frac{e^{-(x/a)}}{x}$.

For the large value of 'a', the above potential reduces to

$$V(x) = \frac{\alpha}{x} \qquad (3)$$

This potential is used to solve the Schrödinger equation in one dimension preferably in positive x-direction using an approximate method and express the solution in the form of a confluent hyper-geometric function. The Schrödinger equation in positive x-direction for the wave function $\psi(x)$ is as follows:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\,\psi(x) \qquad (4)$$

$$\frac{d^2}{dx^2}\psi(x) + \left[\frac{2m}{\hbar^2}E - \frac{2m}{\hbar^2}V\right]\psi(x) = 0 \qquad (5)$$

Substituting the value of the potential, $V(x) = \frac{\alpha}{x}$ in equation (5), it gives,

$$\frac{d^2}{dx^2}\psi(x) + [\frac{2m}{\hbar^2}E - \frac{2m}{\hbar^2}\frac{\alpha}{x}]\psi(x) = 0 \qquad (6)$$

where, $\frac{2m}{\hbar^2}E = k^2$ (7)

And

$$\frac{2m}{\hbar^2}\alpha = \zeta^2 \qquad (8)$$

With the help of the equations (7) and (8), the equation (6) takes the form as follows

$$\frac{d^2}{dx^2}\psi(x) + [k^2 - \frac{\zeta^2}{x}]\psi(x) = 0 \qquad (9)$$

Solving the equation (9) for the wave function $\psi(x)$, the following is obtained as,

$$\psi(x) = Exp\left[-\sqrt{k^2}x + \left(-\sqrt{-k^2} + \sqrt{k^2}\right)x\right] x\, C_2\ {}_1F_1\left[1 + \frac{-4\sqrt{k^2} - 2\left(-2\sqrt{k^2} - \zeta^2\right)}{4\sqrt{-k^2}}, 2, 2\sqrt{-k^2}\,x\right] +$$

$$Exp\left[-\sqrt{k^2}x + \left(-\sqrt{-k^2} + \sqrt{k^2}\right)x\right] x\, C_1\ {}_1F_1\left[1 + \frac{-4\sqrt{k^2} - 2\left(-2\sqrt{k^2} - \zeta^2\right)}{4\sqrt{-k^2}}, 2, 2\sqrt{-k^2}x\right] \qquad (10)$$

where, the term ${}_1F_1\left[1+\frac{-4\sqrt{k^2}-2(-2\sqrt{k^2}-\zeta^2)}{4\sqrt{-k^2}}, 2, 2\sqrt{-k^2}x\right]$ is well known by the name 'confluent hyper-geometric function' and C_1 , C_2 are constants.

Let,

$${}_1F_1\left[1+\frac{-4\sqrt{k^2}-2(-2\sqrt{k^2}-\zeta^2)}{4\sqrt{-k^2}}, 2, 2\sqrt{-k^2}x\right]$$

$$= F\left[1+\frac{-4\sqrt{k^2}-2(-2\sqrt{k^2}-\zeta^2)}{4\sqrt{-k^2}}, 2, 2\sqrt{-k^2}x\right]$$

$$= F(a, b, c) \qquad (11)$$

where, $a = \left[1+\frac{-4\sqrt{k^2}-2(-2\sqrt{k^2}-\zeta^2)}{4\sqrt{-k^2}}\right] = 1+\frac{\zeta^2}{2ik}$,

$b = 2$ and $c = 2\sqrt{-k^2}x = 2ikx$.

In terms of a, b, c the equation (10) now can be written as follows:

$\psi(x) = Exp\left[-\sqrt{k^2}x+\left(-\sqrt{-k^2}+\sqrt{k^2}\right)x\right] x\, C_2\, F(a, b, c) + Exp\left[-\sqrt{k^2}x+\left(-\sqrt{-k^2}+\right.\right.$
k2x x C1 F(a,b,c) (12)

Here, $F(a, b, c)$ is the Whittaker's hyper-geometric function.

Let, $C_1 + C_2 = C_3$, C_3 is also a constant.

Then the equation (12) takes the form as

$\psi(x) = Exp[-ikx]\; x\, C_3\, F(a, b, c)$

$\therefore \psi(x) = x\, C_3\, [\cos kx - i\sin kx]\, F(a, b, c)$ (13)

RESULTS AND DISCUSSION

To find the final expression for the solution of the wave function expressed in equation (13), it requires determining the value of C_3 and $F(a, b, c)$. In order to do that the following transformation of the hyper-geometric function is applied in equation (13),

$F(n, 2, c) = x^{1-d} F(a-d+1, 2-d, c).$ (14)

For, $d = 0$, the equation (14) becomes,

$F(n, 2, c) = x\, F(a+1, 2, c)$ (15)

Let's write now, $n = a - 1 = \frac{\zeta^2}{2ik}$

For special case, where $n = 1$, then equation (15) gives

$F(1,2, c) = c^{-1}$

i.e. $F(1,2, c) = \frac{1}{c}$ (16)

Substituting the value, $c = 2\sqrt{-k^2}x = 2ikx$, in equation (16) we have

$F(1,2, c) = \frac{1}{2ikx}$ (17)

Therefore, for $n = 1$, the solution for the wave function using equation (13) is written as

$\psi(x) = x\, C_3\, [\cos kx - i\sin kx]\;\; F(1,2, c)$ (18)

Considering the real part of the solution, the expression of the wave function becomes: $\psi(x) = x\, C_3 \cos kx\;\; F(1,2, c)$

Now, taking absolute value of $\psi(x)$ and substituting the value of $F(1,2, c)$ of the equation (17) in equation (18), the expression of the wave function can be written as

$\psi(x) = \frac{C_3}{2k}\sin kx$ (19)

where C_3 is normalization constant.

The value of the normalization constant (C_3) is obtained as $C_3 = \sqrt{\frac{2}{\pi}}$.
Finally the expression for the solution of the wave function is obtained as

$$\psi(x) = \sqrt{\frac{2}{\pi}} \sin kx \qquad (20)$$

Now, to calculate the energy eigenvalue of the wave function of equation (20) the following values are recalled:

$n = (a - 1) = \frac{\zeta^2}{2ik}$;

$\frac{2m}{\hbar^2} E = k^2$ and $\frac{2m}{\hbar^2} \alpha = \zeta^2$.

After some algebraic calculation, the expression for the energy eigenvalue (E) can be written as follows

$$E = -\frac{m\alpha^2}{2n^2\hbar^2} \qquad (21)$$

where, $n = (a - 1) = 1,2,3 \dots$.
And in terms of the atomic radius i.e. the screening parameter 'a', the equation (21) can be rewritten as

$$E = -\frac{m\alpha^2}{2(a-1)^2\hbar^2} . \qquad (22)$$

It can be inferred from the equations (21) and (22) that the energy eigenvalue, $E \propto -\alpha^2$; 'a', 'n' and $\hbar$ being constants.
Thus, it is evident from the above expression that the energy eigenvalue of the eigenfunction varies directly as the square of the fine-structure constant (α). Further, it can also be noted from the equation (21) that the energy eigenvalue varies inversely as the square of the diminished screening parameter 'n'.

CONCLUSIONS

In the present work, the analytical solution of Yukawa potential in light of an approximate method is obtained by solving the Schrödinger equation in terms of confluent hyper-geometric function. The essence of the present work lies in the fact that the obtained approximate solution of the Schrödinger equation has been expressed in terms of a '*sine function*' and with different normalization constant instead of the '*cosine function*' in comparison with the work done by Adeyemi *et al.* Later, the energy eigenvalue has been obtained from this eigenfunction. It has been seen that the results obtained in this work are in well agreements with other works. The present work clearly indicates to the fact that the energy eigenvalue of the Yukawa potential depends on fine-structure constant (α) and that of the screening parameter 'a'. It has been observed that the energy eigenvalue obtained from the analytical solution is directly proportional to the fine-structure constant (α) and inversely proportional to the square of the diminished screening parameter 'n'; where $n = (a - 1)$ and 'a' being the screening parameter. It can be expected that the results obtained in this work may have interesting applications in different fields of quantum mechanical systems, atomic and molecular physics.

ACKNOWLEDGEMENT

The author is thankful to the anonymous referees for their useful and constructive suggestions.

REFERENCES

Adeyemi OF, Samson DO, Okwokwo IO and Umar M (2017). Analytical Solution To The Yukawa's Potential Using Confluence Hypergeometry Functions. *International Organization of Scientific Research Journal of Applied Physics* **9**(6) 23-26. DOI: 10.9790/4861-09060022326.

Arda A and Sever R (2012). Non-central potentials, exact solutions and Laplace transform approach. *Journal of Mathematical Chemistry* **50**(6) 1484–1494. https://doi.org/10.1007/s10910-012-9984-y

Deta UA, Suparmi A and Cari C (2013). Approximate solution of Schrödinger equation in D-dimensions for Scarf hyperbolic potential using Nikiforov-Uvarov method. *Advanced Studies in Theoretical Physics* **7**(13) 647 – 656. http://dx.doi.org/10.12988/astp.2013.3549.

Ferraz A, March NH and Flores F (1984). Metal-insulator transition in hydrogen and in expanded alkali metals. *Journal of Physics and Chemistry of Solids* **45**(6) 627-635. https://doi.org/10.1016/0022-3697(84)90055-6.

Ferraz A, Oliveira FA and Amato MA (1987). Dielectric screening, excitation binding energy and the insulator-metal transition in molecular hydrogen. *Solid State Communications* **64**(10) 1321-1323. https://doi.org/10.1016/0038-1098(87)90634-X.

Hemley RJ and Mao HK (1989). Optical Studies of Hydrogen Above 200 Gigapascals: Evidence for Metallization by Band Overlap. *Science* **244**(4911) 1462-1465. DOI:10.1126/science.244.4911.1462.

Ikot AN, Awoga AO and Ita BI (2012). Bound state solutions of exponential – coshine screened coulomb plus Morse potential. *Journal of Atomic and Molecular Sciences* **3**(4) 285 – 296. doi: 10.4208/jams.102511.113011a.

Miranda MG, Sun GH and Dong SH (2010). THE SOLUTION OF THE SECOND P*ö*SCHL-TELLER LIKE POTENTIAL BY NIKIFOROV-UVAROV METHOD. *International Journal of Modern Physics E* - *nuclear physics* **19**(1) 123-129. https://doi.org/10.1142/S0218301310014704.

Rajabi AA (2007). A Method to Solve the Schrödinger Equation for Any Power Hypercentral Potentials. *Communications in Theoretical Physics* **48**(1) 151 – 158. http://dx.doi.org/10.1088/0253-6102/48/1/031

Rajabi AA and Hamzavi M (2013). A new coulomb-shaped potential via generalized parametric Nikiforov-Uvarov method, *Journal of Theoretical and Applied Physics* **7**(17) 1-5. https://doi.org/10.1186/2251-7235-7-17.

Singh RM, Chand F and Mishra SC (2009). The solutions of the Schr*ö*dinger equation for coupled quadratic and quartic potentials in two dimensions. *PRAMANA–journal of physics* **72** (4) 647 – 654. DOI:10.1007/S12043-009-0058-Z.