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Research Article

SUMMABLE SERIES AND SUMMABILITY FACTORS IN NON-ARCHIMEDEAN ANALYSIS

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ABSTRACT
In this paper we study about certain summable series and the corresponding summability factors in complete, non-trivially valued, non-archimedean fields.

Keywords: *Summable Series, Summability Factors, Non-archimedean Fields*

INTRODUCTION AND PRELIMINARIES

In this paper, K denotes a complete, non-trivially valued, non-archimedean field. For the definition of summability factors see (Peyerimhoff, 1969). In the classical case, the following theorems are well-known (see, for instance, Hardy (1949), p. 51).

Theorem A.
$\sum_{n=0}^{\infty} a_n b_n$ converges whenever $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} |b_n - b_{n+1}| < \infty$.

Theorem B.
$\sum_{n=0}^{\infty} a_n b_n$ converges whenever $\sum_{n=0}^{\infty} a_n$ has bounded partial sums if and only if $\lim_{n\to\infty} b_n = 0$.

We shall now prove the analogues of Theorem A and Theorem B in K.

Theorem 1.
$\sum_{n=0}^{\infty} a_n b_n$ converges whenever $\sum_{n=0}^{\infty} a_n$ converges if and only if $\{b_n\}$ is bounded.

Proof.
If $\sum_{n=0}^{\infty} a_n$ converges and $\{b_n\}$ is bounded, then $\lim_{n\to\infty} a_n b_n = 0$ so that $\sum_{n=0}^{\infty} a_n b_n$ converges (see Bachman (1964), p. 25). Conversely, let $\sum_{n=0}^{\infty} a_n b_n$ converge whenever $\sum_{n=0}^{\infty} a_n$ converges. We claim that $\{b_n\}$ is bounded. Suppose not. We can now find a strictly increasing sequence $\{n(i)\}$ of positive integers such that $n(i+1) - n(i) > 1$ and

$$|b_{n(i)}| > i^2, \quad i = 1,2,.... \tag{1}$$

Since K is non-trivially valued, there exists $\pi \in K$ such that $0 < \rho = |\pi| < 1$. For i = 1, 2, ..., we can find a non-negative integer $\alpha(i)$ such that

$$\rho^{\alpha(i)+1} \leq \frac{1}{i^2} < \rho^{\alpha(i)}. \tag{2}$$

Define the series $\sum_{n=0}^{\infty} a_n$, where

$$\begin{aligned} a_n &= \pi^{\alpha(i)+1}, \quad \text{if } n = n(i); \\ &= 0, \quad \text{if } n \neq n(i), i = 1,2,.... \end{aligned}$$

It is clear that $\sum_{n=0}^{\infty} a_n$ converges, since

$$\sum_{n=0}^{\infty} |a_n| = \sum_{i=1}^{\infty} |a_{n(i)}| = \sum_{i=1}^{\infty} \rho^{\alpha(i)+1} \leq \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty,$$

using (2). On the other hand,

$$\begin{aligned} |a_{n(i)} b_{n(i)}| &> i^2 \rho^{\alpha(i)+1}, \quad \text{using (1) and (2)} \\ &= i^2 \rho \cdot \rho^{\alpha(i)} \\ &> i^2 \rho \cdot \frac{1}{i^2}, \quad \text{using (2) again} \\ &= \rho, \quad i = 1,2,... \\ &\not\to 0, \quad i \to \infty, \end{aligned}$$

so that $\sum_{n=0}^{\infty} a_n b_n$ does not converge, which is a contradiction. This completes the proof of the theorem.

Theorem 2.
$\sum_{n=0}^{\infty} a_n b_n$ converges whenever $\sum_{n=0}^{\infty} a_n$ has bounded partial sums if and only if $\lim_{n\to\infty} b_n = 0$.

Proof.
Let $\lim_{n\to\infty} b_n = 0$ and $|s_n| \leq M$, n = 0, 1, 2, ..., where $s_n = \sum_{k=0}^{n} a_k$, n = 0, 1, 2, ... Now, $|a_n| = |s_n - s_{n-1}| \leq \max(|s_n|, |s_{n-1}|) \leq M$, n = 0, 1, 2, ..., so that $\lim_{n\to\infty} a_n b_n = 0$. Consequently $\sum_{n=0}^{\infty} a_n b_n$

converges. Conversely, let $\sum_{n=0}^{\infty} a_n b_n$ converge whenever $\{s_n\}$ is bounded. Suppose $\lim_{n\to\infty} b_n \neq 0$. Then there exist $\varepsilon > 0$ and a strictly increasing sequence $\{n(i)\}$ of positive integers such that

$$\left|b_{n(i)}\right| > \varepsilon, \quad i = 1,2,.... \tag{3}$$

We can now choose a positive integer α such that

$$\rho^{\alpha+1} \leq \frac{1}{\varepsilon} < \rho^{\alpha}, \tag{4}$$

where $0 < \rho = |\pi| < 1$, $\pi \in K$ as before. Define

$$\begin{aligned} a_n &= \pi^{\alpha+1}, \quad \text{if } n = n(i); \\ &= 0, \quad \text{if } n \neq n(i), i = 1,2,.... \end{aligned}$$

It is clear that $\left|a_n\right| \leq \rho^{\alpha+1} \leq \frac{1}{\varepsilon}$ and so $\left|s_n\right| \leq \frac{1}{\varepsilon}$, n = 0, 1, 2, ... Thus $\{s_n\}$ is bounded. However,

$$\begin{aligned} \left|a_{n(i)} b_{n(i)}\right| &> \varepsilon\rho^{\alpha+1}, \quad \text{using (3) and (4)} \\ &= \varepsilon\rho \cdot \rho^{\alpha} \\ &> \varepsilon\rho \cdot \frac{1}{\varepsilon}, \quad \text{using (4) again} \\ &= \rho \\ &\not\to 0, \quad i \to \infty \end{aligned}$$

so that $\sum_{n=0}^{\infty} a_n b_n$ does not converge, which is a contradiction, proving the theorem.

The following definition (see Srinivasan, 1965) is needed in the sequel.

Definition 3.
The sequence $\{x_n\}$ in K is said to be Y-summable to ℓ if

$$\frac{x_n + x_{n-1}}{2} \to \ell, \quad n \to \infty.$$

The infinite series $\sum_{k=0}^{\infty} x_k$ is said to be Y-summable to s, if $\{s_n\}$ is Y-summable to s, where

$$s_n = \sum_{k=0}^{n} x_k, \quad n = 0,1,2,....$$

The next result is a consequence of Theorem 3 of (Natarajan, 2003).

Theorem 4.
If $\sum_{n=0}^{\infty} a_n$ is Y-summable and $\{b_n\}$ converges, then $\sum_{n=0}^{\infty} a_n b_n$ is Y-summable.

Remark 5.
The hypothesis that "$\{b_n\}$ converges" in Theorem 4 cannot be dropped, as the following example illustrates. Let $K = Q_p$, the p-adic field for a prime p and let $\{b_n\} = \{1, -1, 1, -1, ...\}$. It is clear that $\{b_n\}$ does not converge. Let $\sum_{n=0}^{\infty} a_n = 1 - 1 + 1 - 1 + ...$ Note that $\sum_{n=0}^{\infty} a_n$ is Y-summable to $\frac{1}{2}$. However,

$$\begin{aligned} |a_n b_n + a_{n+1} b_{n+1}| &= |2| \\ &\not\to 0, \quad n \to \infty, \end{aligned}$$

so that $\sum_{n=0}^{\infty} a_n b_n$ is not Y-summable (see Srinivasan, 1965).

In the context of Theorem 1 and Theorem 4, the following result is of interest.

Theorem 6.
If $\sum_{n=0}^{\infty} a_n b_n$ is Y-summable whenever $\sum_{n=0}^{\infty} a_n$ is Y-summable, then $\{b_n\}$ is bounded.

Proof.
Let $\sum_{n=0}^{\infty} a_n b_n$ be Y-summable whenever $\sum_{n=0}^{\infty} a_n$ is Y-summable. Suppose $\{b_n\}$ is not bounded. Then we can find a strictly increasing sequence $\{n(i)\}$ of positive integers such that $n(i+1) - n(i) > 1$ and

$$|b_{n(i)}| > i, \quad i = 1,2,.... \tag{5}$$

For i = 1, 2, ..., there exists a positive integer $\alpha(i)$ such that

$$\rho^{\alpha(i)+1} \leq \frac{1}{i} < \rho^{\alpha(i)}, \tag{6}$$

$0 < \rho = |\pi| < 1$, $\pi \in K$ as before. Define

$$\begin{aligned} a_n &= \pi^{\alpha(i)+1}, \quad \text{if } n = n(i); \\ &= 0, \quad \text{if } n \neq n(i), \quad i = 1,2,.... \end{aligned}$$

Note that $\lim_{n\to\infty}(a_n + a_{n+1}) = 0$, using (6) so that $\sum_{n=0}^{\infty} a_n$ is Y-summable. On the other hand,

$$\begin{aligned} \left|a_{n(i)}b_{n(i)} + a_{n(i)+1}b_{n(i)+1}\right| &= \left|a_{n(i)}b_{n(i)}\right| \\ &> i\cdot\rho^{\alpha(i)+1}, \quad \text{using (5) and (6)} \\ &= i\rho\cdot\rho^{\alpha(i)} \\ &> i\rho\cdot\frac{1}{i}, \quad \text{using (6) again} \\ &= \rho \\ &\not\to 0, \quad i\to\infty. \end{aligned}$$

Consequently $\sum_{n=0}^{\infty} a_n b_n$ is not Y-summable, a contradiction, which establishes the theorem.

Remark 7.
Converse of Theorem 6 does not hold in view of Remark 5.

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