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Research Article

A NOTE ON THE MATRIX CLASS (c_0, c)

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ABSTRACT
The present note is a continuation of an earlier paper (Natarajan) of the author. Throughout the present note, entries of sequences, infinite series and infinite matrices are real or complex numbers. We establish an interesting property of the infinite matrix class (c_0, c), where c_0, c respectively denote the Banach spaces of all null and convergent sequences.

Keywords: *A-transform, A-summable, Matrix Class, γ-matrix*

INTRODUCTION AND PRELIMINARIES
Throughout the present note, entries of sequences, infinite series and infinite matrices are real or complex numbers. To make the note self-contained, we recall a few definitions and concepts. c_0, c respectively denote the Banach spaces of all null and convergent sequences under the norm

$$\|x\| = \sup_{k\geq 0} |x_k|, \quad x = \{x_k\} \in c_0, c.$$

Given a sequence $x = \{x_k\}$ and an infinite matrix $A = (a_{nk})$, n, k = 0, 1, 2, …, we define

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0,1,2,...,$$

where, we suppose that the series on the right hand side converge. $A(x) = \{(Ax)_n\}$ is called the A-transform of the sequence $x = \{x_k\}$. If $\{(Ax)_n\} \in c$, we say that A sums $x = \{x_k\}$ or $x = \{x_k\}$ is A-summable. If X, Y are sequence spaces, we write $A \in (X, Y)$ if $\{(Ax)_n\} \in Y$, whenever $x = \{x_k\} \in X$. We write $A \in (c_0, c; P)$ if $A \in (c_0, c)$ and

$$\lim_{n\to\infty} (Ax)_n = \lim_{k\to\infty} x_k = 0, \quad x = \{x_k\} \in c_0,$$

P denoting preservation of limits in c_0 and c.
The following result can be easily proved.

Theorem 1.1 (Stieglitz and Tietz, 1977)
$A = (a_{nk}) \in (c_0, c)$ if and only if

$$\sup_{n\geq 0} \sum_{k=0}^{\infty} |a_{nk}| < \infty; \tag{1.1}$$

and

$$\lim_{n\to\infty} a_{nk} = \delta_k \quad \text{exists}, \ k = 0,1,2,.... \tag{1.2}$$

Further, $A \in (c_0, c; P)$ if and only if (1.1) and (1.2) hold, with $\delta_k = 0$, $k = 0, 1, 2, \ldots$.

We also need the following sequence space in the sequel.

$$\gamma = \left\{ x = \{x_k\} / \sum_{k=0}^{\infty} x_k < \infty \right\}.$$

We write $A = (a_{nk}) \in (\gamma, c; P')$ if $A \in (\gamma, c)$ and

$$\lim_{n\to\infty} (Ax)_n = \sum_{k=0}^{\infty} x_k, \quad x = \{x_k\} \in \gamma,$$

P' denoting preservation of sum in γ and limit in c. We recall that A is called a γ-matrix (see Maddox, 1977). The following result was established by Maddox (see Maddox, 1977, p. 188), which gives a simple necessary condition for a γ-matrix to be stronger than convergence.

Theorem 1.2.
If $A = (a_{nk}) \in (\gamma, c; P')$ sums at least one divergent series, then

$$\lim_{k\to\infty} a_{nk} = 0, \quad n = 0,1,2,\ldots. \tag{1.3}$$

MAIN RESULTS
In this section, we establish an interesting property of the matrix class (c_0, c) similar to the one proved in Theorem 1.2.
In (Natarajan, no date), the author considered the class $(c_0, c)'$ of all infinite matrices $A = (a_{nk})$ such that (1.3) holds. The following result gives a sufficient condition for a (c_0, c) matrix to belong to the class $(c_0, c)'$.

Theorem 2.1.
If $A = (a_{nk}) \in (c_0, c)$ sums at least one non-null sequence, then (1.3) holds.

Proof.
Let $x = \{x_k\} \in c$. Then it is a Cauchy sequence. So

$$x_k - x_{k+p} \to 0, \quad k \to \infty, p = 1,2,\ldots,$$
$$\text{i.e., } \{x_k - x_{k+p}\} \in c_0, \quad p = 1,2,\ldots.$$

Since $A \in (c_0, c)$,

$$\sum_{k=0}^{\infty} a_{nk}(x_k - x_{k+p}) \quad \text{converges}, \quad n = 0,1,2,\ldots; p = 1,2,\ldots,$$
$$\text{i.e., } \sum_{k=0}^{\infty} a_{nk} x_k - \sum_{k=0}^{\infty} a_{nk} x_{k+p} \quad \text{converges}, \quad n = 0,1,2,\ldots; p = 1,2,\ldots,$$

noting that all the series converge absolutely since $\sum_{k=0}^{\infty}|a_{nk}| < \infty$ and $\{x_k\}$ is bounded, n = 0, 1, 2, ...; p = 1, 2, ...
In other words,

$$\sum_{k=0}^{\infty} a_{nk}x_k - \sum_{k=p}^{\infty} a_{n,k-p}x_k \quad \text{converges,}$$

$$\text{i.e., } \sum_{k=p}^{\infty}(a_{nk} - a_{n,k-p})x_k + \sum_{k=0}^{p-1} a_{nk}x_k \quad \text{converges,}$$

$$\text{i.e., } \sum_{k=p}^{\infty}(a_{nk} - a_{n,k-p})x_k \quad \text{converges}, n = 0,1,2,...; p = 1,2,...,$$

for all $x = \{x_k\} \in c$. Choose $x_k = 1$, $k = 0, 1, 2, ...$ Then $\lim_{k\to\infty} x_k = 1$ so that $\{x_k\} \in c$.
Consequently,

$$a_{nk} - a_{n,k-p} \to 0, \quad k \to \infty, n = 0,1,2,...; p = 1,2,....$$

Thus $\{a_{nk}\}_{k=0}^{\infty}$ is a Cauchy sequence of real or complex numbers and so it converges. Let

$$\lim_{k\to\infty} a_{nk} = \lambda_n, \quad n = 0,1,2,..., \lambda_n \in \mathbb{R} \text{ (or } \mathbb{C}), n = 0,1,2,....$$

By hypothesis, $A \in (c_0, c)$ sums at least one non-null sequence $\{b_k\}$ (say). So

$$\sum_{k=0}^{\infty} a_{nk}b_k \quad \text{converges,} \quad n = 0,1,2,....$$

Hence

$$a_{nk}b_k \to 0, \quad k \to \infty, n = 0,1,2,....$$

Consequently

$$a_{nk}b_kd_k \to 0, \quad k \to \infty, n = 0,1,2,... \tag{2.1}$$

for any bounded sequence $\{d_k\}$. We now claim that

$$\lambda_n = 0, \quad n = 0,1,2,....$$

Suppose not. Then

$$\lambda_m \neq 0, \quad \text{for some positive integer m.}$$

Since

$$\lim_{k\to\infty} a_{mk} = \lambda_m \neq 0,$$

there exists a positive integer n_0 such that

$$|a_{mk}| > \frac{|\lambda_m|}{2}, \quad k \geq n_0. \tag{2.2}$$

Now, define $d = \{d_k\}$, where

$$d_k = \begin{cases} \dfrac{1}{a_{mk}}, & k \geq n_0; \\ 0, & 0 \leq k < n_0. \end{cases}$$

Using (2.2), we have

$$|d_k| < \frac{2}{|\lambda_m|}, \quad k = 0,1,2,\ldots,$$

so that $d = \{d_k\}$ is a bounded sequence. Applying (2.1),

$$\begin{aligned} & a_{mk} b_k \frac{1}{a_{mk}} \to 0, \quad k \to \infty, \\ \text{i.e., } & b_k \to 0, \quad k \to \infty, \\ \text{i.e., } & \{b_k\} \quad \text{is a null sequence,} \end{aligned}$$

which is a contradiction. Thus

$$\begin{aligned} & \lambda_n = 0, \quad n = 0,1,2,\ldots, \\ \text{i.e., } & \lim_{k\to\infty} a_{nk} = 0, \quad n = 0,1,2,\ldots, \\ \text{i.e., } & (1.3) \quad \text{holds,} \end{aligned}$$

completing the proof of the theorem.

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