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***Research Article***

# NEW PROPERTIES OF RANDOM WALK, PARABOLA AND GEOMETRY MODELS

***A. Yurkin**
*City of Puschino, Russia*
**Author for Correspondence: alvl1yurkin@rambler.ru*

## ABSTRACT
A general approach for the recurrent description of 1D, 2D and 3D linear and nonlinear random walks is determined and the concept of a random walk step is introduced. It is shown that the form of random walks and the internal structure of 1D, 2D and 3D linear and nonlinear figures (including empty units) constructed as a result of random walks depend on the selection of the random walk step and initial conditions. The case is described when the final 1D, 2D and 3D figures are the same for different sequences of the values of the steps of the random walk; this case reveals a new regularity in nonlinear random walk, similar to the regularity: "The sum does not change from the permutation of the terms." It is shown that 1D envelopes are parabolas; the possibility of constructing paraboloids of revolution and the connection with the models of the geometries of Euclid, Lobachevsky and Riemann is shown.

***Keywords:*** *Parabola, Random Walk*

## INTRODUCTION
Various non-linear phenomena are common in nature. Some of them were described in work of Askar'yan *et al.* (1989).

In the book (Kolmogorov *et al.*, 1995) linear 1D and 2D random walks along a straight line (in the form of Pascal's triangle) and on a plane (in the form of large squares consisting of small squares) are described respectively.

Nonlinear random walks were first shown graphically in work of (Yurkin, 1995). In two works of (Yurkin, 2019 (1, 2)) various generalizations of Pascal's triangle were described and the connection between our constructions and the ancient Egyptian pyramids was shown.

In the three works of (Yurkin, 2019 (3 - 5)) linear and nonlinear 1D, 2D, 3D random walks and empty units (or 0-units, gaps, empty cubes without numbers, "black holes", "islands", "fractions of numbers") were described as a result of these random walks. In these papers 1D nonlinear random walks were presented in the form of generalized arithmetic Pascal triangles. 2D nonlinear random walks as in (Kolmogorov *et al.*,1995) were represented in the form of large squares consisting of small squares. 3D linear and nonlinear random walks have been represented as large octahedron composed of small cubes (Weisstein, no date)). The Yurkin octahedron (Yurkin, 2020) is constructed by 3D linear and nonlinear random walks in space.

In this paper we show on visual geometric models new types and new features of linear and nonlinear random walks in 1D, 2D and 3D spaces (without additional physical concepts such as gravity, speed of light, etc.). In this work we investigate the properties of octahedron composed of small cubes since the octahedron and the cube are dual Platonic figures (Klein, 1926). At the end of the work we draw analogies of our 1D models of random walk with parabolas and paraboloids and models of the geometries of Euclid, Lobachevsky and Riemann.

Figure 1 shows examples of octahedron built in three mutually perpendicular coordinates X, Y and Z.

In Figure 1a, a segment of any length $a$ was laid out in six directions from the origin along the coordinate axes (in the forward-backward, right-left, up-down directions) and then the ends of the segments were connected to each other; we will call this approach the "continuous line-posting approach".

In Figure 1b the octahedron was constructed by sequential random walk (forward-backward, right-left, up-down) in a cubic coordinate grid with a step of length $b$ consisting of small cubes with a height (length) $b$; we will call this approach the "discrete random walk approach".

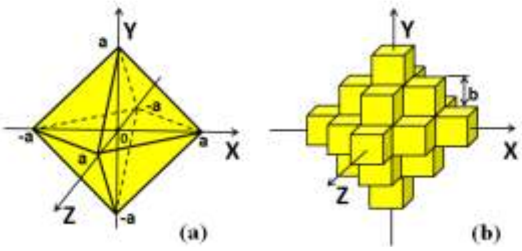


**Figure 1: Constructing an octahedron using 3D coordinates "continuous line segment approach" segment of length $a$ (a); construction of an octahedron using a 3D random "discrete random walk approach" in a cubic lattice with a step of length $b$ (b).**

With the help of the "continuous line segment approach" octahedrons of various sizes can be constructed; such octahedrons do not contain an internal structure.

Using the "discrete random walk" approach it is also possible to construct octahedrons of different sizes but such octahedrons can contain different internal structures.

Figure 2 shows the images of sequences of octahedrons of different sizes built using different approaches: "continuous line segment approach" (a, c) and "discrete random walk approach" (b, d). The linear case with different approaches is shown on (a, b), and the nonlinear case on (c, d).

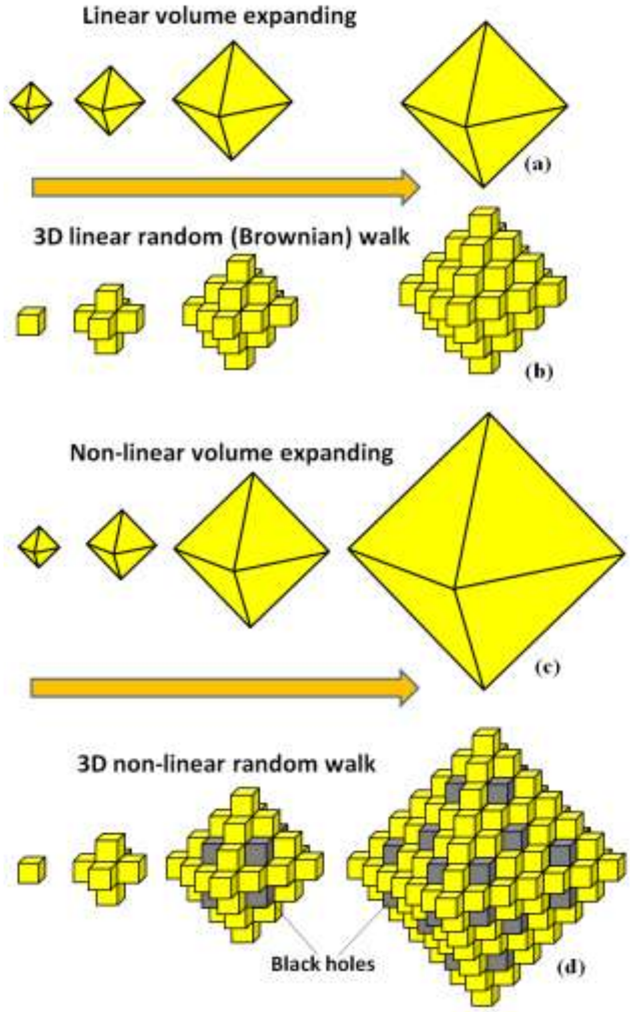


**Figure 2: Sequential process of increasing octahedron size: linear (a) and (b), nonlinear (c) and (d). In the linear case the octahedron is completely filled with small cubes that make up the octahedron (b). In the nonlinear case (with uniform acceleration) inhomogeneities appear in the internal structure: unfilled cubic cells - "Black holes" are marked in gray (d). In what follows for greater rigor we will use the term "Empty units" or "0-units" instead of the term "Black holes".**

In this work in contrast to work (Yurkin, 2019 (4)) we consider only one, the first fraction (1-fraction) of a random walk, the second fraction (2-fraction) in this work we consider zero ("0-units") to simplify the presentation but show in figures as squares without numbers .

## 2. LINEAR RANDOM WALK

### *2.1. Modeling of 1D linear random walk along a straight line*

Figure 3 shows a 1D linear random walk along straight lines arranged horizontally (left-right random walk). Such a random walk is given in the book (Kolmogorov *et al.*, 1995). This is an image of Pascal's triangle.

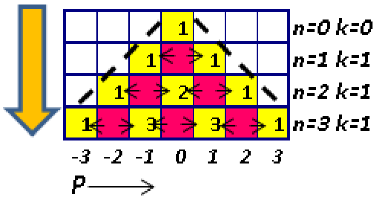


**Figure 3: 1D linear random walk along straight horizontal lines (Pascal's triangle). First three iterations. The envelope lines are shown as dashed straight oblique lines . - The units of 1-fraction, - the units of 2-fraction.**

The sequence of numbers in lines for the 1D case in this example is denoted by $n = 0, 1, 2, \ldots$ .

The step size of the random walk is denoted by $k$.

The numbers characterizing the triangle are denoted by $p$.

$$p = 0, \pm 1, \pm 2, \ldots \tag{1}$$

Denote a number located in the $n$ – line as $\binom{n}{p}$.

The numbers in the line of the arithmetic triangle are binomial coefficients $\binom{n}{p}$ can be found using the recursive 1D expression:

$$\binom{n}{p} = \binom{n}{p+k} + \binom{n}{p-k}, \tag{2}$$

where $k = 1$.

Therefore, formula (2) will take the form:

$$\binom{n}{p} = \binom{n}{p+1} + \binom{n}{p-1}. \tag{3}$$

Then we specify the initial conditions ($n = 0$):

$$\binom{0}{p} = 1 \text{ for } p = 0, \text{ and } \binom{0}{p} = 0 \tag{4}$$

for the other values $p$.

These initial conditions are shown in Figure 3. They provide the appearance of a group of numbers shown in after performing recurrent calculations. The arithmetic triangle shown in Figure 3 contains the unit of 1-fraction and only regular spaces (gaps) which are units of 2-fraction, so we assume that it is completely filled.

Regular gaps can be filled by the 2-fraction of a random walk as shown in Yurkin (2019 (4)).

### *2.2. Modeling a 2D linear random walk on a plane*

Fig. 4 shows a 2D linear random on a plane (random walk in the left-right and forward-backward directions). This image is given in the book (Kolmogorov *et al.*,1995) as an interpretation of the walk of a Brownian particle on a two-dimensional lattice and the number of different paths for different periods of time is calculated.

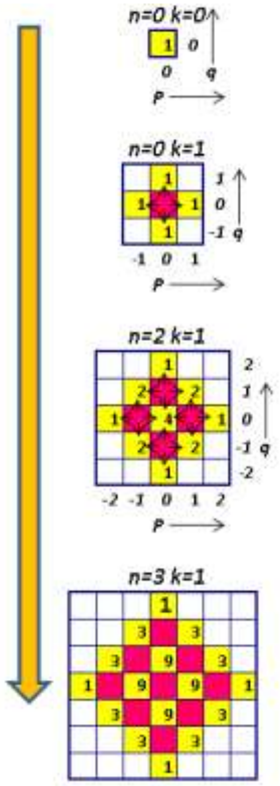


**Figure 4: 2D linear random walk on the plane. First three iterations. 2D Brownian motion model (Kolmogorov *et al.*, 1995). - The units of 1-fraction, - the units of 2-fraction.**

The sequence of numbers of squares for the 2D case in this example is denoted by $n: = 0, 1, 2, \ldots$ The numbers characterizing the squares (the numbers characterizing the position of the small squares of which the big square is composed) are denoted by $p$ and $q$:

$$p = 0, \pm 1, \pm 2, \ldots . \quad q = 0, \pm 1, \pm 2, \ldots . \tag{5}$$

Denote a number located in the $n$ – square as $\begin{pmatrix} n \\ p \\ q \end{pmatrix}$.

The numbers in the linear arithmetic square are binomial coefficients $\begin{pmatrix} n \\ p \\ q \end{pmatrix}$ can be found using the recursive 2D expression:

$$\begin{pmatrix} n \\ p \\ q \end{pmatrix} = \begin{pmatrix} n-1 \\ p \\ q-k \end{pmatrix} + \begin{pmatrix} n-1 \\ p-k \\ q \end{pmatrix} + \begin{pmatrix} n-1 \\ p+k \\ q \end{pmatrix} + + \begin{pmatrix} n-1 \\ p \\ q+k \end{pmatrix}, \tag{6}$$

where $k = 1$.

Therefore formula (6) will take the form:

$$\begin{pmatrix} n \\ p \\ q \end{pmatrix} = \begin{pmatrix} n-1 \\ p \\ q-1 \end{pmatrix} + \begin{pmatrix} n-1 \\ p-1 \\ q \end{pmatrix} + \begin{pmatrix} n-1 \\ p+1 \\ q \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q+1 \end{pmatrix}, \tag{7}$$

Then we specify the numbers of the zero square ($n = 0$) or in other words the initial conditions:

$$\begin{pmatrix} 0 \\ p \\ q \end{pmatrix} = 1 \text{ for } p = 0, q = 0 \text{ and } \begin{pmatrix} 0 \\ p \\ q \end{pmatrix} = 0 \qquad (8)$$

for the other values $p, q$.

These initial conditions are shown in Figure 4. They provide the appearance of a group of numbers after performing recurrent calculations.

This construction process is presented in more detail in (Yurkin, 2019 (3, 4)).

### *2.3. Modeling a 3D linear random walk in space*

Figure 5 shows a 3D linear random in space (random walk in the left-right direction, forward-backward, and up-down). This image can be considered a model of a random walk of a Brownian particle on a three-dimensional lattice and the number of different paths for different time intervals is calculated. The correctness of our constructions of the 3D figure for all iterations can be checked by layer-by-layer consideration of the 2D figures that make up the 3D figure, as was done in (Yurkin, 2019 (3, 4)).

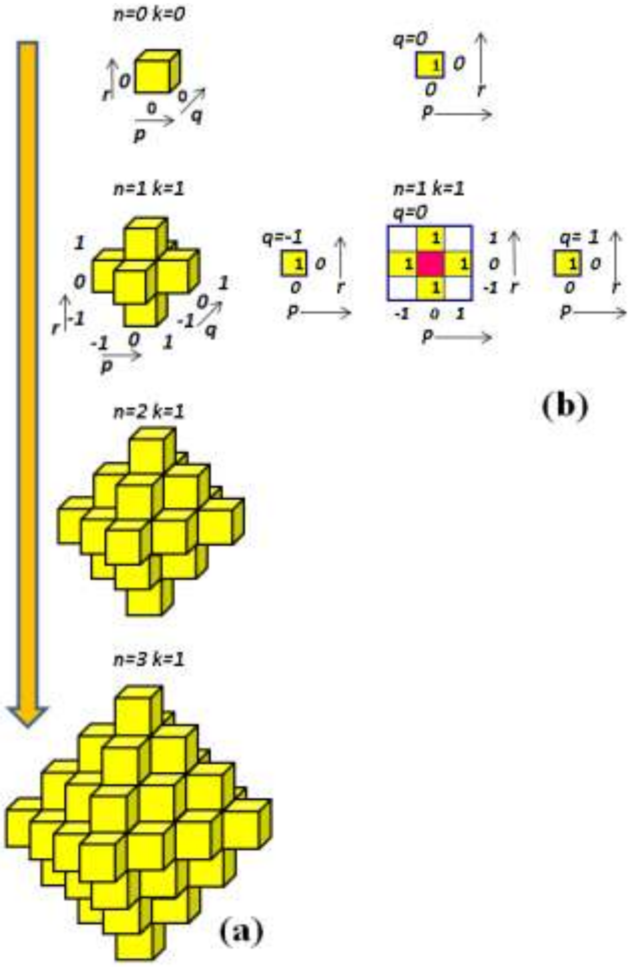


**Figure 5: 3D linear random walk. 3D model of Brownian motion. First three iterations (a). Layered (2D) image of figures at zero and first iterations (b); we do not present the layered image of 2D figures for the rest of the iterations here because of space savings (see Yurkin, 2019 (3, 4)). ; - The units of 1-fraction, - the unit of 2-fraction.**

The octahedrons depicted in Figures 5 do not contain empty units (0-units) in adjacent cells.

The sequence of numbers of octahedrons for the 3D case in this example is denoted by $n := 0, 1, 2, \ldots$ .

The numbers characterizing the octahedron (the numbers characterizing the position of the cubes of which the octahedron is composed) are denoted $p$, $q$ and $r$:

$$p = 0, \pm 1, \pm 2, \dots . \; q = 0, \pm 1, \pm 2, \dots . \; r = 0, \pm 1, \pm 2, \dots . \qquad (9)$$

Denote a number located in the $n$ – octahedron as $\begin{pmatrix} n \\ p \\ q \\ r \end{pmatrix}$.

The numbers in the linear arithmetic octahedron are linear binomial coefficients $\begin{pmatrix} n \\ p \\ q \\ r \end{pmatrix}$ can be found using the recursive 3D expression:

$$\begin{pmatrix} n \\ p \\ q \\ r \end{pmatrix} = \begin{pmatrix} n-1 \\ p \\ q-k \\ r \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q \\ r+k \end{pmatrix} + \begin{pmatrix} n-1 \\ p-k \\ q \\ r \end{pmatrix} + \begin{pmatrix} n-1 \\ p+k \\ q \\ r \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q \\ r-k \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q+k \\ r \end{pmatrix}, \qquad (10)$$

where $k = 1$

Therefore formula (10) will take the form:

$$\begin{pmatrix} n \\ p \\ q \\ r \end{pmatrix} = \begin{pmatrix} n-1 \\ p \\ q-1 \\ r \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q \\ r+1 \end{pmatrix} + \begin{pmatrix} n-1 \\ p-1 \\ q \\ r \end{pmatrix} + \begin{pmatrix} n-1 \\ p+1 \\ q \\ r \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q \\ r-1 \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q+1 \\ r \end{pmatrix}, \qquad (11)$$

Then we specify the zero octahedron ($n = 0$) or in other words the initial conditions:

$$\begin{pmatrix} 0 \\ p \\ q \\ r \end{pmatrix} = 1 \text{ for } p = 0, q = 0, r = 0 \text{ and } \begin{pmatrix} 0 \\ p \\ q \\ r \end{pmatrix} = 0 \qquad (12)$$

for the other values $p, q$ and $r$. These initial conditions are shown in Figure 5. They provide the appearance of a group of numbers after performing recurrent calculations.

This construction process is presented in more detail in (Yurkin, 2019 (3, 4)).

## 3. NONLINEAR RANDOM WALK

### *3.1. Modeling of 1D nonlinear cases of random walk along a straight line*

Fig. 6 shows six cases of 1D nonlinear random walk along straight lines arranged horizontally (random walk in the left-right direction). This is an interpretation of a nonlinear walk of a particle with uniform acceleration (deceleration) along a line. We calculate the number of different paths over different periods of time. This can be interpreted as a random walk along an acceleratingly lengthening or shortening 1D line.

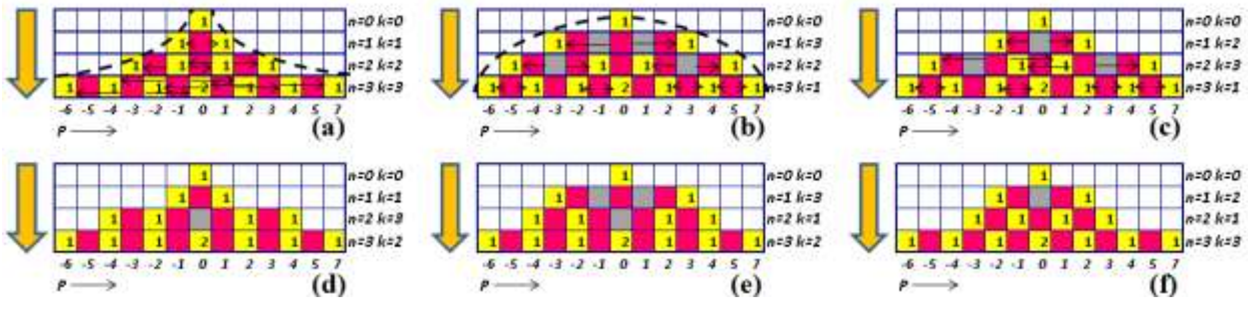


**Figure 6: 1D non-linear random walk along straight horizontal lines (non-linear Pascal triangles). First three iterations. The envelope curves are shown as dashed branches of opposite signs (a, b). ☐ - The units of 1-fraction, ☐ - the 0-units of 1-fraction, ☐ - the units of 2-fraction.**

The sequence of numbers in lines for the 1D cases in these examples is denoted by $n := 0, 1, 2, \dots$.

The numbers characterizing the triangle are denoted by $p$ we take from Eq. (1).

Denote a number located in the $n$ – line as $\binom{n}{p}$.

The numbers in the line of the arithmetic triangle are non-linear binomial coefficients $\binom{n}{p}$ can be found using the recursive 1D expression from Eq. (2).
Where $k =$

| $n =$ | 0 | 1 | 2 | $n_{max} = 3$ |
|---|---|---|---|---|
| a) $k =$ | 0 | 1 | 2 | 3 |
| b) $k =$ | 0 | 3 | 2 | 1 |
| c) $k =$ | 0 | 2 | 3 | 1 |
| d) $k =$ | 0 | 1 | 3 | 2 |
| e) $k =$ | 0 | 3 | 1 | 2 |
| f) $k =$ | 0 | 2 | 1 | 3 |

**Table 1: The step of the random walk $k$ depending on the line number $n$.**

The form of Figures 6 (a - f) depend on the step of the random walk $k = 1, 2, 3$ (all numbers are different) and is determined by permutations of the sequences of numbers 1, 2, 3. The size of the step $k$ of the random walk varies with $n$. $n_{max} = 3$. The number of options for nonlinear random walk (a - f) is equal to the number of permutations (Kolmogorov *et al.*, 1995), i e $n_{max}!$. In our case, $n_{max}! = 3! = 6$.We use formula (2) for our six cases (a - f), where $k$ is taken from Table 1.

Then we specify the initial conditions ($n = 0$) from Eq. (4)

These initial conditions are shown in Figure 6. They provide the appearance of a group of numbers shown in after performing recurrent calculations. The arithmetic triangles shown in Figure 6 are characterized by different sequences $k$ however the last line in Figure 6 is the same for all types of random walks along the straight line (a - f).

The most interesting in our opinion are the nonlinear cases shown in Figure 6 (a, b); their envelope curves are parabolas taken with the opposite sign relative to each other. Straight line envelopes are shown in Figure 3.

An interesting feature of the nonlinear case shown in Figure 6 (a) is that it like the linear case shown in Figure 3 does not contain irregular gaps of more than one cell in the horizontal direction for all iterations. That is there are no 0-units in Figure 6 (a) case.

### *3.2. Modeling of 2D nonlinear random walk in a plane*

Figure 7 shows a 2D nonlinear random walk on a plane (random walk in the left-right and forward-backward directions). This is an interpretation of the nonlinear walk of a particle with uniform acceleration (or deceleration) over a two-dimensional lattice. We calculate the number of different paths over different periods of time. This can be interpreted as a random walk on a rapidly expanding or reducing 2D plane.

The sequence of numbers of squares for the 2D case in this example is denoted by $n = 0, 1, 2, \dots$. The numbers characterizing the squares (the numbers characterizing the position of the small squares of which the big square is composed) are denoted $p$ and $q$ we take from Eq. 5.

Denote a number located in the $n$ – squares as $\begin{pmatrix} n \\ p \\ q \end{pmatrix}$.

The numbers in the linear arithmetic square are non-linear binomial coefficients $\begin{pmatrix} n \\ p \\ q \end{pmatrix}$ can be found using the recursive 2D expression we take from Eq. (6),
where $k$ we take from Table 1 (a - c).

Then we specify the numbers of the zero square ($n = 0$) or in other words the initial conditions we take from Eq. (8).

These initial conditions are shown in Figure 7. They provide the appearance of a group of numbers after performing recurrent calculations.

The process of constructing a 2D random walk (a) is presented in more detail in Yurkin (2019 (3, 4)).

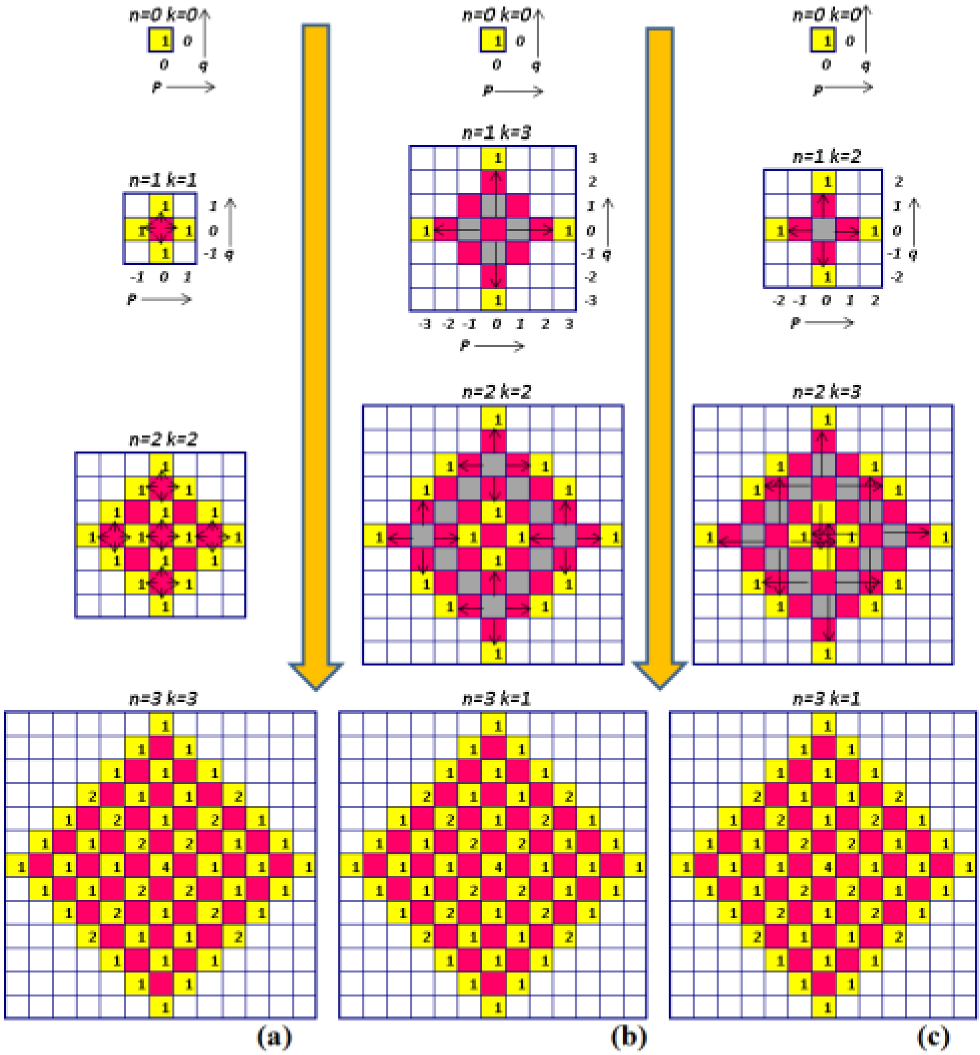


**Figure 7: 2D non-linear random walk in space. First three iterations. [yellow square] - The units of 1-fraction, [grey square] - the 0-units of 1-fraction, [red square] - the units of 2-fraction.**

The arithmetic squares shown in Figure 7 are characterized by different $k$ sequences however the last squares in Figure 7 are the same for random walks on the 2D plane (a - c) similarly for the 1D case (a - c) from Figure 6.

An interesting feature of the nonlinear case shown in Figure 7 (a) is that it like the linear case shown in Figure 4 does not contain irregular gaps of more than one cell in the horizontal or vertical directions for all iterations. That is there are no 0-units in this case.

### *3.3. Modeling of 3D nonlinear random walk in space*

Fig. 8 shows 3D non-linear random in space (random walk in the left-right direction, forward-backward and up-down). This image can be considered a model of the random walk of a particle on a three-dimensional lattice. We calculate the number of different paths for different time intervals. This can be interpreted as a random walk in an acceleratingly expanding or reducing 3D space.

The correctness of our constructions of the 3D figure for all iterations can be checked by layer-by-layer consideration of the 2D figures that make up the 3D figure as was done in (Yurkin, 2019 (3, 4)).

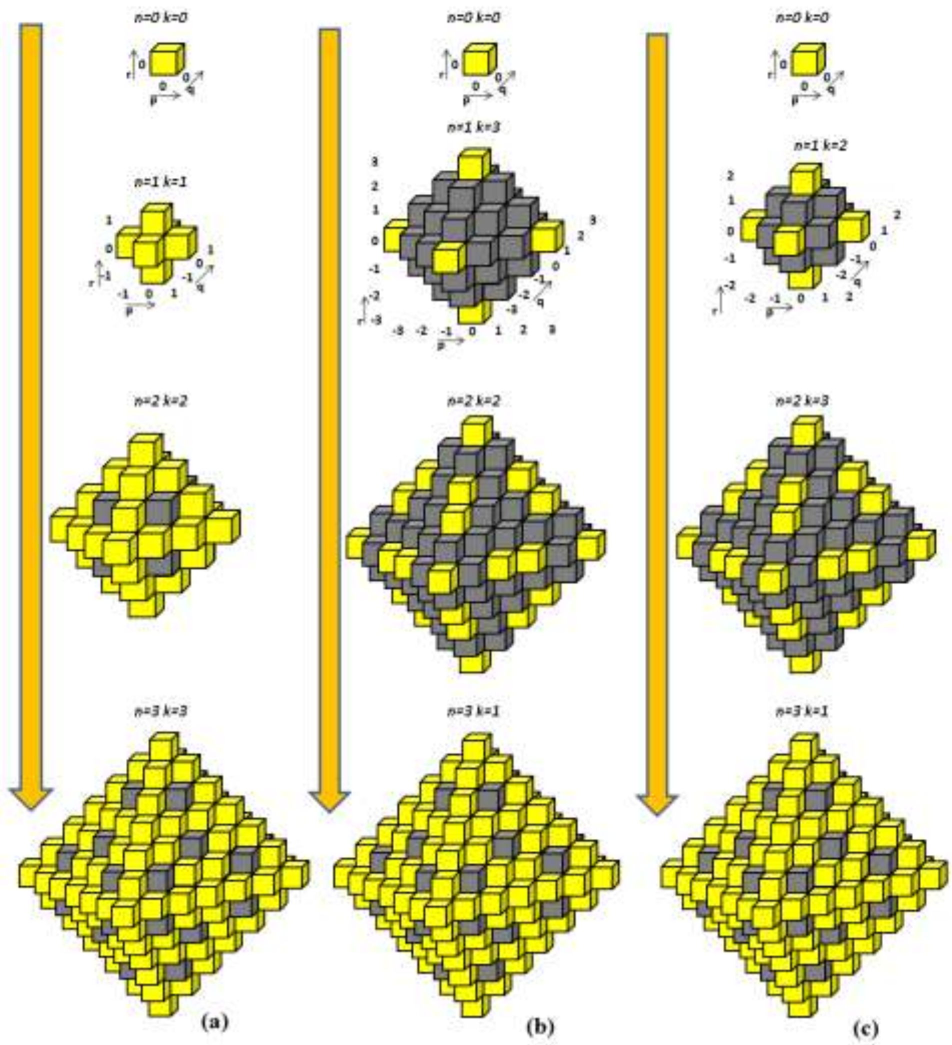


**Figure 8: Nonlinear random walk in 3D space. First three iterations. Layered detailed (2D) images (a) of figures are presented in works (Yurkin, 2019 (3, 4)). - The units of 1-fraction, -the 0-units of 1-fraction.**

The octahedrons depicted in Figures 8 contain 0-units in adjacent cells.

The sequence of numbers of octahedrons for the 3D case in this example is denoted by $n = 0, 1, 2, \ldots$ .

The numbers characterizing the octahedron (the numbers characterizing the position of the cubes of which the octahedron is composed) are denoted $p$, $q$ and $r$ we take from Eq. (9).

Denote a number located in the $n$ – octahedron as $\begin{pmatrix} n \\ p \\ q \\ r \end{pmatrix}$.

The numbers in the non-linear arithmetic octahedron are non-linear binomial coefficients $\begin{pmatrix} n \\ p \\ q \\ r \end{pmatrix}$ can be found using the recursive 3D expression we take from Eq. (10) where $k$ we take from Table 1 (a - c).

Then we specify the zero octahedron ($n = 0$) or in other words the initial conditions from Eq. (12).

These initial conditions are shown in Figure 8. They provide the appearance of a group of numbers after performing recurrent calculations.

The process of constructing a 3D random walk (a) is presented in more detail in Yurkin (2019 (3, 4)).

The arithmetic octahedrons shown in Figure 8 are characterized by different $k$ sequences however the last octahedrons in Figure 8 are the same for all kinds of random walks in 3D space (a - c) similarly for 1D and 2D cases (a - c) from Figure 6 and Figure 7. That is 0-units are missing in this case.

### *3.4. Some other examples of modeling 3D random walk in space*

Figure 9 shows other examples of 3D random walk patterns in space (left-right, forward-backward, and up-down random walk). They differ from the examples given in the previous section 3.3 in different values of the steps of the random walk and initial conditions. The correctness of our constructions of a 3D figure for all iterations can be checked by layer-by-layer consideration of the 2D figures that make up the 3D figure as was done in (Yurkin, 2019 (3, 4)).

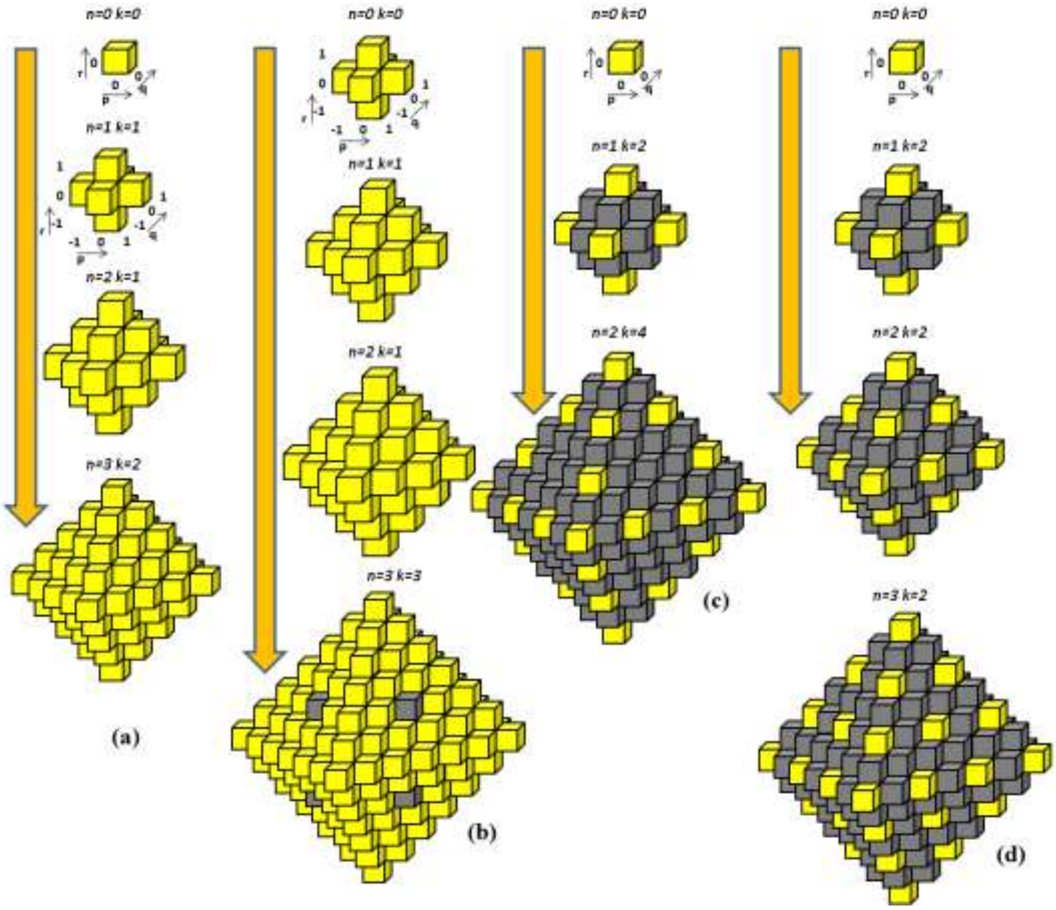


**Figure 9: Examples of 3D random walk in space. (a - c) non-linear random walk cases, (d) linear case. - The units of 1-fraction, - the 0-units of 1-fraction.**

The sequence of numbers of octahedrons for the 3D case in this example is denoted by $n$ as in paragraph 3.3. The numbers are denoted by $p$, $q$ and $r$ $n$ as in paragraph 3.3 too, from Eq. (9).

The numbers in the arithmetic octahedron are non-linear binomial coefficients $\begin{pmatrix} n \\ p \\ q \\ r \end{pmatrix}$ can be found using the same recursive 3D expression from Eq. (10).

Where $k =$

| $n =$ | 0 | 1 | 2 | 3 | 4 | 5 | ... |
|---|---|---|---|---|---|---|---|
| a) $k =$ | 0 | 1 | 1 | 2 | 3 | 4 | ... |
| b) $k =$ | 0 | 1 | 1 | 3 | 4 | 5 | ... |
| c) $k =$ | 0 | 2 | 4 | 6 | 8 | 10 | ... |
| d) $k =$ | 0 | 2 | 2 | 2 | 2 | 2 | ... |

**Table 2: The step of the random walk $k$ is depending on the line number $n$.**

Then we specify for examples (a, c) and (d) the initial conditions from Eq. (12).

Then we specify for example (b) the initial conditions:

$$\begin{pmatrix}0\\p\\q\\r\end{pmatrix} = 1 \text{ for } p=0, q=0, r=1; \quad \begin{pmatrix}0\\p\\q\\r\end{pmatrix} = 1 \text{ for } p=0, q=0, r=-1;$$

$$\begin{pmatrix}0\\p\\q\\r\end{pmatrix} = 1 \text{ for } p=0, q=1, r=0; \quad \begin{pmatrix}0\\p\\q\\r\end{pmatrix} = 1 \text{ for } p=0, q=-1, r=0;$$

$$\begin{pmatrix}0\\p\\q\\r\end{pmatrix} = 1 \text{ for } p=1, q=0, r=0; \quad \begin{pmatrix}0\\p\\q\\r\end{pmatrix} = 1 \text{ for } p=-1, q=0, r=0; \text{ and } \begin{pmatrix}0\\p\\q\\r\end{pmatrix} = 0 \qquad (13)$$

for the other values $p, q$ and $r$.

These initial conditions are shown in Figure 9. They provide the appearance of a group of numbers after performing recurrent calculations. Example (a) does not contain 0-units; example (b) contains less 0-units than the example in Figure 8 (b, c); example (c) contains more 0-units than example in Figure 8 (a). Examples (a - c) show a nonlinear random walk with different step $k$; and example (d) show a linear random walk (similar to that shown in Figure 5), but with a constant double step $k = 2$.

In (Yurkin, 2019 (4)). we also defined several initial conditions.

The arithmetic octahedrons shown in Figure 9 are characterized by different sequences of $k$ steps and / or different initial conditions. Similar examples can be further cited.

**4. RANDOM WALK, GEOMETRY MODELS, PARABOLAS AND PARABOLOIDS**

Analogies can be drawn between the above-described random walks and the construction of parabolas. Figure 10 (a, b, c) shows separately the envelopes in the form of parabolas (bold curves (a, b)) taken from Figure 6 (a) and Figure 6 (b) and envelopes in the form of straight lines (bold lines (c)) taken from Figure 3 respectively. In Figure 10 (a) and Figure 10 (b) the branches of the parabolas have mutually opposite signs and are directed in opposite directions. The dotted lines show the extensions of the branches of the parabolas (a, b). The arrows schematically show the production of surfaces of revolution that is paraboloids (Mathematical encyclopedic dictionary, 1988) when the parabolas rotate around the axis.

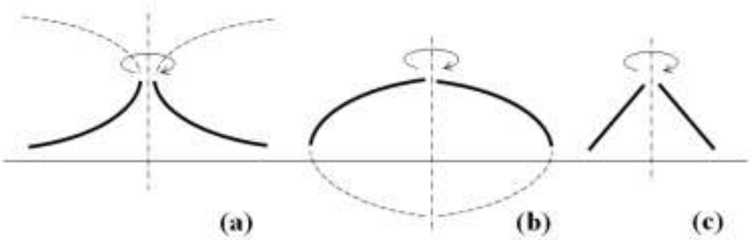


**Figure 10: Envelopes (in the form of bold solid lines) images of nonlinear and linear 1D random walks along a straight line. In Figure 6 (a), Figure 6 (b) and Figure 3 above; these envelopes are shown with dotted lines but the image scale is the same.**

Figure 11 shows the same shapes as in Figure 10 but for clarity the horizontal scale is reduced. On (a) the branches of the parabola during rotation form a figure close to an imaginary sphere (hyperboloid of revolution) (Mathematical encyclopedic dictionary, 1988) on the surface of which the hyperbolic geometry is fulfilled and the geometry of Lobachevsky is modeled. On (b) the branches of the parabola during rotation form a figure close to an ellipsoid or a sphere on the surface of which spherical geometry is performed and the Riemann geometry is modeled. On (c) the branches of straight lines during rotation form a figure close to a cone on the side surface (plane) of which the Euclid geometry is modeled.

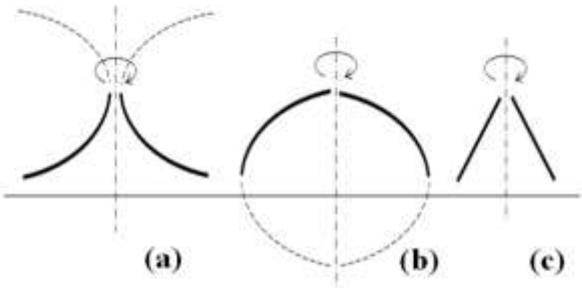


**Figure 11: A repetition of Figure 10 but for clarity the scale is reduced along the horizontal.**

We drew an analogy between the 1D random walks described in this work in Figure 6a, Figure 6 and, Figure 3 and the models of the geometries of Lobachevsky, Riemann and Euclid, respectively. We assume that such an analogy exists between 2D random walk in Figure 7a, Figure 7b and Figure 4 and the models of the geometries of Lobachevsky, Riemann and Euclid, respectively; and between 3D random walk in Figure 8a, Figure 8b and Figure 5 and models of Lobachevsky, Riemann and Euclid geometries, respectively.

The geometries of Euclid, Lobachevsky and Riemann are widely known. Ordinary 2D Euclidean geometry can be considered linear it is depicted (modeled) on an ordinary plane. The Lobachevsky and Riemann geometries (hyperbolic and spherical, respectively) can be considered nonlinear. 2D models of Lobachevsky geometry are modeled in Euclidean space on a saddle surface or on the surface of a horn or within a circle or on a hyperboloid surface. A 2D model of Riemann geometry in Euclidean space is modeled on a spherical surface (Mathematical encyclopedic dictionary, 1988).

## CONCLUSION

Based on the above the following conclusions can be drawn.

- Two approaches for constructing octahedrons are outlined: 1)"continuous line-posting approach" and "discrete random walk approach". Octahedrons of various sizes can be constructed using the "continuous line-segment approach" but such octahedrons do not contain internal structure. 2) "discrete random walk approach" is stated. Using the "discrete random walk" approach it is also possible to construct octahedrons of different sizes but such octahedrons can contain different internal structures.
- Defined a general approach for recurrent description of 1D, 2D and 3D linear and nonlinear random walks and introduced the concept of the step $k$ of a random walk.
- The nonlinear random walk depicted in Figure 8a can be interpreted as a random walk in an accelerating 3D space. The nonlinear random walk depicted in Figure 8b can be interpreted as a random walk in an acceleratingly decreasing 3D space.
- This paper provides various examples of 1D, 2D and 3D linear and nonlinear random walks and shows that 0-units of 1-fraction during a random walk can occur in both nonlinear and linear cases. We have shown that the shape of random walks and the internal structure of 1D, 2D and 3D linear and nonlinear constructed figures (as a result of random walks) depend on the selection of the random walk step and initial conditions.
- In this work, we showed that if the number of nonlinear random walks $n$ is finite and equal to $n_{max}$, and the step $k$ of the random walk is expressed by the numbers of a natural series in any sequence for example $k = 1, 2, ...\ n_{max} = 3$, then the number variants for a nonlinear random walk is equal to the number of permutations $n_{max}!$ for example $n_{max}! = 3! = 6$. In this case the final 1D, 2D and 3D figures are the same for different sequences of $k$ values. This pattern found in nonlinear random walk is similar to the pattern: "The sum does not change from the change of places of the terms."

- In this work new visual analogies between our 1D models of random walk with parabolas and paraboloids are given a visual connection with the models of the geometries of Euclid, Lobachevsky and Riemann is shown. An assumption is made about the visual connection of our 2D and 3D models of random walk with the models of the geometries of Euclid, Lobachevsky and Riemann.
- In this work we investigated new properties of only 1-fraction (as in Kolmogorov *et al.,* 1995) of a random walk. It is also possible to extend our research further to 2-fraction and sub-fractions of 1-fractions and sub-fractions of 2-fraction as in (Yurkin, 2019 (4)).
- We hope that our models can be useful in cosmological (Yurkin, 2019 (4)), (Tozzi *et al.*, 2019 (1)), (Whiting, 2004) research, as well as in biological research (Yurkin *et al.*, 2017), (Tozzi *et al.*, 2019 (2)), (Yurkin *et al.*, 2020), and in other fields of science and technology.

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