International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jpms.htm
2019 Vol. 9 (1) January-March, pp. 20-36/Parihar and Mathur
Research Article

TO CONTEMPLATE AND INVESTIGATE DISTINCT METHODS OF LINEAR FRACTIONAL PROGRAMMING PROBLEMS

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ABSTRACT
Various methods to solve linear fractional programming problem is discussed in this paper. LFP problem with only two unknown variables can be solved graphically subject to constraints. The equations and the above method are elaborated in detail via mathematical equations and inequalities. In order to optimize some absolute criteria, Simplex Method comes in picture. For example Profit gained by some company, number of full time employees etc fall under this category. The simplex Method includes a function Q(x) which is maximized followed by constraints and iterations discussed painstakingly. In the end the sum of linear plus LFP problem is investigated theoretically.

Keywords: *fractional programming, feasible and infeasible, Optimal Vertex, global maximum*

1.1 SOLUTION OF LFP

There are many well known methods to solve linear fractional programming problems. Some of them are as follows:

(i) Graphical Method
(ii) Simplex Method
(iii) Charnes & Cooper's Transformation
(iv) Dinkelbach Algorithm
(v) Wolfe's Parametric Method
(vi) Kanti Swarup Method
(vii) Ratio Algorithm, etc.

We now go on to discuss how any LFP problem can be solved by these methods with the help of numerical example.

1.1(a) The Graphical Method

We discuss how any LFP problem with only two variables can be solved graphically. Let us consider the following LFP problem with two unknown variables:

Optimize $$Q(x) = \frac{N(x)}{D(x)} = \frac{n_1x_1 + n_2x_2 + n_0}{d_1x_1 + d_2x_2 + d_0} \quad \ldots(1.1.1)$$

Subject to, $$a_{i1}x_1 + a_{i2}x_2 \le b_i,\ i = 1, 2, \ldots, m \quad \ldots(1.1.2)$$

and . $$x_1 \ge 0,\ x_2 \ge 0 \quad \ldots(1.1.3)$$

(1) The Single Optimal Vertex

Let us suppose that constraints (1.1.2) and (1.1.3) define feasible set S shown by shading in figure -1.1.

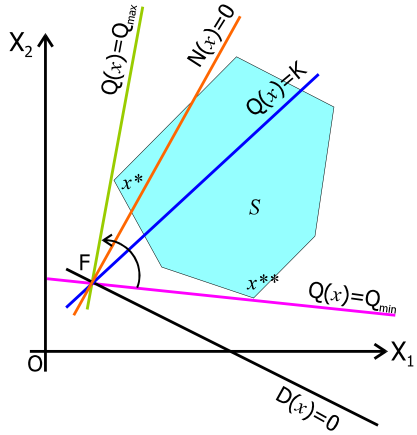

Figure 1.1: Two-variable LFP problem-Single Optimal Vertex.

Let $Q(x) = K$, where K is an arbitrary constant.

or $$\frac{n_1x_1 + n_2x_2 + n_\circ}{d_1x_1 + d_2x_2 + d_\circ} = K \qquad \text{[from (1.1.1)]}$$

or, $$(n_1 - Kd_1)x_1 + (n_2 - Kd_2)x_2 + (n_0 - Kd_0) = 0$$

For any real value of K, the above equation represents all the points on a straight line in the two dimensional plane. If this so-called level-line (or isoline) intersects the set of feasible solutions S, the points of intersection are the feasible solutions that give the value K to the objective function $Q(x)$. Changing the value of K translates the entire line to another line that intersects the previous line in focus point (point F in figure-1.1) with coordinates defined as solution of system

$$\left.\begin{aligned} n_1x_1 + n_2x_2 &= -n_0 \\ d_1x_1 + d_2x_2 &= -d_0 \end{aligned}\right\} \qquad \ldots(1.1.4)$$

In other words, in the focus point F, straight lines with equations and intersect one another.

If lines $N(x) = 0$ and $D(x) = 0$ are not parallel with one another, then the determinant of system (1.1.4) is not equal to zero and the system has a unique solution (coordinates of focus point F). In the other case, if lines $N(x) = 0$ and $D(x) = 0$ are parallel with one another, the determinant of system (1.1.4) is equal to zero and the system has no solution. It means that there is no focus point and all level-lines are also parallel with one another.

Now we return to the case, when level-lines are not parallel with one another. Taking an arbitrary value of K, we draw a line $Q(x) = K$ shown in figure-1.1.

Let us rewrite equality $Q(x) = \dfrac{n_1x_1 + n_2x_2 + n_\circ}{d_1x_1 + d_2x_2 + d_\circ} = K$ as follows:

$$x_2 = -\frac{n_1 - Kd_1}{n_2 - Kd_2} x_1 - \frac{n_0 - Kd_0}{n_2 - Kd_2}$$

If 'k' be the slope of the above line (level line), then

$$k = \frac{n_1 - Kd_1}{n_2 - Kd_2}$$

Therefore, in such a case, the slope of level-line $Q(x) = K$ depends on value K of objective function $Q(x)$, and is a monotonic function on K because

$$\frac{dk}{dK} = \frac{d_1 n_2 - d_2 n_1}{(n_2 - Kd_2)^2}$$

This implies that the sign of $\frac{dk}{dK}$ does not depend on the value of K, so we can write

$$\text{sign}\left\{\frac{dk}{dK}\right\} = \text{sign}\{d_1 n_2 - d_2 n_1\} = \text{const.}$$

It means that when rotating level-line around its focus point F in positive direction (i.e. counter clockwise), the value of objective function $Q(x)$ increases or decreases depending on the sign of expression $(d_1 n_2 - d_2 n_1)$.

Obviously, figure-1.1 represents the case when rotating level-line in positive direction leads to growth of value $Q(x)$. When rotating level-line around its focus point F the line $Q(x) = K$ intersects feasible set S in two vertices (extreme points) x^* and x^{**}.

In the point x^*, objective function $Q(x)$ takes its maximal value over set S and in the point x^{**}, it takes its minimal value.

(2) Multiple Optimal Solutions

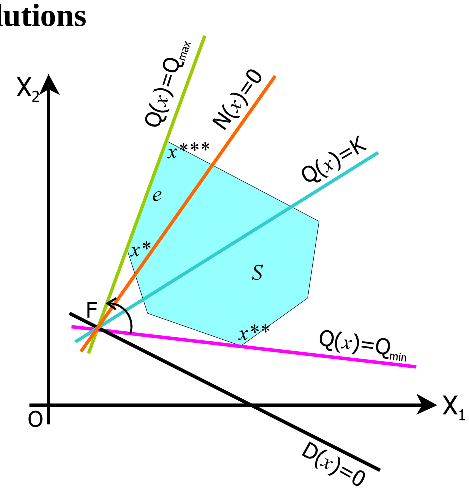

Figure 1.2: Two-variable LFP problem-Multiple Optimal Solutions.

Multiple Optimal Solutions may occur that when rotating level-line on its focus point F the level-line $Q(x) = K$ captures some edge of set S (edge e in figure-1.2). In this case the problem has an infinite number of optimal solutions (all points x of edge e) that may be represented as a linear combination of two vertex points x^* and x^{***}:

$$x = \lambda x^* + (1-\lambda)x^{***}, \quad 0 \le \lambda \le 1$$

(3) Mixed Cases

If feasible set S is unbounded and an appropriate unbounded edge concurs with extreme level-line (figure-1.3), then the problem has an infinite number of optimal solutions, one of them in vertex x^* and others are non-vertex points of unbounded edge. We should note here that among these non vertex points, there is one infinite point too. This is why we say in this case that the problem has 'mixed' solutions, i.e. finite optimal solution(s) and asymptotic one(s).

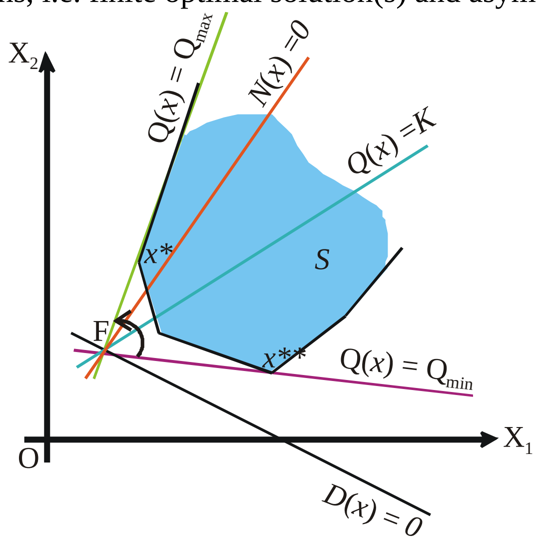

Figure 1.3: Two-variable LFP problem-Mixed Cases.

(4) Asymptotic Case

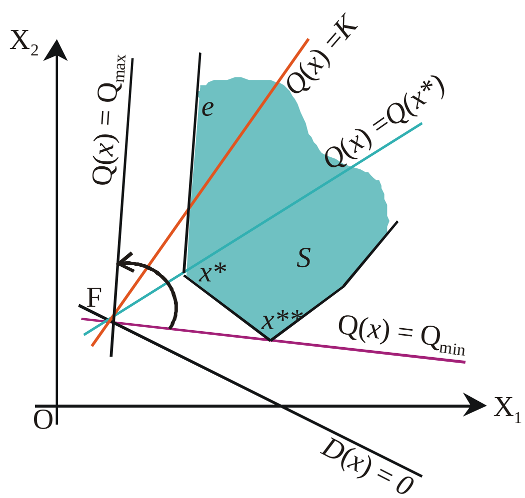

Figure-1.4: Two-variable LFP problem- Asymptotic case

Suppose the constraints (1.1.2) and (1.1.3) define an unbounded feasible set S shown in figure-1.4. It may occur that when rotating level-line, after an extreme vertex (vertex x^* in figure-1.4) we can rotate the level-line a bit more in the same direction because the intersection between level-line and feasible set S is still not empty (line $Q(x) = K$ in figure-1.4). In this case we can rotate level-line until it becomes parallel with the appropriate unbounded edge (edge e in figure-1.4).

If such a case occurs we should compute the value of objective function $Q(x)$ in infinite point x of the unbounded edge e, i.e. the following limit:

$$\lim_{\substack{x\to\infty \\ x\in e}} Q(x)$$

Depending on the value of this limit, the maximal (minimal) value of objective function $Q(x)$ may be finite or infinite.

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To illustrate the Graphical method, we consider the following numerical examples:
Example: Graphical Method (Bounded feasible set)

Optimize $$Q(x) = \frac{x_1 + x_2 + 5}{3x_1 + 2x_2 + 15}$$

Subject to, $3x_1 + x_2 \le 6$, $3x_1 + 4x_2 \le 12$

and $x_1 \ge 0,\ x_2 \ge 0$.

Solution Approach: First, we have to construct a feasible set. The convex set S of all feasible solutions for this problem is shown as the shaded region in figure-1.5.

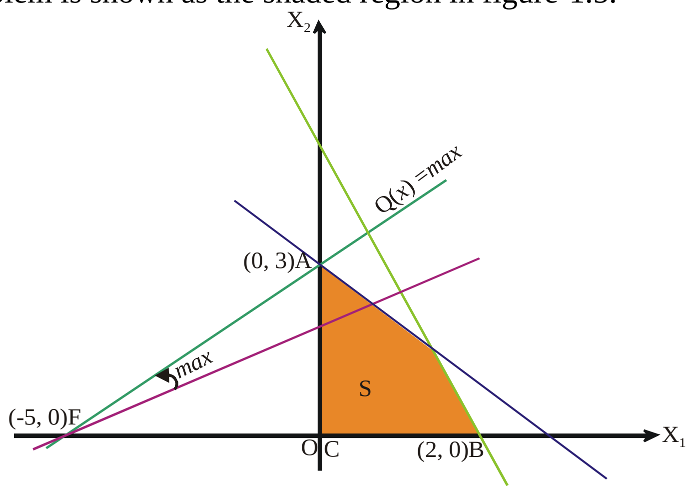

Figure 1.5: Graphical example - Bounded feasible set.

Then, to determine coordinates of the focus point F, we solve the system

$$x_1 + x_2 = -5$$
$$3x_1 + 2x_2 = -15$$

which gives us $F = (-5, 0)$. Level-lines being rotated around focus point F give the following extremal points

$$A = (0,3),\ B = (2,0) \text{ and } O = (0,0)\ ,$$

with objective values
$Q(A) = 8/21,\; Q(B) = 7/21,$ and $Q(\text{O}) = 5/15$,
resectively. So, the objective function $Q(x)$ reaches its maximal value in the point $A = (0,3)$, while the minimization problem has multile optimal solutions: two extremal points $B = (2,0)$ and $\text{O} = (0,0)$, and all points x representable as a linear combination of B and O.

Example: Graphical Method (Asymptotic Case)

Optimize $$Q(x) = \frac{x_1 - 2x_2 + 1}{x_1 + x_2 + 4}$$

Subject to, $x_1 + \;\; x_2 \geq 2,\; x_1 - 2x_2 \leq 4$

and $x_1 \geq 0,\; x_2 \geq 0$.

Set S of all feasible solutions for this problem is shown in figure-1.6.

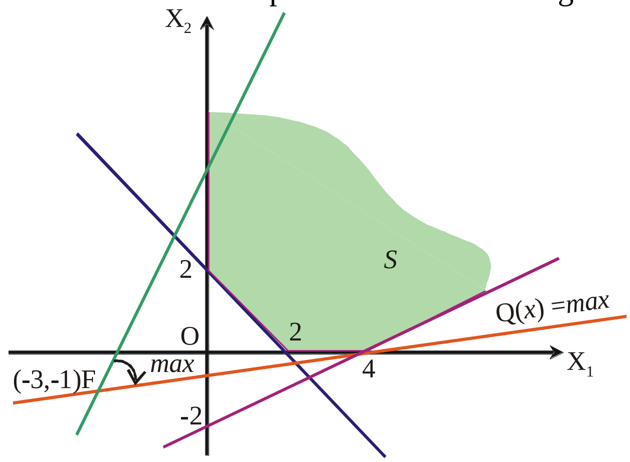

Figure 1.6: Graphical example - Unbounded feasible set.

On solving the system, $x_1 - 2x_2 = -1,\; x_1 + \;\; x_2 = -4$

and $x_1 \geq 0,\; x_2 \geq 0$

we obtain a focus point $F(-3,-1)$. Then rotating level lines around focus point F in both directions (i.e. clockwise and counter clockwise) we realize that the maximization problem has an optimal solution on the point whose co-ordinates are $(4, 0)$ where $Q(x) = \frac{5}{8}$, and the minimization problem has an asymptotic optimal solution in point whose co-ordinates are $(0, \infty)$ on the axes OX_2

$$\min_{x \in S} Q(x) = \lim_{x_2 \to \infty} Q(0, x_2) = -2$$

1.1(b) The Simplex Method

In 1947, George Dantzig [8] developed an efficient method, the simplex algorithm, for solving linear programming problems. Since the development of the simplex method, LP has been used to solve optimization problems anywhere where there appears a necessity of optimizing some absolute criteria. It might be, for example, cost of trucking, profit gained by some company, number of full-time employees, cost of nutrition rations, etc.

Later, in 1960, Bela Martos [10], [22] upgraded the simplex method for solving LFP problems formulated in the following standard form

Maximize $$Q(x) = \frac{N(x)}{D(x)} = \frac{\sum_{j=1}^{r} n_j x_j + n_0}{\sum_{j=1}^{r} d_j x_j + d_0} \quad ...(1.1.5)$$

Subject to, $$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, ..., m \quad ...(1.1.6)$$

and $$x_j \geq 0, \quad j = 1, 2, ..., r \quad ...(1.1.7)$$

where $D(x) > 0$ for all $x = (x_1, x_2, ..., x_r)^T$, which satisfy constraints (1.1.6)–(1.1.7). We assume fessible set S is a regular set, i.e. is non-empty and bounded.

The following simplex methods can be used to solve the LFP problem in which its objective funtion must be maximized. The solution of a minmization LFP problem may be obtained in the same way by substituting the original minimization problem with its appropriate maximization equivalent.

(i) The Standard Simplex Algorithm
(ii) The Big-M method
(iii) The Two-Phase Simplex Method
(iv) The Bounded-Variables Simplex Method

The Standard Simplex Algorithm is an iterative procedure (step by step) for finding the optimal solution to the problem. To illustrate the Simplex method we consider the following LFP problem.

Example: Simplex Algorithm

Maximize $$Q(x) = \frac{x_1 + x_2 + 5}{3x_1 + 2x_2 + 15} = \frac{Q^{(1)}}{Q^{(2)}}$$

Subject to, $3x_1 + x_2 \leq 6$, $3x_1 + 4x_2 \leq 12$

And $x_1 \geq 0, x_2 \geq 0$.

Solution Approach: On introducing the slack variables and, the problem is reduced into the standard form as follows:

Maximize $$Q(x) = \frac{x_1 + x_2 + 5}{3x_1 + 2x_2 + 15} = \frac{Q^{(1)}}{Q^{(2)}}$$

Subject to, $$3x_1 + x_2 + x_3 + 0x_4 = 6,$$
$$3x_1 + 4x_2 + 0x_3 + x_4 = 12$$

And $x_1, x_2, x_3, x_4 \geq 0$

Iteration 1

			$n_j \rightarrow$	1	1	0	0	
			$d_j \rightarrow$	3	2	0	0	
Basic Variables	d_B	n_B	x_B	x_1	x_2	$x_3(\beta_1)$	$x_4(\beta_2)$	Min $\left(\frac{x_B}{x_2}\right)$
x_3	0	0	6	3	1	1	0	$6/3$
x_4	0	0	12	3	$\boxed{4}$	0	1	$12/3$
$Q^{(1)} = n'_B x_B + n_\circ = 5$				-1	-1	0	0	$\leftarrow \Delta_j^{(1)}$
$Q^{(2)} = d'_B x_B + d_\circ = 15$				-3	-2	0	0	$\leftarrow \Delta_j^{(2)}$
$Q = \frac{Q^{(1)}}{Q^{(2)}} = \frac{1}{3}$				0	-5 $\uparrow$	0	0 $\downarrow$	$\leftarrow \Delta_j$

Computation of Iteration 1

$$Q^{(1)} = n'_B x_B + n_\circ = 0 + 5 = 5 \text{ and } Q^{(2)} = d'_B x_B + d_\circ = 0 + 15 = 15$$

Here initial basic feasible solution is $x_1 = 0,\ x_2 = 0$.

Now we find $\Delta_j^{(1)} = n'_B x_j - n_j$

and $\Delta_j^{(2)} = d'_B x_j - d_j$

$\therefore$ $\Delta_1^{(1)} = n'_B x_1 - n_1 = (0, 0)'\,(3, 3) - 1 = -1$

$$\Delta_2^{(1)} = n'_B x_2 - n_2 = (0, 0)'\,(1, 4) - 1 = -1$$

$$\Delta_1^{(2)} = d'_B x_1 - d_1 = (0, 0)'\,(3, 3) - 3 = -3$$

$$\Delta_2^{(2)} = d'_B x_2 - d_2 = (0, 0)'\,(1, 4) - 2 = -2$$

Also, $\Delta_j = Q^{(2)}(Q_j^{(1)} - n_j) - Q^{(1)}(Q_j^{(2)} - d_j)$

$$\therefore \Delta_1 = Q^{(2)}(Q_j^{(1)} - n_1) - Q^{(1)}(Q_j^{(2)} - d_1)$$

$$= 15(0-1) - 5\,(0-3) = 0$$

and $\Delta_2 = Q^{(2)}(Q_2^{(1)} - n_1) - Q^{(1)}(Q_2^{(2)} - d_2)$

$$= 15(0-1) - 5\,(0-2) = -5$$

We observe Δ_2 that is minimum. Therefore, x_2 is entering vector. For outgoing vector, we calculate

$$\min\left[\frac{x_{Bi}}{x_{ij}}\right], \text{ where } \quad j=1,\ldots,4$$

$$\therefore \qquad \min\left[\frac{x_{Bi}}{x_{ij}}\right]=\min\left[\frac{6}{1},\frac{12}{4}\right]=\frac{12}{4}=3 \qquad [\text{for } j=2\ ]$$

Corresponds to x_{22}, x_4 is outgoing vector and 4 is the key element in the basis. So, on introducing x_2 and dropping $x_4\ (\beta_2)$, we get the following iteration table

Iteration 2

			$n_j \rightarrow$	1	1	0	0	
			$d_j \rightarrow$	3	2	0	0	
Basic Variables	d_B	n_B	x_B	x_1	$x_2(\beta_2)$	$x_3(\beta_1)$	x_4	Min $\left(\frac{x_B}{x_2}\right)$
x_3	0	0	3	$9/4$	0	1	$-1/4$	
x_2	2	1	3	$3/4$	1	0	$1/4$	
$Q^{(1)}=n_B' x_B+n_\circ=3+5=8$				$-1/4$	0	0	$1/4$	$\leftarrow \Delta_j^{(1)}$
$Q^{(2)}=d_B' x_B+d_\circ=6+15=21$				$-3/2$	0	0	$1/2$	$\leftarrow \Delta_j^{(2)}$
$Q=\frac{Q^{(1)}}{Q^{(2)}}=\frac{8}{21}$				$27/4$	0	0	$5/4$	$\leftarrow \Delta_j$

Computation

Since all $\Delta_j \geq 0$, therefore the solution is optimal.

Also, $x_1=0,\ x_2=3,\ x_3=3$ and $x_4=0$.

$\therefore$ Maximize $Q(x)=8/21$

1.1(c) Charnes & Cooper's Transformation

In 1962, A. Charnes and W.W. Copper [16] showed that any linear fractional programming problem with a bounded set of feasible solutions may be converted to a linear programming problem. General form of the LFP problem is

$$\text{Optimize} \qquad Q(x)=\frac{N(x)}{D(x)}=\frac{\sum_{j=1}^{r} n_j x_j+n_\circ}{\sum_{j=1}^{r} d_j x_j+d_\circ}, \quad D(x)>0 \qquad \ldots(1.1.8)$$

Subject to, $$\sum_{j=1}^{r} a_{ij}x_j \le b_i, \quad i = 1, 2, \ldots, m \qquad \ldots(1.1.9)$$

and $$x_j \ge 0, \quad j = 1, 2, \ldots, r \qquad \ldots(1.1.10)$$

Let us introduce the following new variables:

$$t_j = \frac{x_j}{D(x)} \text{ and } t_0 = \frac{1}{D(x)}$$

where $$D(x) = \sum_{j=1}^{r} d_j x_j + d_0 \qquad \ldots(1.1.11)$$

Using these new variables $t_j, \quad j = 0,1,\ldots,r$, we can rewrite the original objective function $Q(x)$ in the following form

Optimize $$L(t) = \sum_{j=0}^{r} n_j t_j \qquad \ldots(1.1.12)$$

Since $D(x) > 0, \ \forall x \in S$, we can multiply constraints (1.1.9) and (1.1.10) by $\frac{1}{D(x)}$, so we obtain the following constraints

$$-b_i t_0 + \sum_{j=1}^{r} a_{ij} t_j \le 0, \ i = 1, 2, \ldots, m \qquad \ldots(1.1.13)$$

and $$t_j \ge 0, \quad j = 0, 1, \ldots, r \qquad \ldots(1.1.14)$$

The connection between the original variables x_j and the new variables t_j will be completed if we multiply equality (1.1.11) by the same value $\frac{1}{D(x)}$, and then append the new constraint to the new problem

$$\sum_{j=0}^{r} d_j t_j = 1 \qquad \ldots(1.1.15)$$

Here the new problem (1.1.12)–(1.1.15) will be referred to as a linear analogue of the LFP problem.

The above transformation of variables also establishes a "$\text{one} \leftrightarrow \text{one}$" connection between the original LFP problem (1.1.8)–(1.1.10) and its linear analogue (1.1.12)–(1.1.15) .

Since feasible set S is bounded, function $D(x)$ is linear and $D(x) > 0, \ \forall x \in S$, the following lemma and theorem plays the role of foundation of Charnes & Cooper's transformation.

The connection between the optimal solutions of the original LFP problem and its linear analogue seems to be very useful and at least from the point of view of theory allows to substitute the original LFP problem with its linear analogue and in this way to use LP theory and methods. However, in practice, this approach based on the Charnes & Cooper's transformation may not always be utilized. The problems arise when we should transform the LFP problem with some special structure of constraints, for example transportation problem, or assignment problem or any other problem with a fixed structure of constraints, and would like to apply appropriate special methods and algorithms. Indeed, if in the original LFP problem we have r unknwon variables and m main constraints, then in its linear analogue we obtain $r + 1$ variables and $m + 1$ constraints. Moreover, in the right hand side of (1.1.13) we have no vector b. Instead of the original vector b we have a vector of zeros.

Example: Charnes & Cooper's Transformation

Maximize $$Z = \frac{2x_1 + 6x_2}{x_1 + x_2 + 1}$$

Subject to,
$$x_1 + x_2 + x_3 + 0x_4 = 4$$
$$3x_1 + x_2 + 0x_3 - x_4 = 6$$
$$x_1 - x_2 + 0x_3 + 0x_4 = 0$$

and $$x_1, x_2, x_3, x_4 \geq 0.$$

Solution Approach:

Since $x_1,\ x_2 \geq 0$, therefore denominator of the objective function will always be nonnegative.

Let $t = \dfrac{1}{x_1 + x_2 + 1}$ and $y_i = tx_i$, where $i = 1, \ldots, 4$, then the objective function of the above problem reduces as follows

Maximize $$Z = \frac{2\left(\frac{y_1}{t}\right) + 6\left(\frac{y_2}{t}\right)}{1/t}$$
$$= 2y_1 + 6y_2$$

Similarly the constraints of the above problem will be reduced as follows

Constraint-I $\quad x_1 + x_2 + x_3 = 4$

$\Rightarrow \quad \dfrac{y_1}{t} + \dfrac{y_2}{t} + \dfrac{y_3}{t} = 4 \quad [\because y_i = tx_i$, where $i = 1, \ldots, 4$.]

$\Rightarrow \quad y_1 + y_2 + y_3 - 4t = 0$

Constraint-II $\quad 3x_1 + x_2 - x_4 = 6$

$\Rightarrow \quad 3y_1 + y_2 - y_4 - 6t = 0$ [$\because y_i = tx_i$,where $i = 1, \ldots, 4$.]

Constraint-III $\quad x_1 - x_2 = 0$

$\Rightarrow \quad y_1 - y_2 = 0 \quad [\because y_i = tx_i$, where $i = 1,...,4$.]

Also, we have, $\quad t = \dfrac{1}{x_1 + x_2 + 1}$

$\Rightarrow \quad t\,(x_1 + x_2 + 1) = 1$

$\Rightarrow \quad tx_1 + tx_2 + t = 1$

$\Rightarrow \quad y_1 + y_2 + t = 1 \quad [\because y_i = tx_i$, where $i = 1,...,4$]

Therefore, the given problem reduces into the Linear Programming problem(Standard form for Simplex Method) as follows:

Maximize $\quad Z = 2y_1 + 6y_2 + 0y_3 + 0y_4 + 0t - MA_1 - MA_2 - MA_3$

Subject to
$$y_1 + y_2 + y_3 - 4t = 0$$
$$3y_1 + y_2 - y_4 - 6t + A_1 = 0$$
$$y_1 - y_2 + A_2 = 0$$
$$y_1 + y_2 + t + A_3 = 1$$

and $\quad y_j \geq 0, \quad j = 1, ..., 4$

Also $\quad t > 0,\ A_1, A_2, A_3 > 0$

and $\quad A_1, A_2, A_3$ are artificial variables.

Initial Simplex Tableau

		$C_j \rightarrow$	2	6	0	0	0	$-M$	$-M$	$-M$
C_B	X_B	b	y_1	y_2	y_3	y_4	t	A_1	A_2	A_3
0	y_3	0	1	1	1	0	–4	0	0	0
$-M$	A_1	0	3	1	0	–1	–6	1	0	0
$-M$	A_2	0	$\boxed{1}$	–1	0	0	0	0	1	0
$-M$	A_3	1	1	1	0	0	1	0	0	1
$Z_j - C_j \rightarrow$			$-5M - 2$ $\uparrow$	$-M - 2$	0	M	$5M$	0	0	0

Final Simplex Tableau

		$C_j \rightarrow$	2	6	0	0	0	$-M$	$-M$	$-M$
C_B	X_B	b	y_1	y_2	y_3	y_4	t	A_1	A_2	A_3
0	y_3	$\frac{2}{5}$	0	0	$\frac{8}{5}$	1	0	-	-	-
6	y_2	$\frac{2}{5}$	0	1	$\frac{1}{10}$	0	0	-	-	-
2	y_1	$\frac{2}{5}$	1	0	$\frac{1}{10}$	0	0	-	-	-
0	t	$\frac{1}{5}$	0	0	$\frac{-1}{5}$	0	1	-	-	-
$Z_j - C_j \rightarrow$			0	0	$\frac{2}{5}$	0	0	-	-	-

Computation

Since all $Z_j - C_j \geq 0$. The optimal solution is

$$y_1 = \frac{2}{5},\ y_2 = \frac{2}{5},\ y_3 = 0,\ y_4 = \frac{2}{5} \text{ and } t = \frac{1}{5}\ .$$

Therefore, $$x_1 = \frac{y_1}{t} = \frac{2/5}{1/5} = 2, \quad x_2 = \frac{y_2}{t} = \frac{2/5}{1/5} = 2, \quad x_3 = \frac{y_3}{t} = \frac{0}{1/5} = 0,$$

$$x_4 = \frac{y_4}{t} = \frac{2/5}{1/5} = 2 \quad \text{and} \quad t = \frac{1}{x_1 + x_2 + 1} = \frac{1}{5} [\because x_1 = 2, x_2 = 2]$$

Hence, the optimal solution is

$$Z = \frac{2x_1 + 6x_2}{x_1 + x_2 + 1} = \frac{2 \times 2 + 6 \times 2}{5} = \frac{16}{5}.$$

1.1(d) Dinkelbach's Algorithm

One of the most popular and general strategies for fractional programming (not necessary linear) is the parametric approach described by W. Dinkelbach. In the case of linear-fractional programming, this method reduces the solution of a problem to the solution of a sequence of linear programming problems.

Consider the common LFP problem (1.1.1)–(1.1.3) and function

$$F(\lambda) = \max_{x \in S} \{N(x) - \lambda D(x)\}, \quad \lambda \in R,$$

where S denotes the feasible set of (1.1.1)–(1.1.3).

Dinkelbach's Algorithm

Step 1: Take $x^{(0)} \in S$, compute $\lambda(1) = \dfrac{N\left(x^{(0)}\right)}{D\left(x^{(0)}\right)}$, and let $k=1$;

Step 2: Determine $x^{(k)} = \arg\max\limits_{x \in S} \left\{ N(x) - \lambda^{(k)} D(x) \right\}$;

Step 3: If $F\left(\lambda^{(k)}\right) = 0$, then $x^* = x^{(k)}$ is an optimal solution; Stop;

Step 4: Let $\lambda^{(k+1)} = \dfrac{N\left(x^{(k)}\right)}{D\left(x^{(k)}\right)}$; let $k = k+1$; go to Step 2;

Example: Dinkelbach's Algorithm

$$\text{Maximize} \quad Q(x) = \frac{N(x)}{D(x)} = \frac{x_1 + x_2 + 5}{3x_1 + 2x_2 + 15}$$

$$\text{Subject to,} \quad \begin{array}{rcrcl} 3x_1 & + & x_2 & \le & 6, \\ 3x_1 & + & 4x_2 & \le & 12, \end{array}$$

$$\text{and} \quad . \ x_1 \ge 0, x_2 \ge 0 \qquad \text{...(1.1.16)}$$

Solution Approach

Step 1: Since vector $x = (0,0)^T$ satisfies all constraints of the problem, we may take it as a strating point $x^{(0)} \in S$. So, for $x^{(0)} = (0,0)^T$, we obtain

$$\lambda^{(1)} = \frac{N\left(x^{(0)}\right)}{D\left(x^{(0)}\right)} = \frac{5}{15} = \frac{1}{3},$$

Step 2: Now, we have to solve the following linear programming problem

$$N(x) - \lambda^{(1)} D(x) = N(x) - \frac{1}{3} D(x) = \frac{1}{3} x_2 \to \text{Max}.$$

subject to constraints in (1.1.16).
Solving this problem, we obtain

$$x^{(1)} = (0,3)^T, F\left(\lambda^{(1)}\right) = 1.$$

Step 3: Since $F\left(\lambda^{(1)}\right) \ne 0$, we have to perform

Step 4: We have to calculate

$$\lambda^{(2)} = \frac{N\left(x^{(1)}\right)}{D\left(x^{(1)}\right)} = \frac{1 \times 3 + 5}{2 \times 3 + 15} = \frac{8}{21},$$

then to put $k = k+1 = 2$ and the process repeated till
Step 2: Solve the following LP problem:

$$N(x) - \lambda^{(2)} D(x)$$
$$= \left(1 - \frac{8}{21}\times 3\right)x_1 + \left(1 - \frac{8}{21}\times 2\right)x_2 + \left(5 - \frac{8}{21}\times 15\right)$$
$$= -\frac{1}{7}x_1 + \frac{5}{21}x_2 - \frac{5}{7} \to \text{Max}.$$

subject to constraints in (1.1.16)

Solving the above problem, we obtain $x^{(2)} = (0,3)^T$ with $F\left(\lambda^{(2)}\right) = 0$.

Step 3: Since $F\left(\lambda^{(2)}\right) = 0$, vector $x^* = x^{(2)}$ is the optimal solution. Therefore, the procedure is stopped.

In accordance with the algorithm, the optimal solution of our LFP problem is $x^* = (0,3)^T$ with optimal objective value $Q(x^*) = 8/21$.

1.2 LINEAR PLUS LINEAR FRACTIONAL PROGRAMMING PROBLEM

The sum of a linear and linear-fractional function is investigated in terms of quasi convexity and quasi-concavity. The optimization problem

$$\text{Sup}\left\{ x \in S \,\middle|\, q(x) = a^T x + \frac{b^T x}{c^T x} \right\},\ S \subseteq \mathbb{R}^n \text{ convex, } c^T x > 0.$$

which arises when a compromise between absolute and relative terms is to be maximized. For linear programs, *b* = 0 and linear fractional programs, *a* = 0.

The above optimization problem can often be solved by a convex programming procedure if a local maximum is a global maximum and a simplex like procedure can be applied if a local maximum attained at an extreme point of S.

REFERENCES

Aggarwal, S. P. and Arora, S (1974). A special Class of Non-Linear Fractional Functional Programming Problems. SCIMA, *Journal of Management Science and Applied Cybernetics* **3**, 30-39.

Aggarwal, S. P. and Saxena, P. C (1975). Duality Theorems for Non-Linear fractional Programs. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)* **55**, 523-525.

Aggarwal, S. P. and Saxena, P. C (1979). A Class of Fractional Functional Programming Problems. *New Zealand Operational Research* 7, 79-90.

Aggarwal, S. P., and Swarup, K (1966). Fractional Functionals Programming with a Quadratic Constraint. *Operations Research* 14, 950-956.

Agrawal, S. C (1975). On Integer Solutons to Linear Fractional Functional Programming Problems. *Acta Ciencia Indica* 1, 203-208.

Agrawal, S. C (1976). On Integer Solution to Linear Fractional Functionals by a Branch and Bound Technique. *Acta Ciencia Indica* **2**, 75-78.
Agrawal, S. C (1977). An Alternative Method on Integer Solutions to Linear Fractional Functionals by a Branch and Bound Technique. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)* **57**, 52-53.
Avriel, M. and Williams, A. C (1970). Complementary Geometric programming. *SIAM Journal of Applied Mathematics*, **19**,125-141
Bajalinov, Erik B (2003). Linear-Fractional Programming: Theory, Methods, Applications and Software. *Kluwer Academic Publishers*.
Barros, Ana I (1965). Discrete and Fractional Programming Techniques for Location Models. *Kluwer Academic Publisher. the Hungarian Academy of Sciences, Budapest*.
Bector, C. R (1968). Programming Problems with Convex Fractional Functions. *Operations Research* 16,383-391.
Billy, E. G (2011). Introduction to Operations Research: A Computer Oriented Algorithmic Approach. *Tata McGraw-Hill Publishing Company Ltd., New Delhi.*
Bitran, G. R. and Novaes, A. J (1973). Linear Programming with a Fractional Objective Function. *Operations Research* **21**, 22-29.
Borwein, J. M (1976). Fractional Programming without Differentiability.” *Mathematical Programming* **11**, 283 – 290.
Büher, W (1975). A Note on Fractional Interval Programming.” *Zeitschrift für Operations Research* **19**, 29-36.
Chadha, S. S (1971). A Linear Fractional Functional Program with a Two Parameter Objective Functions. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)* **51**, 479-481.
Chadha, S. S (1999). A Linear Fractional Program with homogeneous Constraints. *OPSEARCH* **36**, 390-398.
Chandra, S (1967). Linear Fractional Functionals Programming. *Journal of the Operations Research Society of Japan* 10, 1-2.
Charnes, A., and Cooper, W. W (1961). Industrial Applications of Linear Programming Problem. *Wiley, New York*.
Charnes, A., and Cooper, W. W (1962). Programming with Linear Fractional Functionals. *Naval Research Logistics Quarterly*, **9**, 181-186.
Craven, B. D (1989). Fractional Programming.” *Heldermann verlag, Berlin.*
Dantzig, G.B (1964). Linear Programming Methods and Applications. *Princeton University Press, New Jersey*.
Gass, S.I (1985). Linear Programming Methods and Applications. *McGraw-Hill Book Company, New York*.
Ibaraki, T (1976). Algorithms for Quadratic Fractional Programming Problems” *Journal of the Operations Research Society of Japan* **19**, 174-191.
Jain, S. and Lachhwani, K (2008). Solution of Fuzzy Bilevel Linear Programming Problem. *Ganita Sandesh*, **22**(1), 9-14.
Jain, S. and Mangal, A (2004). Solution of generalized fractional programming problem. *Journal of Indian Academy of Mathematics*, **26**. No. 1, 15-21.

Jain, S. and Mangal, A (2004). Modified Fourier elimination technique for fractional programming problem. *Acharya Nagarjuna International Journal of Mathematics and Information Technology*, 1, 121 – 131.
Kanchan, P. K (1976). Linear Fractional Functional Programming” *Acta Ciencia Indica* **2**, 401 – 405.
Martos, B (1964). Hyperbolic Programming. Naval Research Logistic Quarterly 11, 135-155; originally published in Math. Institute of Hungarian Academy of Sciences (Hungarian) **5**, 1960, 383-406.
Misra, S. and Das, C (1981). The sum of a Linear and Linear Fractional Function and a Three Dimensional Transportation Problem. *OPSEARCH* **18**, 139-157.
Rommelfangre, H (1996). Fuzzy Linear programming and Applications. *European Journal of Operational Research* **92**, 512-527
Schaible, S (1995). Fractional programming. In R. Horst and P.M. Pardalos, editors, Handbook of Global Optimization, pages 495-608. *Kluwer Academic Publishers. Dordrecht.*
Schaible, S. and Shi, J (2003). Fractional Programming: The Sum-of-Ratios Case. *Optimization Methods and Software,* **18** 219-229.
Stancu-Minasian, I.M (1997). Fractional Programming: Theory, Methods and Applications. *Kluwer Academic Publishers*.
Swarup, K (1965). Programming with Quadratic Fractional Functionals” *OPSEARCH* 2, 23-30.
Tantawy, S. F (2007). A New Method for Solving Linear Fractional Programming Problems. *Australian Journal of Basic and Applied Sciences*, **1**(2) 105-108.
Wolf, H (1985). A Parametric Method for Solving Linear Fractional Programming Problem. *Operations Research*, **33**, 835-841.
Zionts, S (1968). Programming with Linear Fractional Functionals. *Naval Research Logistics Quarterly* **15**, 449-451.