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Research Article

DEVISING THE INTRICACIES OF LINEAR FRACTIONAL PROGRAMMING PROBLEMS

[1]P. R. Parihar and [2]Pankaj Mathur*
[1]Department of Mathematics, S.P.C. Government College, Ajmer (Raj)
[2]Department of Mathematics, Government College, Tonk (Raj.)
**Author for Correspondence: pmathur_8@yahoo.com*

ABSTRACT
There are distinct fields in real world which face the problem of optimization like in the area of financial and corporate planning has to optimize its debt/equity ratio. Similarly in health care and hospital planning requires the need to optimize its nurse/patient ratio and there are many more realms which deal with the same quandary. Optimization could be modelled as a fractional programming problem which is a generalization of linear programming problem.

The mathematical formation of LFP is discussed in detail by optimizing an arbitrary function Q(x) with some constraints. The feasible and infeasible concept of LFP is also explained in the paper.

Further it is discovered that the LFP problem is said to be in standard form if all the constraints are equations and all the unknown variables are nonnegative. For the general form of LFP the constraints are <= inequalities and rest is same. The intricacies of the topic is well elaborated with the help of formulas and mathematical expressions.

***Keywords:** fractional programming, feasible and infeasible.*

1.1 INTRODUCTION

The modern theory of optimization that we see today essentially began with the discovery of the simplex method for linear programming problem (LPP) in the late 1940's. In the modelling of the real-world problems like financial and corporate planning, production planning, marketing and media selection, university planning and student admissions, health care and hospital planning, air force maintenance units, bank branches, etc. frequently may be faced up with decision to optimise debt/equity ratio, profit/cost, inventory/sales, actual cost/standard cost, output/employee, student/cost, nurse/patient ratio etc. with respect to some constraints.

In 1960's, it was realized that the above problems could not be modelled as a linear programming problem where as they could be model as an Fractional Programming Problem and its applications.

1.2 FORMULATION OF LFP

In 1960, Hungarian Mathematician Bèla Martos [19] formulated and considered a so-called *hyperbolic programming* problem, which in the English language special literature is referred as a *linear-fractional programming problem (LFP)*. The common problem of LFP may be formulated as follows:

Given objective function

$$Q(x) = \frac{N(x)}{D(x)} = \frac{\sum_{j=1}^{r} n_j x_j + n_\circ}{\sum_{j=1}^{r} d_j x_j + d_\circ}$$

which must be maximized (or minimized),

Subject to, $\quad \sum_{j=1}^{r} a_{ij} x_j \le b_i, \quad i = 1, 2, ..., m_1$

$$\sum_{j=1}^{r} a_{ij}x_j \geq b_i, \quad i = m_1+1, m_1+2, ..., m_2$$

$$\sum_{j=1}^{r} a_{ij}x_j = b_i, \quad i = m_2+1, m_2+2, ..., m$$

and $$x_j \geq 0, \quad j = 1, 2, \ldots, r$$

… (1.2.1).

where $m_1 \leq m_2 \leq m$. Here and in what follows we suppose that $D(x) \neq 0$, $\forall x = \{x_1, x_2, \ldots, x_r\} \in S$, where S denotes a set of feasible solutions defined by constraints given in (1.1.1).

Since the denominator $D(x) \neq 0$, so we can assume $D(x) > 0$. In the case of $D(x) < 0$, we can multiply numerator $N(x)$ and denominator $D(x)$ of objective function $Q(x)$ by '–1'.

Here we will consider such linear fractional programming problems that satisfy condition $D(x) > 0$. We also suppose that the constraints

$$\sum_{j=1}^{r} a_{ij}x_j \leq b_i, \quad i = 1, 2, ..., m_1$$

$$\sum_{j=1}^{r} a_{ij}x_j \geq b_i, \quad i = m_1+1, m_2+2, 2, ..., m_2$$

$$\sum_{j=1}^{r} a_{ij}x_j = b_i, \quad i = m_2+1, m_2+2, ..., m.$$

in system (1.2.1) are linearly independent and so the rank of matrix $A = [a_{ij}]_{m \times r}$ is equal to *m*.

Therefore, in an LFP problem, our aim is to find such a vector of decision variables $x_j \geq 0, \quad j = 1, 2, \ldots, r$, which

(i) maximizes (or minimizes) function $Q(x)$, called **objective function**,
(ii) satisfies a set of **main constraints**

$$\sum_{j=1}^{r} a_{ij}x_j \leq b_i, \quad i = 1, 2, ..., m_1$$

$$\sum_{j=1}^{r} a_{ij}x_j \geq b_i, \quad i = m_1+1, m_2+2, ..., m_2$$

$$\sum_{j=1}^{r} a_{ij}x_j = b_i, \quad i = m_2+1, m_2+2, ..., m.$$

and sign restrictions $x_j \geq 0, \quad j = 1, 2, \ldots, r$, at the same time.

1.3 DEFINITIONS

We introduce the main conceptions that will be used throughout the rest of the work.

1.3.1 If given vector $x = (x_1, x_2, ..., x_r)$ satisfies constraints

$$\sum_{j=1}^{r} a_{ij} x_j \le b_i, \quad i = 1, 2, ..., m_1$$

$$\sum_{j=1}^{r} a_{ij} x_j \ge b_i, \quad i = m_1 + 1, m_1 + 2, ..., m_2$$

$$\sum_{j=1}^{r} a_{ij} x_j = b_i, \quad i = m_2 + 1, m_2 + 2, ..., m.$$

and $x_j \ge 0, \quad j = 1, 2, \dots, r$, we will say that vector x is a feasible solution of LFP problem (1.2.1).

1.3.2 If given vector $x = (x_1, x_2, ..., x_r)$ is a feasible solution of maximization (minimization) LFP problem (1.2.1), and provides maximal (minimal) value for objective function $Q(x)$ over the feasible set S, we say that vector x is an optimal solution of maximization (minimization) linear - fractional programming problem (1.2.1).

1.3.3 A maximization (minimization) LFP problem is solvable, if its feasible set S is not empty and objective function $Q(x)$ has finite upper (lower) bound on S.

1.3.4 If the feasible set S is empty, we say that the LFP problem is infeasible.

1.3.5 If the objective function $Q(x)$ of a maximization (minimization) LFP problem has no upper (lower) finite bound, we say the LFP problem is unbounded.

1.4 RELATIONSHIP WITH LINEAR PROGRAMMING

It is obvious that if $x = (x_1, x_2, ..., x_r)$ and $d_\circ \ne 0$, then LFP problem (1.2.1) becomes an LP problem. This is a reason why we say that an LFP problem (1.2.1) is a generalization of the following LP problem

Given objective function

$$Q(x) = \frac{N(x)}{D(x)} = \frac{\sum_{j=1}^{r} n_j x_j + n_\circ}{d_\circ} = \sum_{j=1}^{r} \frac{n_j}{d_\circ} x_j + \frac{n_\circ}{d_\circ}$$

which must be maximized (or minimized),

Subject to,

$$\sum_{j=1}^{r} a_{ij} x_j \le b_i, \quad i = 1, 2, ..., m_1$$

$$\sum_{j=1}^{r} a_{ij} x_j \ge b_i, \quad i = m_1 + 1, m_1 + 2, ..., m_2$$

$$\sum_{j=1}^{r} a_{ij} x_j = b_i, \quad i = m_2 + 1, m_2 + 2, ..., m$$

and $x_j \ge 0, \quad j = 1, 2, \dots, r$, where $m_1 \le m_2 \le m$.

In this case, optimization of the original objective function $Q(x)$ may be substituted with optimization of linear function $\dfrac{N(x)}{d_\circ}$ correspondingly on the same feasible set S. It is to be noted that if $d_\circ = 1,\ Q(x) = N(x) = \sum_{j=1}^{r} n_j x_j + n_\circ$.

There are also a few special cases when the original LFP problem may be replaced with an appropriate LP problem:

(1) If $n_j = 0,\ j = 1, 2, ,\ldots r$, the objective function

$$Q(x) = \frac{N(x)}{D(x)} = \frac{n_o}{\sum_{j=1}^{r} d_j x_j + d_o}$$

may be replaced with function $D(x)$. In this case maximization (minimization) of the original objective function $Q(x)$ must be substituted with minimization (maximization) of a new objective function $D(x)$ on the same feasible set S.

(2) If vectors $n = (n_1, n_2, \ldots, n_r)$ and $d = (d_1, d_2, \ldots, d_r)$ are linearly dependent, that is there exists such $\mu \neq 0$, that $n = \mu d$, then objective function

$$Q(x) = \frac{N(x)}{D(x)}$$

$$= \frac{\sum_{j=1}^{r} \mu d_j x_j + n_o}{\sum_{j=1}^{r} d_j x_j + d_o}$$

$$= \frac{\sum_{j=1}^{r} \mu d_j x_j + \mu d_\circ + n_\circ - \mu d_\circ}{\sum_{j=1}^{r} d_j x_j + d_\circ}$$

$$= \frac{\mu\left(\sum_{j=1}^{r} d_j x_j + d_\circ\right) + n_\circ - \mu d_\circ}{\sum_{j=1}^{r} d_j x_j + d_\circ}$$

$$= \mu + \frac{n_\circ - \mu d_\circ}{\sum_{j=1}^{r} d_j x_j + d_\circ}$$

may be replaced with function $D(x)$. Obviously, in this case, maximization (minimization) of the original objective function $Q(x)$ must be substituted with

* minimization (maximization) of $D(x)$, if $n_o - \mu d_o > 0$,

* maximization (minimization) of $D(x)$, if $n_o - \mu d_o < 0$.

Note: If $n_o - \mu d_o = 0$, $Q(x) = \mu$, which means that Q(x) = const. $\forall\ x \in S$.

1.5 STANDARD FORM OF THE LFP PROBLEM

An LFP problem is said to be in standard form if all constraints are equations and all unknown variables are nonnegative, that is

Optimize $$Q(x) = \frac{N(x)}{D(x)} = \frac{\sum_{j=1}^{r} n_j x_j + n_o}{\sum_{j=1}^{r} d_j x_j + d_o}, \quad D(x) > 0$$

Subject to $$\sum_{j=1}^{r} a_{ij} x_j = b_i, \quad i = 1, 2, \ldots, m$$

and $$x_j \geq 0, \quad j = 1, 2, \ldots, r.$$

1.6 GENERAL FORM OF THE LFP PROBLEM

An LFP problem is said to be in general form if all constraints are "<" (less than) inequalities and all unknown variables are nonnegative, that is

Optimize $$Q(x) = \frac{N(x)}{D(x)} = \frac{\sum_{j=1}^{r} n_j x_j + n_o}{\sum_{j=1}^{r} d_j x_j + d_o}, \quad D(x) > 0$$

Subject to, $$\sum_{j=1}^{r} a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots, m$$

and $$x_j \geq 0, \quad j = 1, 2, \ldots, r.$$

We have seen that LFP problems may have both equality and inequality constraints. They may also have unknown variables that are required to be nonnegative and variables that are allowed to be unrestricted in sign.

It is obvious that standard and general forms of LFP problems are special cases of a LFP problem formulated in (1.2.1). Indeed, if in the common LFP problem (1.2.1) we put $m_1 = m_2 = 0$, then we get a standard LFP problem. But if $m_1 = m$, then we have general LFP problem.

To convert one form to another, we use to following converting procedures

(1) '>' (greater than) $\rightarrow\ '\leq'$ (less than)
Both sides of the constraint must be multiplied by '–1'.

(2) '$\leq$' (less than) $\rightarrow$ '=' (equal)

Define for " constraint a nonnegative slack variable s_i ($s_i \geq 0 -$slak variable for i-th constraint) and put this variable into the left-side of the constraint, where it will play a role of difference between the left and right sides of the original i-th constraint. Also add the sign restrictions $s_i \geq 0$ to the set of constraints. So

$$\sum_{j=1}^{r} a_{ij}\ x_j \leq b_i \longrightarrow \left[\begin{array}{l} \sum_{j=1}^{r} a_{ij}\ x_j + s_i = b_i, \\ \qquad\qquad s_i \geq 0. \end{array}\right.$$

(3) Unrestricted in sign variable $x_j \rightarrow$ restricted in sign
nonnegative variable(s).

For each urs(unrestricted sign) variable x_j, we begin by defining two new nonnegative variables x'_j and x''_j. Then substitute $x'_j - x''_j$ for x_j in each constraint and in objective function. Also add the sign restrictions $x'_j \geq 0$ and $x''_j \geq 0$ to the the set of constraints.

Because all three forms of an LFP problem [the most common form (1.2.1), standard from, and general form] may be easily converted to one another. Instead of an LFP problem in form (1.2.1), sometimes we will consider its equivalent LFP problem in standard or in general form. Obviously, such substitution does not lead to any loss of generality, but allows us to simplify our consideration.

1.7 MATRIX FORM OF THE LFP PROBLEM

On introducing the following notations,

$$A_j = (a_{1j}, a_{2j}, \ldots, a_{mj})^T,\ b = \left(b_1, b_2, \ldots, b_m\right)^T,\ x = (x_1, x_2, \ldots, x_r)^T,$$

$$n = (n_1, n_2, \ldots, n_r)^T \text{ and } d = (d_1, d_2, \ldots, d_r)^T,$$

where $j = 1, 2, \ldots, r$, we can re-formulate the LFP problem in a matrix form as follows:

Standard problem

Optimize $Q(x) = \dfrac{N(x)}{D(x)} = \dfrac{n^T x + n_\circ}{d^T x + d_\circ},\quad D(x) > 0$

Subject to, $\sum_{j=1}^{r} A_j\ x_j = b$

and $x \geq 0$.

General problem

Optimize $Q(x) = \dfrac{N(x)}{D(x)} = \dfrac{n^T x + n_\circ}{d^T x + d_\circ},\quad D(x) > 0,$

Subject to, $$\sum_{j=1}^{r} A_j\, x_j \le b$$

and $x \ge 0$.

Note: From the theory of Mathematical programming, we know that

$$\min_{x \in S} F(x) \equiv \max_{x \in S} (-F(x))$$

which means that to convert a minimization LFP problem to a maximization one, we should multiply its objective function by '–1'. So, there is no reason to consider both cases (i.e. maximization and minimization) separately.

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