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Research Article

GENERATE FUZZY CONCEPTS BASED ON JOIN-IRREDUCIBLE ELEMENTS

Hua Mao and *Zhen Zheng
Department of Mathematics and Information Science, Hebei University,
Baoding 071002, China
**Author for Correspondence: 373380431@qq.com*

ABSTRACT

The construction of fuzzy concept lattices is one of the major matters of fuzzy formal concept analysis. However, the construction of efficiency of fuzzy concept lattices is still exponential. Irreducible elements push forward a immense influence on improving construction efficiency of fuzzy concept lattices. Hence, this paper firstly proposes a weakened object topology graph for finding join-irreducible elements. After that, it provides a novel way to compute join-irreducible elements, and then generate all crisp-fuzzy concepts.

Keywords: *fuzzy concept lattices, irreducible elements, join-irreducible elements*

INTRODUCTION

Concept lattice is a branch of lattice theory and also an immediate outcome of formal concept analysis (FCA), which was put forward by Ganter and Wille (1982) in the early 1980s. Concept lattice is a concept hierarchical structure established by analyzing and disposing the binary relationship between objects and attributes of a data set. At present, concept lattice has been successfully applied to mathematics and computer science, and still owns a great deal of potential application value.

A formal concept is a composition of extension and intension. However, in the real world, the public's cognition is fuzzy rather than crisp for the great majority of things. With the rapid development of technology, some scholars proposed a fuzzy formal concept, which indicates the vague relationship among the objects and attributes. Such as Burusco and Fuentes (1994) firstly defined fuzzy concepts using implication operators based on fuzzy closure operators, and Krajci (2003) presented one-sided fuzzy concepts. Researchers have employed fuzzy concepts to fuzzy classification and fuzzy decision, though the approaches of constructing fuzzy concept lattices are less productive. Consequently, the construction of fuzzy concept lattices is count for research.

It is generally known that every element can be depicted by the join (meet) of join-irreducible (meet-irreducible) elements. Thus, in a lattice, all the join-irreducible elements and meet-irreducible elements are the most fundamental elements. Similarly, in a concept lattice, object concepts and attribute concepts are the most fundamental elements, where each formal concept is either the intersection of attribute concepts or the union of object concepts. In other words, each object concept and attribute concept corresponds to a meet-irreducible element and a join-irreducible element in a concept lattice. So it is very significant to study irreducible elements in a concept lattice. On the one hand, usually, irreducible elements can be used to explore the attribute characteristics in attribute reduction, such as Shao and Leung (2016) proposed a granular reduct methods based on join-irreducible elements in a fuzzy formal context. On the other hand, irreducible elements also make contributions to the construction of concept lattices. For instance, Li and Shao (2017) built the lattice from meet-irreducible attribute concepts by using generators directly. In fuzzy formal contexts, Zhang (2018) raised a batch-mode algorithm for directly constructing fuzzy concept lattice based on union and intersection operations, yet the construction of efficiency of fuzzy concept lattices is still exponential.

Accordingly, in a fuzzy formal context, using irreducible elements to generate fuzzy concepts can greatly improve the construction of efficiency of fuzzy concept lattices. Inspired by attribute topology, proposed by Zhang (2014), this paper presents a weakened object topology graph, in order to find join-irreducible elements in a fuzzy formal context. Furtherly, we can obtain all fuzzy concepts by the join of the join-irreducible elements.

The remainder of this paper is organized as follows. Firstly, we briefly review some basic notions of fuzzy formal contexts and graph theory. Secondly, we will define the notion of a weakened object topology graph and discuss how to find join-irreducible elements from a weakened object topology graph, and then generate all fuzzy concepts. Finally, we conclude the paper and outline the future work.

Preliminaries

In this section, we review some notions and properties about fuzzy formal contexts and graph theory. For more details, please refer to Krajci (2003), Bondy and Murty (2008).

Definition 1 (1) (Krajci, 2003) Let *U* and *B* as the set of objects and attributes respectively, *R* is a fuzzy relation on their Cartesian product, i.e. $R: A\times B\rightarrow [0, 1]$. A table with rows and columns corresponding to objects and attributes is used for representing this fuzzy relation. Then the value $R(a, b)$ expresses the grade in which the object *b* have the attribute *a*.

(2) (Krajci, 2003) Let (U, A, R) be a fuzzy formal context. Define a mapping: $f: \mathrm{P}(U)\rightarrow\Gamma(A)$ which assigns to every crisp set *X* of objects , a fuzzy set $f(X)$ of attributes, a value in a point $a\in A$ of which is

$$f(X)(a)=\wedge_{x\in X} R(x,a), a\in A \tag{1}$$

i.e. this function assigns to every attribute the greatest value so that all objects from X have this attribute at least in such grade. Conversely, define a mapping $g:\Gamma(A)\rightarrow \mathrm{P}(U)$, which assigns to every function $\rho: A\rightarrow[0,1]$ a set

$$g(\rho)=\{x\in U \mid \forall a\in A, \rho(a)\leq R(x,a)\} \tag{2}$$

i.e. such objects which have all attributes at least in the grade set by function ρ (in other words, the function of their fuzzy-membership to objects dominates over ρ) .

It is easy to see that operators f and g form a Galois connection between $\mathrm{P}(U)$ and $\Gamma(A)$, and the following properties can be obtained.

Property 1 (Krajci, 2003) Let (U, A, R) be a formal context $X, X_1, X_2, X_i\in \mathrm{P}(U)$, and $B, B_1, B_2, B_i\in\Gamma(A)$, $i\in J$ (J is an index set). Then

(i) $X_1\subseteq X_2 \Rightarrow f(X_2)\subseteq f(X_1), B_1\subseteq B_2 \Rightarrow g(B_2)\subseteq g(B_1);$

(ii) $X \subseteq g \circ f(X), B \subseteq f \circ g(B)$;

(iii) $f(X) = f \circ g \circ f(X), g(B) = g \circ f \circ g(B)$;

(iv) $f(\bigcup_{i \in J} x_i) = \bigcap_{i \in J} f(X_i), g(\bigcup_{i \in J} B_i) = \bigcap_{i \in J} g(B)$.

Definition 2 (1) (Krajci, 2003) Let (U, A, R) be a formal fuzzy context, a pair $(X, B) \in P(U) \times \Gamma(A)$ satisfying $X = g(B)$ and $B = f(X)$ is called crisp-fuzzy concept of (U, A, R).

(2) (Krajci, 2003) For a set of objects $X \subseteq P(U)$ and a fuzzy set of attributes $B \in f(A)$, from Property1 (iii), both $(g \circ f(X), f(X))$ and $(g(B), f \circ g(B))$ are crisp-fuzzy concepts. In particular, $(g \circ f(X), f(X))$ is a crisp-fuzzy concept for each $x \in U$ and is called an object concept.

For two crisp-fuzzy concept (X_1, B_1) and (X_2, B_2) to make $(X_1, B_1) \le (X_2, B_2)$, if and only if $X_1 \subseteq X_2$ (or equivalently, $B_2 \subseteq B_1$).

Example 1 A fuzzy formal context *K*= (*U*, *A*, *R*) with $U = \{x_1, x_2, x_3, x_4, x_5\}, A = \{a, b, c, d, e\}$ and fuzzy relation is described in Table 1. According to Definition 1(2), all the corresponding crisp-fuzzy concepts are showed in Table 2. In addition, Fig.1 is the crisp-fuzzy concept lattice *L* (*U*, *A*, *R*).

Table 1 A fuzzy formal context (*U*, *A*, *R*)

R	a	b	c	d	e
x_1	0.5	0.6	1.0	1.0	1.0
x_2	0.7	0.7	0.9	0.9	0.9
x_3	0.7	0.9	1.0	0.9	0.9
x_4	0.5	0.5	0.1	0.1	0.1
x_5	0.7	1.0	1.0	0.9	0.9

Table 2 All crisp-fuzzy concepts generated from Table 1

Crisp-fuzzy concepts	
C_1	$(x_1x_2x_3x_4x_5, a^{0.5} + b^{0.5} + c^{0.1} + d^{0.1} + e^{0.1})$
C_2	$(x_1x_2x_3x_5, a^{0.5} + b^{0.6} + c^{0.9} + d^{0.9} + e^{0.9})$
C_3	$(x_2x_3x_5, a^{0.7} + b^{0.7} + c^{0.9} + d^{0.9} + e^{0.9})$
C_4	$(x_1x_3x_5, a^{0.5} + b^{0.6} + c^{1.0} + d^{0.9} + e^{0.9})$
C_5	$(x_3x_5, a^{0.7} + b^{0.9} + c^{1.0} + d^{0.9} + e^{0.9})$
C_6	$(x_1, a^{0.5} + b^{0.6} + c^{1.0} + d^{1.0} + e^{1.0})$
C_7	$(x_5, a^{0.7} + b^{1.0} + c^{1.0} + d^{0.9} + e^{0.9})$
C_8	$(\phi, a^{1.0} + b^{1.0} + c^{1.0} + d^{1.0} + e^{1.0})$

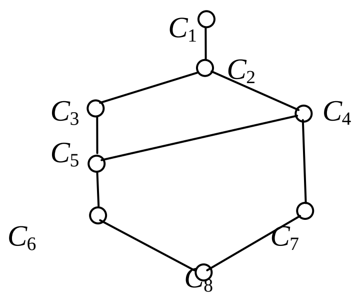

Fig.1 crisp-fuzzy concept lattice *L*(*U*,*A*,*R*)

Definition 3 (1) (Bondy and Murty, 2008) A graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of nonempty set $V(G)$ of a vertexes, a set $E(G)$, disjoint from $V(G)$, of edges, and an function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertexes of G. If e is an edge and u and v are vertexes such that $\psi_G(e) = uv$, then e is said to join u and v; The vertexes *u* and *v* are called the ends of *e*.

(3) (Bondy and Murty, 2008) A directed graph *D* is an ordered triple (*V*(*D*),*A*(*D*),*d*(*v*)) consisting of a

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nonempty set $V(D)$ of vertices, a set $A(D)$, disjoint from $V(D)$, of arcs, and ordered pair of vertices of D. If a is an arc and u and v are vertices such that $d(v)(a)=(u, v)$,
then a is said to join u to v; u is the tail of a, and v is its head.

Definition 4 (B.Ganter and R.Wille, 1999) Let L be finite lattice and $v \in L$ .We denote $v_* = \vee\{x \in L | x < v\}$, v_* is said to be join-irreducible if $v \neq v_*$.

Lemma 1 (Ganter and Wille, 1999) Let L be finite lattice. Every element in L is a join of some join-irreducible elements.

It should be noted that every crisp-fuzzy concept (X, B) in the concept lattice L (U, A, R) can be represented as a join of object concepts of its extension, that is

$$(X,B)=\vee_{x\in X}(g\circ f(x), f(x)).$$

Generating all crisp-fuzzy concepts by the join-irreducible elements

In this section, we propose a method to find the join-irreducible elements by a weakened object topology graph, and further we can generate all crisp-fuzzy concepts.

Weakened object topology graph in a fuzzy formal context

Definition 5 Let $K=(U, A, R)$ be a fuzzy formal context. A weakened object topology graph, denoted as G, is defined as follows. The vertex set $V(G)$ is U and the edges set $E(G)$ is $x_i x_j$ ($x_i, x_j \in U, i \neq j$), where $x_i x_j$ is defined as:

(i)For $\forall a \in A, R(x_i, a) = R(x_j, a)$, then there is a double-arrow x_i to x_j, that is $x_i \leftrightarrow x_j$;

(ii)For $\forall a \in A, R(x_i, a) \leq R(x_j, a)$, then there is a single-arrow x_i to x_j, that is $x_i \to x_j$.

Definition 6 In a weakened object topology graph G, for any $x_i \in V(G)$, its granule of knowledge can be derive from the arrow relation, denoted as

$T^+(x_i) = \{x_j \in V(G) | x_i \to x_j\}$, $T(x_i) = \{x_j \in V(G) | x_i \leftrightarrow x_j\}$.

Example 2 Continuing from Example 1, by the Definitions 5 and 6, we can obtain a weakened object topology graph G (U, A, R) in fuzzy formal context $K=(U, A, R)$, which is shown in Fig.2.

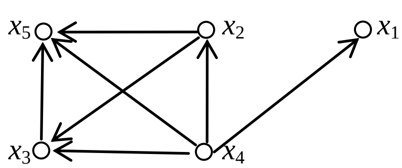

Fig 2 A weakened object topology graph G (U, A, R)

The granule of knowledge for each object $x_i \in V(G)$ is expressed as follows:

$T^+(x_1) = \phi$, $T^+(x_2)=\{x_3, x_5\}$, $T^+(x_3)=\{x_5\}$, $T^+(x_4)=\{x_1, x_2, x_3, x_5\}$, $T^+(x_5) = \phi$.

$T(x_1) = \{x_1\}$, $T(x_2)=\{x_2\}$, $T(x_3)=\{x_3\}$, $T(x_4)=\{x_4\}$, $T(x_5) = \{x_5\}$

How to find the join-irreducible elements

Irreducible elements play a crucial role in the construction of fuzzy formal concept lattices. In this section, we discuss the property of the join-irreducible elements obtained from a weakened object topology graph in a fuzzy formal context (see Theorems 1 and 2). In addition, we debate how to determine whether a crisp-fuzzy concept is a join-irreducible elements in a fuzzy concept lattice (see Theorems 3 and 4).

Theorem 1 Let *K*= (*U*, *A*, *R*) be a fuzzy formal context, *G* is the corresponding weakened object topology graph, for $\forall a \in A, x \in U$, then

$f(T^+(x_i))(a) = f(x_i)(a) \qquad f(T(x_i))(a) = f(x_i)(a)$

Proof: In the light of Definitions 4 and 5, we know that $T^+(x_i) = \{x_j \in V(G) | x_i \to x_j\}$, then $\forall a \in A, R(x_i, a) \le R(x_j, a)$, we can obtain

$f(T^+(x_i))(a) = \wedge_{x_j \in T^+(x_i)} R(x_j, a) = f(x_i)(a) \ \forall a \in A$ by Definition 1 and Property1.

Similarly, we can get $f(T(x_i))(a) = f(x_i)(a)$

Theorem 2 Let *K*= (*U*, *A*, *R*) be a fuzzy formal context, $x_i \in U$ then

$(T^+(x_i) \cup T(x_i), f(T^+(x_i)))$ is a crisp-fuzzy concept and $T^+(x_i) \cup T(x_i) = g \circ f(x_i)$.

Proof: Firstly, it can be deduced from theorem 1 that $g \circ f(T^+(x_i) \cup T(x_i)) = g \circ (f(T^+(x_i)) \wedge f(T(x_i))) = g \circ f(x_i)$. According to Property1 (ii), we elicit

$$T^+(x_i) \cup T(x_i) \subseteq g \circ f(T^+(x_i) \cup T(x_i)) = g \circ f(x_i) \qquad \text{(i)}$$

Secondly, for any $x \in g \circ f(T^+(x_i) \cup T(x_i))$, we have $f(x_i) = f \circ g \circ f(T^+(x_i) \cup T(x_i)) \le f(x)$ on the basis of Property1 (i) and (iii). In other words, this means, for $\forall a \in A$,

$f(x_i) = R(x_i, a) \le R(x, a) = f(x)$ .Therefore $x \in T^+(x_i)$ and we can get

$$T^+(x_i) \cup T(x_i) \supseteq g \circ f(T^+(x_i) \cup T(x_i)) \qquad \text{(ii)}$$

In summary, based on (i) and (ii), we can confirm that

$g \circ f(T^+(x_i) \cup T(x_i)) = g \circ f(x_i) = T^+(x_i) \cup T(x_i)$

and then $(T^+(x_i) \cup T(x_i), f(T^+(x_i)))$ is a crisp-fuzzy concept.

It is known from Theorems 1 and 2 that the relationship is introduced between a granule of knowledge and a pair of operators *f* and *g*. For the sake of determining whether a crisp-fuzzy concept is a join-irreducible element, we deduce the Theorems 3 and 4 as follows.

Theorem 3 Let *K*= (*U*, *A*, *R*) be a fuzzy formal context, $x_i \in U$ if $|T^+(x_i)| \le 1$, and then the object concept $(T^+(x_i) \cup T(x_i), f(x_i))$ is a join-irreducible element in the fuzzy formal concept lattice.

Proof: Suppose that $|T^+(x_i)| \le 1$, then $|T^+(x_i)| = 0$ or $|T^+(x_i)| = 1$.

If $|T^+(x_i)| = 0$, it means that there is no object $y \in U$ in $T^+(x_i)$, hence,

$(T^+(x_i) \cup T(x_i), f(x_i))$ is a join-irreducible element according to Lemma 1.

If $|T^+(x_i)| = 1$, it means that there is only one object $y \in U$ in $T^+(x_i)$, and it can be expressed as $(T^+(y) \cup T(y), f(y))$. By the Definitions 5 and 6, we have $f(x_i) \le f(y)$, that is

$(T^+(x_i) \cup T(x_i), f(x_i)) \le (T^+(y) \cup T(y), f(y))$. Thus, it can be affirm that

$(T^+(x_i) \cup T(x_i), f(x_i))$ is a join-irreducible element by the Definition 6.

Theorem 4 Let *K*= (*U*, *A*, *R*) be a fuzzy formal context, $x_i \in U$ if

$|T^+(x_i)| = |\{y_1, y_2, \cdots, y_n\}| > 1$ and $\wedge_{j=1}^{n} f(y_j) \ne f(x_i)$, then the object concept

$(T^{+}(x_i)\cup T(x_i), f(x_i))$ is a join-irreducible element in the fuzzy formal concept lattice.

Proof: Suppose that $|T^{+}(x_i)|=|\{y_1,y_2,\cdots,y_n\}|>1$ and $\overset{n}{\underset{j=1}{\wedge}} f(y_j)\neq f(x_i)$.

Owing to $|T^{+}(x_i)|=|\{y_1,y_2,\cdots,y_n\}|>1$, we have $T^{+}(x_i)=\{y_1,y_2,\cdots,y_n\}$ and $f(y_1)>f(x_i)$, $f(y_2)>f(x_i),\cdots,f(y_n)>f(x_i)$. So $\forall j=1,\cdots,n\ (T^{+}(x_i)\cup T(x_i), f(x_i))\leq(T^{+}(y_j)\cup T(y_j), f(y_j))$ can be obtained.

In other words, $\overset{n}{\underset{j=1}{\wedge}} f(y_j)\neq f(a)$ it means

$(T^{+}(x_i)\cup T(x_i), f(x_i))\neq\overset{n}{\underset{j=1}{\wedge}}(T^{+}(y_j)\cup T(y_j), f(y_j))$. According to the Definitions 5 and 6,

it is evident that the object concept $(T^{+}(x_i)\cup T(x_i), f(x_i))$ is a join-irreducible element.

The Theorems 3 and 4 provide a way for finding the join-irreducible elements from a weakened object topology graph. Then, by Lemma 1 we can generate all crisp-fuzzy concepts. The procedure for computing the join-irreducible elements is shown in Algorithm 1.

Algorithm 1 Computing the join-irreducible elements of a fuzzy formal context

Input: A fuzzy formal context *K*= (*U*, *A*, *R*).
Output: Join-irreducible elements set *JI*.
1: Initialize $JI=\phi$;
2: For $\forall a\in A$
3: If $R(x_i,a)=R(x_j,a)$, then $x_i\leftrightarrow x_j$, end if
4: If $R(x_i,a)\leq R(x_j,a)$, then $x_i\rightarrow x_j$, end if
5: End for;
6: For each $x_i\in U$, compute $T^{+}(x_i), T(x_i)$;
7: If $|T^{+}(x_i)|\leq 1$, then $JI=JI\cup(T^{+}(x_i)\cup T(x_i), f(x_i))$ end if ;
8: If $|T^{+}(x_i)|=|\{y_1,y_2,\cdots,y_n\}|>1$ and $\overset{n}{\underset{j=1}{\wedge}} f(y_j)\neq f(x_i)$, then
$JI=JI\cup(T^{+}(x_i)\cup T(x_i), f(x_i))$, end if;
9: Return *JI*.

Example 3 We use the context in Table 1 to examine Algorithm 1.

Continuing from Example 2, we can find the join-irreducible elements according to Algorithm 1. Firstly, we calculate $|T^{+}(x_1)|=|\phi|=0$, $|T^{+}(x_2)|=|\{x_3,x_5\}|=2$

$|T^{+}(x_3)|=|\{x_5\}|=1$, $|T^{+}(x_4)|=|\{x_1,x_2,x_3,x_5\}|=4$, $|T^{+}(x_5)|=|\phi|=0$. Owing to $|T^{+}(x_1)|=0$, $|T^{+}(x_3)|=1$, $|T^{+}(x_5)|=0$, we can infer that $(T^{+}(x_1)\cup T(x_1), f(x_1))$,

$(T^{+}(x_5)\cup T(x_5), f(x_5))$, $(T^{+}(x_3)\cup T(x_3), f(x_3))$ are the join-irreducible elements, due to $|T^{+}(x_2)|=2>1$, $|T^{+}(x_4)|=4>1$ and

$f(x_3)\wedge f(x_5)\neq f(x_2)$, $f(x_1)\wedge f(x_2)\wedge f(x_3)\wedge f(x_5)\neq f(x_4)$ we can conclude that

$(T^{+}(x_2)\cup T(x_2), f(x_2))$, $(T^{+}(x_4)\cup T(x_4), f(x_4))$ are the join-irreducible elements.

The next, we can generate all crisp-fuzzy concepts in the light of Lemma 1, all crisp-fuzzy concepts are shown in Table 2 and the crisp-fuzzy concept lattice is presented in Fig.2.

Conclusion

In this paper, we put forward an idea of constructing a fuzzy concept lattice by using join-irreducible elements, and present how to find the join-irreducible elements from a weakened objects-topology graph. Similarly, irreducible elements have also a vital position in attribute reduction, and then we will devote to attribute reduction of fuzzy formal concept analysis in the future.

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REFERENCES

Bondy JA and Murty USR (2008). *Graph Theory.* Springer, London.

Burusco A and Fuentes R (1994). The study of the L-fuzzy concept lattices. *Math-ware and Soft Computing* **3** 209-218.

Ganter B and Wille R (1999). *Formal concept analysis: Mathematical foundations*, Springer-Verlag. New York.

Krajci S (2003). Cluster based efficient generation of fuzzy concepts. *Neural Network World* **13**(5) 521-530.

Li X and Mingwen Shao (2017). Constructing lattice based on irreducible concepts. *International Journal of Machine Learning & Cybernetics* **8**(1) 1-14.

Shao MW and Leung Y (2016). Granular reducts of formal fuzzy contexts. *Knowledge-Based Systems* **114** 156-166.

T. Zhang (2014). Deep first formal concept search. *The Scientific World Journal* **8** 275679.

Z. Zhang (2018). Constructing L-fuzzy concept lattices without fuzzy Galois closure operation. *Fuzzy Sets and Systems* **333** 71-86.