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Research Article

A MEIR-KEELER TYPE FIXED POINT THEOREM

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ABSTRACT
We are improving the result of Badshah and Gagrani, (2007) by removing the assumption of continuity, relaxing compatibility to weak compatibility property and replacing the completeness of the space with a set of four alternative conditions for four functions satisfying an inequality.

Keywords: *Meir-Keeler Type Contractive Condition, Coincidence Point, Weak Compatibility Property*

INTRODUCTION

During the last decade, a large body of literature has grown on common fixed points of compatible maps satisfying various contractive conditions. The most general results of this type deal with common fixed points of four mappings, say A, B, S, T of a Metric space (X. d), and use either a Meir-Keeler type (ε, δ) contractive condition of the form

(1) Given $\in > 0$ there exists $\delta > 0$ such that

$\varepsilon \leq \max \{ d(Sx, Ty), d(Ax, Sx), d(By, Ty),$

$[d(Sx, By) + d(Ax, Ty)]/2 \} < \varepsilon + \delta$

$\Rightarrow d(Ax, By) < \varepsilon$

or a ϕ - contractive condition of the from

(2) $d(Ax, By) \leq \phi(\max \{ d(Sx, Ty), d(Ax, Sx), d(By, Ty),$

$[d(Ax, Ty) + d(By, Sx)] / 2\}$

Where $\phi : R^+ \to R^+$ is such that $\phi(t) < t$ for each $t > 0$ or some generalized version of these conditions which is applicable to sequences of mapping. The contractive condition (2) does not ensure the existence of a fixed point unless some additional condition is assumed on the function ϕ. The following conditions on the function ϕ, which were introduced by various authors, are known to ensure a common fixed point under the contractive condition (2).

(i) $\phi(t)$ is non decreasing and t/t-$\phi(t)$ is non-increasing (Carbone *et al.*, 1989).

(ii) $\phi(t)$ is non decreasing and $\lim_n \phi^n(t) = 0$ for each $t > 0$ (Jachymski, 1994).

(iii) ϕ is upper, semi-continuous, and $\phi(t) < t$, for each $t > 0$ (Boyd and Wong, 1969) or equivalently.

It is now known (Jachymski, 1995) that if any of the conditions (i), (ii), (iii) or (iv) is assumed on ϕ, then a ϕ-contractive condition (2) implies an analogous.

(ε, δ)-contractive condition (1) and both the contractive conditions hold simultaneously. Similarly, a Meir-keelar type (ε, δ)- contractive condition does not ensure the existence of a fixed point.

The following example illustrates that an (ε, δ) contractive condition of type (1) neither ensures the existence of a fixed point nor implies an analogous ϕ contractive condition (2)

Example: (Pant *et al.*, 2001) Let X = [0, 2] and d be the Euclidean mteric on X. Define

f: $X \to X$ by $f(x) = \left(\frac{1+x}{4}\right)$

If $x > 1$; $f(x) = 0$ if $x \geq 1$.

Then, it satisfies the contractive condition

$\varepsilon \leq \max \{d(x, y), d(x, fx), d(y, fy),$

$[d(x, fy) + d(y, fx)]/2\} < \varepsilon + \delta$

$\Rightarrow d(fx, fy) < \varepsilon$

with $\delta(\varepsilon) = 1$, for $\varepsilon \geq 1$ and $\delta(\varepsilon) = 1-\varepsilon$, for $\varepsilon < 1$ but f does not have a fixed point. Also f does not satisfy the contractive condition.
$d(fx, fy) \leq \phi \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy) + d(y, fx)] /2\}$
Since the desire function $\phi(t)$ cannot be defined at $t = 1$.
Hence, the two type of condition (1) and (2) are independent of each other. Thus, to ensure the existence of common fixed point under the contractive condition (1), the following conditions on the function δ have been introduced and used by various authors.
(v) δ is non decreasing (Pant *et al.*, 2001)
(vi) δ is lower semi-continuous (Jungck, 1986)
Jachymski (1994) has shown that the (ε, δ) contractive condition (1) with a non-decreasing δ implies a ϕ-contractive condition (2). Also, Pant *et al.*, (2001) have shown that (ε, δ)-contractive condition (2) with a lower semi continuous δ, implies a ϕ-contractive condition (2). Thus, we see that if additional conditions are assumed on δ then the (ε, δ)-contractive condition (1) implies an analogous ϕ-contractive condition (2) and both the contractive conditions hold simultaneously.
It is thus, clear that contractive condition (1) & (2) hold simultaneously whenever (1) or (2) is assumed with an additional condition on δ or ϕ respectively. It follows, therefore, that the known common fixed point theorems can be extended and generalized if instead of assuming one of the contractive condition (1) or (2) with additional condition on δ and ϕ, we assume contractive condition (1) together with the following condition of form.
$d(Ax, By) < k[d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) + d(Sx, By) + d(Ax, Ty)]$;
for $0 \leq k \leq 1/3$

RESULTS AND DICUSSION

Main Result

Recently, Badshah and Gagrani (2007) have proved following common fixed point theorem for four mappings.
Theorem 1: Let (A, S) and (B, T) be compatible pairs of self mappings of a complete Metric space (X, d) such that
(a) $A(X) \subset T(X)$, $B(X) \subset S(X)$
(b) Given $\varepsilon > 0$ there exist a $\delta > 0$ such that for all x, y in X,
$\varepsilon \leq \max k\{d(Sx, Ty), d(Ax, Sx), d(By, Ty),$
$[d(Sx, By) + d(Ax, Ty)]/2\} < \varepsilon + \delta$
$\Rightarrow d(Ax, By) < \varepsilon$ and
(c) $d(Ax, By) \leq \max\{d(Sx, Ty), \{d(Ax, Sx) + d(By, Ty)]/2,$
$k[d(Sx, By) + d(Ax, Ty)]\}$; for $l \leq k < 2$.
If one of the mappings A, B, S and T is continuous then A, B, S and T have unique common fixed point.
Now, we are improving Theorem 1 by removing the assumption of continuity, relaxing compatibility to weak compatibility property and replacing the completeness of the space with a set of four alternative conditions for four functions satisfying an inequality-
To prove our theorem, we shall use the following lemma of Jachymski (1994).
Lemma 2 – Let A, B, S and T be self maps of a metric space (X, d) such that
$A(X) \subset T(X)$ and $B(X) \subset S(X)$. Assume that further that given $\in > 0$, there exist a
$\delta > 0$ such that, for all x, y in X
$\varepsilon < M(x, y) < \varepsilon + \delta$ implies that $d(Ax, By) \leq \varepsilon$ (3)
For all x, y in X with $M(x, y) > 0$, $d(Ax, By) < M(x, y)$ (4)
Where, $M(x,y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty) + d(By, Sx)/2\}$
Then, for each x_0 in X any sequence $\{y_n\}$, being an S, T-iteration of x_0 under A and B is a cauchy sequence.

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Let $\{A_i, i = 1, 2, 3\ldots\}$, S and T be self mapping of a Metric space (X, d). In the sequel we shall denote

$M_{1i} = (x, y) = \max \{d(Sx, Ty)\ d(A_1x, Sx), d(A_iy, Ty) + [d(A_1x, Ty) + d(A_iy, Sx)]/2\}$

Theorem 2: Let $\{A_i = 1, 2, 3,\ldots\}$, S and T be self mappings of a metric space (X, d) such that

(a) $A_1(X) \subset T(X)$, $A_i(X) \subset S(X)$, $i > 1$

(b) Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$\varepsilon \leq M_{12}(x, y) < \varepsilon + \delta$

$\Longrightarrow d(A_1x, A_2y) < \varepsilon$

(c) $d(A_1x, A_iy) \leq \phi_i \max \{d(Sx, Ty)\ d(A_1x, Sx) + d(A_iy, Ty)]/2 + [d(Sx, A_iy) + d(A_1x, Ty)]\}$,

Where, $\phi_i : R^+ \rightarrow R^+$ is such that $\phi_i(t) < t$ for each $t > 0$.

If one of $A_1(X)$, $B(X)$, $S(X)$ is a complete subspace of X, then

(i) A_1 and S have a coincidence point,

(ii) A_i, $i > 1$ and T have a coincidence point.

Moreover, if the pair (A_1, S) and (A_i, T), $i > 1$ and weakly compatible then all the A_i, S and T have a unique common fixed point.

Proof: Let x_0 be any point in X. Define sequence $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$y_{2n} = A_ix_{2n} = Tx_{2n+1}$

$y_{2n+1} = A_2x_{2n+1} = Sx_{2n+2}$

This can be done by virtue of (a). We claim that $\{y_n\}$ is a cauchy sequence. Two cases arise. Either $y_n = y_{n+1}$ for some n or $y_n \neq y_{n+1}$ for each n.

If $y_n = y_{n+1}$ for some n then, as shown by **Carbone** *et al.* (1989). $y_n = y_{n+k}$ for each $k \geq 1$. For instance suppose that $y_{2m} = y_{2m+1}$. Then,

$y_{2m+1} = y_{2m+2}$. Otherwise, using (b) we get

$d(y_{2m+1}, y_{2m+2}) < M_{12}(x_{2m+2}, x_{2m+1}) = d(y_{2m+1}, y_{2m+2})$

which is a contradiction

Hence, $y_{2m+1} = y_{2m+2}$ implies that $y_{2m+2} = y_{2m+3}$

Proceeding in this manner it follows that $y_{2m} = y_{2m+k}$ for each $k \geq 1$ and $\{y_n\}$ is a cauchy sequence. Let us, therefore, consider the case when $y_n \neq y_{n+1}$ for each n. In this case, using (b) we obtain.

$d(y_{2m}, y_{2n+1}) < d(y_{2n-1}, y_{2n})$

$d(y_{2n-1}, y_{2n}) < d(y_{2n-2}, y_{2n-1})$

Thus, $\{d(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive numbers and, therefore, tends to a limit $r \geq 0$. If possible, suppose $r > 0$. Then, given $\delta > 0$ there exists a positive integer N such that for each $n \geq N$, we have

$r < d(y_{2n}, y_{2n+1}) = M_{12}(x_{2n+2}, x_{2n+1}) < r + \delta$ (5)

Selecting δ in (5) in accordance with (b), for each $n \geq N$, we get

$d(y_{2n+2}, y_{2n+1}) = d(A_1x_{2n+2}, A_2x_{2n+1}) < r$ this, in turn, gives

$d(y_{2n+3}, y_{2n+2}) < d(y_{2n+1}, y_{2n+2}) < r$, contradicting (5) hence,

$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$

We now show that $\{y_n\}$ is a cauchy sequence. suppose it is not. Then, there exists an $\in > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $d(y_{n_i}, y_{n_i}+1) > 2\in$. Select δ in (b) so that

$0 < \delta \leq \in$.

Since $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, there exists an integer N such that

$d(y_n, y_{n+1}) < \delta/6$; whenever $n \geq N$.

Let $n_i \geq N$, then there exist integers m_i satisfying $n_i < m_i < n_{i+1}$

Such that $d(y_{n_i}, y_{m_i}) \geq \in + (\delta/3)$. If not, then

$d(y_{n_i}, y_{n_{i+1}}) \leq d(y_{n_i}, y_{n_{i+1}}-1) + d(y_{n_{i+1}}-1, y_{n_{i+1}})$
$< \varepsilon + (\delta/3) + (\delta/6) < 2 . \varepsilon,$

a contradiction. Without loss of generality, we can assume n_i, to be odd. Let be the smallest even integer such that $d(y_{n_i}, y_{m_i}) \geq \varepsilon + (\delta/3)$. Then,

$d(y_{n_i}, y_{m_i}-2) < \varepsilon + (\delta/3)$ and

$\varepsilon + (\delta/3) \leq d(y_{n_i}, y_{m_i}) \leq + d(y_{n_i}, y_{m_i}-2) + d(y_{m_i}-2, y_{m_i}-1) + (y_{m_i}-1, y_m)$
$< \varepsilon + (\delta/3) + (\delta/6) + (\delta/6) = \varepsilon + (2\delta/3)$ (2)

Also $d(y_{n_i}, y_{m_i}) \leq M_{12}(X_{n_{i+1}}, X_{m_{i+1}}) < \in + (2\delta/3) + (\delta/6) < \in + \delta$

that is, $\in + (\delta/3) \leq M_{12}(X_{n_{i+1}}, X_{m_{i+1}}) < \in + \delta$.

In view of (b), this yields $d(y_{n_i}, y_{m_{i+1}}) < \in$. But then

$d((y_{n_i},) \leq d(y_{n_i}, y_{m_{i+1}}) + d((y_{n_i}, y_{m_{i+1}}) + d(y_{m_{i+1}}, y_{m_i})$
$< (\delta/6) + \in + (\delta/6) = \in + (\delta/3)$

Which is a contradiction (4)

Hence, $\{y_n\}$ is a cauchy sequence. In X Now, suppose that T (X) is a complete subspace of X. Then, the subsequence $y_{2n} = Tx_{2n+1}$ is a cauchy sequence in T (X) and hence, has a limit u.

Now, we show that $A_i x_{2n+1} \rightarrow u$ for each $i > 1$.

If $\lim_{n\rightarrow\infty} A_i x_{2n+1} \neq u$ for some $i > 1$. Then, either $\lim_{n\rightarrow\infty} A_i x_{2n+1} = w \neq u$ or $\lim_{n\rightarrow\infty} A_i x_{2n+1}$ does not exist.

In the later case either sequence $\{A_i x_{2n+1}\}$ is unbounded or has at least two limit points. However, in each of these cases there exists a subsequence $\{A_i x_{2m+1}\}$ and a number $r > 0$ such that $d(A_i x_{2m}, A_i x_{2m+1}) \geq r$, $d(A_i x_{2m+1}, u) \geq r/6$

while by virtue of (4),

$d(A_1 x_{2m}, Sx_{2m}) < r/6$, $d(A_1 x_{2m+1}, Tx_{2m+1}) < r/6$, $d(Sx_{2m}, Tx_{2m+1}) > r/6$, for all m sufficiently large. Using (b) and (c), for all large m, we get,

$d(A_1 x_{2m}, A_i x_{2m+1}) < \max\{d(Sx_{2m}, Tx_{2m+1}), d(A_i x_{2m}, Sx_{2m})\, d(A_i x_{2m+1}, Tx_{2m+1})$
$[d(Sx_{2m}, A_i x_{2m+1}) + d(A_i x_{2m}, Tx_{2m+1})]\}$

$d(A_1 x_{2m}, A_i x_{2m+1}) < \max\{r/6, r/6, d(A_i x_{2m+1}, Tx_{2m+1})\, [0 + d(A_i x_{2m+1}, Sx_{2m})]/2\}$
$\leq \max\{d(A_i x_{2m+1}, Tx_{2m+1})\, [d(A_i x_{2m+1}, Ax_{2m}) + d(A_i x_{2m}, Sx_{2m})]/2\}$
$\leq \max\{d(A_i x_{2m+1}, A_1 x_{2m})\, [d(A_i x_{2m+1}, Ax_{2m}) + r/6]/2\}$
$= d(A_1 x_{2m}, A_1 x_{2m+1})$

Which is a contradiction. Hence,

$\lim A_i x_{2n+1} = u$, $i > 1$, $n \rightarrow \infty$

Let $v = T^{-1} u$, then $Tv = u$. Since y_{2n} is convergent, then y_{2n} is convergent to u and y_{2n+1} also converge to u.

For any $k > 1$,

Setting $x = x_{2n}$ and $y = v$ in (c) we have

$d(A_i x_{2n}, A_k v) \leq \phi\, k \max\{d(Sx_{2n}, Tv), [d(A_1 x_{2n}, Sx_{2n}) +$
$d(A_k v, Tv]/2, [d(Sx_{2n}, A_k v) + d(A_1 x_{2n}, Tv)]\}$

Letting n tend to infinity, we obtain

$d(u, A_k v) \leq \phi\, k \max\{d(u, u), [d(u, u) + d(A_k v, u]/2, [d(u, A_k v) + d(u, u)]\}$

$d(u, A_k v) \leq \phi\, k \max d(u, A_k v)$

which is a contradiction. Hence, $u = A_k v$.
There T and A_k have a coincidence point.
Since $A_k (X) \subset S (X)$, $u = A_k v$ implies that $u \in S (X)$
Let $w \in S^{-1} u$, then $Sw = u$. Setting $X = w$ and $y = x_{2n+1}$, we obtain by (C)
$d (A_1 w, A_k x_{2n+1}) \leq \phi$ k max {d (Sw, Tx_{2n+1}), [d (A_1w, Sw) +
d ($A_k x_{2n+1}$, Tx_{2n+1}]/2, [d (Sw, $A_k x_{2n+1}$) + d ($A_1 w_i$, Tx_{2n+1})]}
Letting $n \to \infty$
$d (A_1 w, u) \leq \phi$ k max { d (u, u), [d (A_1w, u) +
d (u, u]/2, d (u, u) + d (u, u) + (A_iw, u)]}
$d (A_1 w, u) \leq \phi$ k d ($A_1 w_i$, u)
We have $A_i w = u$. Hence, A_i and s have a coincidence point. If one assumes that S (X) is complete, then analogous arguments establish the existence of a coincidence point.
The remaining two cases are essentially the same as the previous cases. Indeed if A_i (X) is complete by (a)
$u \in A_i (X) \subset T (X)$
Then, (d) and (e) are completely established
By u = Tv = Bv and by the weak compatibility of (A_i T), i > 1 we have
$A_i u = A_i Tv = TA_i v = Tu$
By u = Sw = Aw and by the weak compatibility of (A, S) we have
$A_1 u = A_1 Sw = SA_1 w = Su$
By (c) we have successively.
$d (A_1 w, A_k u) \leq \phi$ k max {d (Sw, Tu), [d (A_1 w, Sw) +
d (A_ku, Tu]/2 , [d (Sw, A_ku) + d ($A_i w_i$, Tu)]}
$d (u, A_k u) \leq \phi$ k max {d (u, A_1u), [d (u, u) + d (A_iu, A_iu)]/2 d (u, A_iu) + d (u, A_iu)]}
$= \phi$ k max {d (u, A_iu), 0, 2d (u, A_iu)}
$d (u, A_k u) \leq \phi_k$ 2 d (u, A_iu)
which implies that
$u = A_k u$
Similarly, one can show that $u = A_1 u$. Thus,
$u = A_i u = Tu = A_i u = Su$
The common fixed point of A_i {i = 1, 2, 3..............} S, T is u.

Uniqueness

Suppose that A, B, S, T have two common fixed point u and u'. Then, by (c) we have successively
$d (A_1 u, A_k u') \leq \phi$ k max {d (Su, Tu′), [d (A_1u, Su) + d (A_k u′, T u′)]/2 d (Su, A_k u′) +
d (A_iu, T u′)] }
$d (u, u') \leq \phi$ k max {d (u, u′), [d (u, u) + d (u′, u′)]/2 d (u, u′) + d (u, u′)]}
$d (u, u') \leq \phi_k$ 2 d (u, u′)
which implies that
$u = u'$
which is a contradiction
Hence, A_i {i = 1, 2, 3..............} S and T have a unique common fixed point u.
We now give an example to illustrate the above theorem
Example: Let X = [2, 20] and d be the usual metric on X. Define A_i S,T $X \to X_1$ i = 1, 2, …………. as follows
$A_1 x = 2$, for each x
$Sx = x$ is $x \leq 8$, $Sx = 8$ if $8 < x \leq 14$ $Sx = (x + 10) / 3$ if $x > 14$.
$Tx = 2$ if $x = 2$ or $x \geq 5$, $Tx = 12 + x$ if $2 < x < 4$
$Tx = 9 + x$ if $4 \leq x < 5$
$A_2 x = 2$ if $x < 4$ or $x \geq 5$, $A_2 x = 3 + x$ if $4 \leq x < 5$

and for each $i > 2$

$A_i x = 2$ if $x = 2$ or $x \geq 4$, $A_i x = (30 + x)/4$ if $2 < x < 4$

Then $\{A_i\}$, S and T satisfy all the conditions of the above theorem and have a unique common fixed point $x = 2$.

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