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Research Article

NEW PROPERTIES OF THE NATARAJAN METHOD OF SUMMABILITY FOR DOUBLE SEQUENCES

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ABSTRACT
In this paper, we study some properties of the $(M, \lambda_{m,n})$ method of summability introduced earlier by the author in (Natarajan (to appear)).

Keywords: *Double Sequence, Double Series, $(M, \lambda_{m,n})$ Method (or the Natarajan Method) of Summability, Cauchy Product, Inclusion, Equivalence, Iteration Product*

INTRODUCTION AND PRELIMINARIES
To make the paper self-contained, we recall the following (Natarajan, 2014):
Definition 1.1:
Let $\{x_{m,n}\}$ be a double sequence. We say that

$$\lim_{m+n\to\infty} x_{m,n} = x,$$

if for every $\varepsilon > 0$, the set

$$\{(m, n) \in \mathbb{N}^2 : |x_{m,n} - x| \geq \varepsilon\}$$

is finite, $\mathbb{N}$ being the set of positive integers. In such a case, x is unique and x is called the limit of the double sequence $\{x_{m,n}\}$. We also say that $\{x_{m,n}\}$ converges to x.
Definition 1.2:
Let $\{x_{m,n}\}$ be a double sequence. We say that

$$\sum_{m,n=0}^{\infty,\infty} x_{m,n} = s,$$

if

$$\lim_{m+n\to\infty} s_{m,n} = s,$$

where,

$$s_{m,n} = \sum_{k,\ell=0}^{m,n} x_{k,\ell}, \quad m, n = 0,1,2,....$$

In such a case, we say that the double series $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ converges to s.
Remark 1.3:
If $\lim_{m+n\to\infty} x_{m,n} = x$, then the double sequence $\{x_{m,n}\}$ is bounded.
It is easy to prove the following results.
Theorem 1.4:

$$\lim_{m+n\to\infty} x_{m,n} = x,$$

if and only if
(i) $\lim_{m\to\infty} x_{m,n} = x, \quad n = 0, 1, 2, ...;$
(ii) $\lim_{n\to\infty} x_{m,n} = x, \quad m = 0, 1, 2, ...;$
and
(iii) for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$|x_{m,n} - x| < \varepsilon, \quad m, n \geq N,$
which we write as

$$\lim_{m,n\to\infty} x_{m,n} = x$$

(Note that this is Pringsheim's definition of limit of a double sequence).

Theorem 1.5:

If the double series $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ converges, then,

$$\lim_{m+n\to\infty} x_{m,n} = 0.$$

However, the converse is not true.

Definition 1.6:

$\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ is said to converge absolutely, if $\sum_{m,n=0}^{\infty,\infty} |x_{m,n}|$ converges.

Note that if $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ converges absolutely, it converges. However, the converse is not true.

Some Properties of the (M, $\lambda_{m,n}$) Method or the Natarajan Method

Definition 2.1:

Given a 4-dimensional infinite matrix $A = (a_{m,n,k,\ell})$, m, n, k, ℓ = 0, 1, 2, ... and a double sequence $\{x_{k,\ell}\}$, k, ℓ = 0, 1, 2, ..., by the A-transform of $x = \{x_{k,\ell}\}$, we mean the sequence $A(x) = \{(Ax)_{m,n}\}$, where,

$$(Ax)_{m,n} = \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell} x_{k,\ell}, \quad m, n = 0,1,2,...,$$

assuming that the double series on the right converge. If $\lim_{m+n\to\infty} (Ax)_{m,n} = s,$ we say that the double sequence $x = \{x_{k,\ell}\}$ is A-summable or summable A to s, written as,

$x_{k,\ell} \to s(A).$

If $\lim_{m+n\to\infty} (Ax)_{m,n} = s,$ whenever $\lim_{k+\ell\to\infty} x_{k,\ell} = s,$ we say that the 4-dimensional infinite matrix A is "regular".

The following important theorem on the regularity of a 4-dimensional infinite matrix was proved by Natarajan (2014).

Theorem 2.2 (Silverman-Toeplitz):

The 4-dimensional infinite matrix $A = (a_{m,n,k,\ell})$ is regular if and only if

$$\sup_{m,n\geq 0} \sum_{k,\ell=0}^{\infty,\infty} |a_{m,n,k,\ell}| < \infty; \tag{2.1}$$

$$\lim_{m+n\to\infty} a_{m,n,k,\ell} = 0, \quad k, \ell = 0,1,2,...; \tag{2.2}$$

$$\lim_{m+n\to\infty} \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell} = 1; \tag{2.3}$$

$$\lim_{m+n\to\infty} \sum_{k=0}^{\infty} |a_{m,n,k,\ell}| = 0, \quad \ell = 0,1,2,...; \tag{2.4}$$

and

$$\lim_{m+n\to\infty} \sum_{\ell=0}^{\infty} |a_{m,n,k,\ell}| = 0, \quad k = 0,1,2,.... \tag{2.5}$$

(Natarajan (to appear)) introduced the (M, $\lambda_{m,n}$) method for double sequences and extended some of the results of the (M, λ_n) method for simple sequences.

Definition 2.3:

Let $\{\lambda_{m,n}\}$ be a double sequence such that $\sum_{m,n=0}^{\infty,\infty} |\lambda_{m,n}| < \infty$. The $(M, \lambda_{m,n})$ method is defined by the 4-dimensional infinite matrix $(a_{m,n,k,\ell})$, where,

$$a_{m,n,k,\ell} = \begin{cases} \lambda_{m-k,n-\ell}, & 0 \le k \le m, 0 \le \ell \le n; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.4:

We say that $(M, \lambda_{m,n})$ is included in $(M, \mu_{m,n})$ (or $((M, \mu_{m,n})$ includes $(M, \lambda_{m,n}))$, written as,
$(M, \lambda_{m,n}) \subseteq (M, \mu_{m,n})$ (or $(M, \mu_{m,n}) \supseteq (M, \lambda_{m,n})$),
If $s_{k,\ell} \to \sigma(M, \lambda_{m,n})$ implies that $s_{k,\ell} \to \sigma(M, \mu_{m,n})$ too.
The methods $(M, \lambda_{m,n})$, $(M, \mu_{m,n})$ are said to be equivalent if $(M, \lambda_{m,n}) \subseteq (M, \mu_{m,n})$ and vice versa.
It is easy to prove the following result.

Theorem 2.5: (see Natarajan (to appear))
The method $(M, \lambda_{m,n})$ is regular if and only if

$$\sum_{m,n=0}^{\infty,\infty} \lambda_{m,n} = 1. \tag{2.6}$$

Analogous to Theorem 176 of Hardy (1949), we have the following result in the context of double sequences and double series.

Theorem 2.6:

If $\lim_{m+n\to\infty} a_{m,n} = 0$ and $\sum_{m,n=0}^{\infty,\infty} |b_{m,n}| < \infty$, then

$$\lim_{m+n\to\infty} c_{m,n} = 0,$$

where,

$$c_{m,n} = \sum_{k,\ell=0}^{m,n} a_{m-k,n-\ell} b_{k,\ell}, \quad m, n = 0,1,2,....$$

Proof.

Since $\{a_{m,n}\}$, $\{b_{m,n}\}$ are convergent, they are bounded and so,

$$|a_{m,n}| \le M, \quad |b_{m,n}| \le M, \quad M > 0,\ m, n = 0, 1, 2, ... \tag{2.7}$$

Since $\sum_{m,n=0}^{\infty,\infty} |b_{m,n}| < \infty$, given $\varepsilon > 0$, there exist positive integers M_1, N_1 such that

$$\sum_{m>M_1, n>N_1}^{\infty,\infty} |b_{m,n}| < \frac{\varepsilon}{4M}. \tag{2.8}$$

Since, for fixed $k, \ell = 0, 1, 2, ...$,

$$\lim_{m+n\to\infty} a_{m-k,n-\ell} = 0,$$

we can choose positive integers $M_2 > M_1$, $N_2 > N_1$ such that for $m > M_2$, $n > N_2$, we have,

$$\sum_{\substack{0\le k\le M_1 \\ 0\le \ell\le N_1}} |a_{m-k,n-\ell}| < \frac{\varepsilon}{4M}; \tag{2.9}$$

$$\sum_{\substack{0\le k\le M_1 \\ N_1+1\le \ell\le n}} |a_{m-k,n-\ell}| < \frac{\varepsilon}{4M}; \tag{2.10}$$

and

(2.11) $$\sum_{\substack{M_1+1\le k\le m \\ 0\le \ell\le N_1}} \left|a_{m-k,n-\ell}\right| < \frac{\varepsilon}{4M}.$$

Then, for $m > M_2$, $n > N_2$,

$$\left|c_n\right| = \left|\sum_{k,\ell=0}^{m,n} a_{m-k,n-\ell} b_{k,\ell}\right|$$

$$= \left|\sum_{\substack{0\le k\le M_1 \\ 0\le \ell\le N_1}} a_{m-k,n-\ell} b_{k,\ell} + \sum_{\substack{0\le k\le M_1 \\ \ell > N_1}} a_{m-k,n-\ell} b_{k,\ell} + \sum_{\substack{k>M_1 \\ 0\le \ell\le N_1}} a_{m-k,n-\ell} b_{k,\ell} + \sum_{\substack{k>M_1 \\ \ell > N_1}} a_{m-k,n-\ell} b_{k,\ell}\right|$$

$$\le \sum_{\substack{0\le k\le M_1 \\ 0\le \ell\le N_1}} \left|a_{m-k,n-\ell}\right| \left|b_{k,\ell}\right| + \sum_{\substack{0\le k\le M_1 \\ \ell > N_1}} \left|a_{m-k,n-\ell}\right| \left|b_{k,\ell}\right| + \sum_{\substack{k>M_1 \\ 0\le \ell\le N_1}} \left|a_{m-k,n-\ell}\right| \left|b_{k,\ell}\right| + \sum_{\substack{k>M_1 \\ \ell > N_1}} \left|a_{m-k,n-\ell}\right| \left|b_{k,\ell}\right|$$

$$< M\frac{\varepsilon}{4M} + M\frac{\varepsilon}{4M} + M\frac{\varepsilon}{4M} + M\frac{\varepsilon}{4M}$$

$$= \varepsilon, \quad \text{using } (2.7), (2.8), (2.9), (2.10) \text{ and } (2.11).$$

It now follows that

$$\lim_{m+n\to\infty} c_{m,n} = 0,$$

completing the proof of the theorem. □

We now have the following results on the cauchy multiplication of $(M, \lambda_{m,n})$-summable double sequences and double series.

Theorem 2.7:

If $\sum_{m,n=0}^{\infty,\infty} \left|a_{m,n}\right| < \infty$ and $\{b_{m,n}\}$ is $(M, \lambda_{m,n})$-summable to B, then $\{c_{m,n}\}$ is $(M, \lambda_{m,n})$-summable to AB, where,

$$c_{m,n} = \sum_{k,\ell=0}^{m,n} a_{m-k,n-\ell} b_{k,\ell}, \quad m,n = 0,1,2,...$$

and

$$\sum_{m,n=0}^{\infty,\infty} a_{m,n} = A.$$

Proof.

Let, $\{t_{m,n}\}$, $\{\tau_{m,n}\}$ be the $(M, \lambda_{m,n})$-transforms of $\{b_{m,n}\}$, $\{c_{m,n}\}$ respectively,

$$\text{i.e., } t_{m,n} = \sum_{k,\ell=0}^{m,n} \lambda_{m-k,n-\ell} b_{k,\ell},$$

$$\tau_{m,n} = \sum_{k,\ell=0}^{m,n} \lambda_{m-k,n-\ell} c_{k,\ell}, \quad m,n = 0,1,2,....$$

We can work to see that

$$\tau_{m,n} = \sum_{k,\ell=0}^{m,n} a_{m-k,n-\ell}(t_{k,\ell} - B) + B\left(\sum_{k,\ell=0}^{m,n} a_{k,\ell}\right), \quad m,n = 0,1,2,...,$$

where, $\lim_{k+\ell\to\infty} t_{k,\ell} = B$. Since $\sum_{m,n=0}^{\infty,\infty} \left|a_{m,n}\right| < \infty$ and $\lim_{k+\ell\to\infty} (t_{k,\ell} - B) = 0$, using Theorem 2.6, it follows that

$$\lim_{m+n\to\infty}\left[\sum_{k,\ell=0}^{m,n} a_{m-k,n-\ell}(t_{k,\ell} - B)\right] = 0,$$

so that

$$\lim_{m+n\to\infty} \tau_{m,n} = B\left(\sum_{m,n=0}^{\infty,\infty} a_{m,n}\right)$$
$$= AB,$$

i.e., $\{c_{m,n}\}$ is $(M, \lambda_{m,n})$-summable to AB, completing the proof of the theorem. □

It is easy to prove the following result on similar lines.

Theorem 2.8:

If $\sum_{m,n=0}^{\infty,\infty} |a_{m,n}| < \infty$, $\sum_{m,n=0}^{\infty,\infty} b_{m,n}$ is $(M, \lambda_{m,n})$-summable to B, then $\sum_{m,n=0}^{\infty,\infty} c_{m,n}$ is $(M, \lambda_{m,n})$-summable to AB, where,

$$c_{m,n} = \sum_{k,\ell=0}^{m,n} a_{m-k,n-\ell} b_{k,\ell}, \quad m, n = 0,1,2,...$$

and

$$\sum_{m,n=0}^{\infty,\infty} a_{m,n} = A.$$

As in the case of the Natarajan method (M, λ_n) for simple sequences (Natarajan, 2013), we can prove the following result, using Theorem 2.6.

Theorem 2.9:

Let $(M, \lambda_{m,n})$, $(M, \mu_{m,n})$ be regular methods. Then, $(M, \lambda_{m,n})$ $(M, \mu_{m,n})$ is also regular, where, we define, for $x = \{x_{m,n}\}$,

$((M, \lambda_{m,n})\ (M, \mu_{m,n}))(x) = (M, \lambda_{m,n})\ ((M, \mu_{m,n})(x))$.

We can prove the following results too.

Theorem 2.10:

For given regular methods $(M, \lambda_{m,n})$, $(M, \mu_{m,n})$ and $(M, t_{m,n})$,

$(M, \lambda_{m,n}) \subseteq (M, \mu_{m,n})$

if and only if

$(M, t_{m,n})\ (M, \lambda_{m,n}) \subseteq (M, t_{m,n})\ (M, \mu_{m,n})$.

In view of Theorem 3.6 of (Natarajan (to appear)), we can reformulate Theorem 2.10 as follows:

Theorem 2.11:

Given the regular methods $(M, \lambda_{m,n})$, $(M, \mu_{m,n})$ and $(M, t_{m,n})$, the following statements are equivalent:

(i) $(M, \lambda_{m,n}) \subseteq (M, \mu_{m,n})$;

(ii) $(M, t_{m,n})\ (M, \lambda_{m,n}) \subseteq (M, t_{m,n})\ (M, \mu_{m,n})$;

and

(iii) $\sum_{m,n=0}^{\infty,\infty} |k_{m,n}| < \infty$ and $\sum_{m,n=0}^{\infty,\infty} k_{m,n} = 1$,

where,

$$\frac{\mu(x)}{\lambda(x)} = k(x) = \sum_{m,n=0}^{\infty,\infty} k_{m,n} x^m y^n;$$

$$\lambda(x) = \sum_{m,n=0}^{\infty,\infty} \lambda_{m,n} x^m y^n;$$

and

$$\mu(x) = \sum_{m,n=0}^{\infty,\infty} \mu_{m,n} x^m y^n.$$

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