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Research Article

SOME PROPERTIES OF THE NATARAJAN METHOD OF SUMMABILITY FOR DOUBLE SEQUENCES IN ULTRAMETRIC FIELDS

***P N Natarajan**
Old No. 2/3, New No. 3/3, Second Main Road, R.A. Puram,Chennai 600 028, India
**Author for Correspondence*

ABSTRACT
Throughout the present paper, K denotes a complete, non-trivially valued, ultrametric field. Entries of double sequences, double series and 4-dimensional infinite matrices are in K. In this paper, we study some properties of the $(M, \lambda_{m,n})$ method of summability introduced earlier by the author in Natarajan (2014).

Keywords: *Ultrametric Field, Double Sequence, Double Series, $(M, \lambda_{m,n})$ Method (or the Natarajan Method) of Summability, Cauchy Product, Inclusion, Equivalence, Iteration Product*

INTRODUCTION AND PRELIMINARIES
Throughout the present paper, K denotes a complete, non-trivially valued, ultrametric field. Entries of double sequences, double series and 4-dimensional infinite matrices are in K.
To make the paper self-contained, we recall the following (Natarajan and Srinivasan (2002):

Definition 1.1: Let $\{x_{m,n}\}$, m, n = 0, 1, 2, ... be a double sequence in K. Let $x \in K$. We say that

$$\lim_{m+n\to\infty} x_{m,n} = x,$$

if for every $\varepsilon > 0$, the set

$$\{(m,n) \in \mathbb{N}^2 : |x_{m,n} - x| \geq \varepsilon\}$$

is finite, $\mathbb{N}$ being the set of positive integers. In such a case, x is unique and x is called the limit of the double sequence $\{x_{m,n}\}$. We also say that $\{x_{m,n}\}$ converges to x.

Definition 1.2: Let $\{x_{m,n}\}$ be a double sequence in K and $s \in K$. We say that

$$\sum_{m,n=0}^{\infty,\infty} x_{m,n} = s,$$

if,

$$\lim_{m+n\to\infty} s_{m,n} = s,$$

where,

$$s_{m,n} = \sum_{k,\ell=0}^{m,n} x_{k,\ell}, \quad m,n = 0,1,2,....$$

In such a case, we say that the double series $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ converges to s.

Remark 1.1: If $\lim_{m+n\to\infty} x_{m,n} = x$, then the double sequence $\{x_{m,n}\}$ is bounded.

It is easy to prove the following results.

Theorem 1.1:

$$\lim_{m+n\to\infty} x_{m,n} = x,$$

if and only if

(i) $\lim_{m\to\infty} x_{m,n} = x$, n = 0, 1, 2, ...;

(ii) $\lim_{n\to\infty} x_{m,n} = x$, m = 0, 1, 2, ...;

and
(iii) for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_{m,n} - x| < \varepsilon$, $m, n \geq N$,
which we write as

$$\lim_{m,n\to\infty} x_{m,n} = x$$

(Note that this is Pringsheim's definition of limit of a double sequence).

Theorem 1.2: The double series $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ converges if and only if

$$\lim_{m+n\to\infty} x_{m,n} = 0.$$

4-Dimensional Regular Matrices and Silverman-Toeplitz Theorem

Definition 2.1: Given a 4-dimensional infinite matrix $A = (a_{m,n,k,\ell})$, $a_{m,n,k,\ell} \in K$, $m, n, k, \ell = 0, 1, 2, \ldots$ and a double sequence $\{x_{k,\ell}\}$, $x_{k,\ell} \in K$, $k, \ell = 0, 1, 2, \ldots$, by the A-transform of $x = \{x_{k,\ell}\}$, we mean the sequence $A(x) = \{(Ax)_{m,n}\}$, where,

$$(Ax)_{m,n} = \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell} x_{k,\ell}, \quad m,n = 0,1,2,\ldots,$$

assuming that the double series on the right converge. If $\lim_{m+n\to\infty} (Ax)_{m,n} = s$, we say that the double sequence $x = \{x_{k,\ell}\}$ is summable A or A-summable to s, written as,
$x_{k,\ell} \to s(A)$.
If $\lim_{m+n\to\infty} (Ax)_{m,n} = s$, whenever $\lim_{k+\ell\to\infty} x_{k,\ell} = s$, we say that the 4-dimensional infinite matrix A is ``regular".
The following theorem, proved in Natarajan and Srinivasan (2002), gives a characterization of a 4-dimensional infinite matrix $A = (a_{m,n,k,\ell})$ to be regular, in terms of the entries of the matrix.

Theorem 2.1: (Silverman-Toeplitz) The 4-dimensional infinite matrix $A = (a_{m,n,k,\ell})$ is regular if and only if

(2.1) $$\sup_{m,n,k,\ell\geq 0} |a_{m,n,k,\ell}| < \infty;$$

(2.2) $$\lim_{m+n\to\infty} a_{m,n,k,\ell} = 0, \quad k,\ell = 0,1,2,\ldots;$$

(2.3) $$\lim_{m+n\to\infty} \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell} = 1;$$

(2.4) $$\lim_{m+n\to\infty} \sup_{k\geq 0} |a_{m,n,k,\ell}| = 0, \quad \ell = 0,1,2,\ldots;$$

and

(2.5) $$\lim_{m+n\to\infty} \sup_{\ell\geq 0} |a_{m,n,k,\ell}| = 0, \quad k = 0,1,2,\ldots.$$

Some Properties of the (M, $\lambda_{m,n}$) Method (or the Natarajan Method)

In Natarajan (2014), the author introduced the (M, $\lambda_{m,n}$) method for double sequences and extended some of the results of the (M, λ_n) method for simple sequences.

Definition 3.1: Let $\{\lambda_{m,n}\}$ be a double sequence in K such that

$$\lim_{m+n\to\infty} \lambda_{m,n} = 0.$$

The (M, $\lambda_{m,n}$) method is defined by the 4-dimensional infinite matrix $(a_{m,n,k,\ell})$, where,

$$a_{m,n,k,\ell} = \begin{cases} \lambda_{m-k,n-\ell}, & 0 \leq k \leq m, 0 \leq \ell \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.2: We say that (M, $\lambda_{m,n}$) is included in (M, $\mu_{m,n}$) (or (M, $\mu_{m,n}$) includes (M, $\lambda_{m,n}$)), written as (M, $\lambda_{m,n}$) $\subseteq$ (M, $\mu_{m,n}$) (or (M, $\mu_{m,n}$) $\supseteq$ (M, $\lambda_{m,n}$)),
if
$s_{k,\ell} \to \sigma(M, \lambda_{m,n})$ implies that $s_{k,\ell} \to \sigma(M, \mu_{m,n})$
too.

The methods $(M, \lambda_{m,n})$, $(M, \mu_{m,n})$ are said to be "equivalent", if
$(M, \lambda_{m,n}) \subseteq (M, \mu_{m,n})$ and vice versa.
It is easy to prove the following result.

Theorem 3.1: Natarajan (2014) The method $(M, \lambda_{m,n})$ is regular if and only if

(3.1) $$\sum_{m,n=0}^{\infty,\infty} \lambda_{m,n} = 1.$$

Analogous to Theorem 1 of Natarajan (1978), we have the following result in the context of double sequences.

Theorem 3.2: Natarajan and Sakthivel (2008) If $\lim\limits_{m+n\to\infty} a_{m,n} = 0$ and $\lim\limits_{m+n\to\infty} b_{m,n} = 0,$ then,

$$\lim_{m+n\to\infty} c_{m,n} = 0,$$

where,

$$c_{m,n} = \sum_{k,\ell=0}^{m,n} a_{m-k,n-\ell} b_{k,\ell}, \quad m,n = 0,1,2,....$$

Proof: Since $\lim\limits_{m+n\to\infty} a_{m,n} = 0,\ \lim\limits_{m+n\to\infty} b_{m,n} = 0,$ there exists $M > 0$ such that
$|a_{m,n}| < M$, $|b_{m,n}| < M$, $m, n = 0, 1, 2, ...$
Given $\varepsilon > 0$, choose positive integers M_1, N_1 such that

$$|a_{m,n}| < \frac{\varepsilon}{M}, |b_{m,n}| < \frac{\varepsilon}{M}, \quad m > M_1, n > N_1.$$

Since $\lim\limits_{m+n\to\infty} a_{m-k,n-\ell} = 0,$ for every fixed k, $\ell = 0, 1, 2, ...$, we can choose positive intergers $M_2 > M_1$, $N_2 > N_1$ such that for $m > M_2$, $n > N_2$,

$$\sup_{\substack{0\le k\le M_1\\0\le\ell\le N_1}} |a_{m-k,n-\ell}| < \frac{\varepsilon}{M};$$

$$\sup_{\substack{0\le k\le M_1\\N_1+1\le\ell\le n}} |a_{m-k,n-\ell}| < \frac{\varepsilon}{M};$$

and

$$\sup_{\substack{M_1+1\le k\le m\\0\le\ell\le N_1}} |a_{m-k,n-\ell}| < \frac{\varepsilon}{M}.$$

Thus, for $m > M_2$, $n > N_2$,

$$\begin{aligned}
|c_{m,n}| &= \left|\sum_{k,\ell=0}^{m,n} a_{m-k,n-\ell} b_{k,\ell}\right| \\
&= \left|\sum_{\substack{0\le k\le M_1\\0\le\ell\le N_1}} a_{m-k,n-\ell} b_{k,\ell} + \sum_{\substack{0\le k\le M_1\\\ell> N_1}} a_{m-k,n-\ell} b_{k,\ell} + \sum_{\substack{k> M_1\\0\le\ell\le N_1}} a_{m-k,n-\ell} b_{k,\ell} + \sum_{\substack{k> M_1\\\ell> N_1}} a_{m-k,n-\ell} b_{k,\ell}\right| \\
&\le \max\left[\sup_{\substack{0\le k\le M_1\\0\le\ell\le N_1}} |a_{m-k,n-\ell}||b_{k,\ell}|, \sup_{\substack{0\le k\le M_1\\\ell> N_1}} |a_{m-k,n-\ell}||b_{k,\ell}|, \sup_{\substack{k> M_1\\0\le\ell\le N_1}} |a_{m-k,n-\ell}||b_{k,\ell}|, \sup_{\substack{k> M_1\\\ell> N_1}} |a_{m-k,n-\ell}||b_{k,\ell}|\right] \\
&\le \max\left[\frac{\varepsilon}{M}M, \frac{\varepsilon}{M}M, \frac{\varepsilon}{M}M, \frac{\varepsilon}{M}M\right] \\
&= \varepsilon.
\end{aligned}$$

In other words,

$$\lim_{m+n\to\infty} c_{m,n} = 0,$$

completing the proof of the theorem. □

We now have the following results on the Cauchy multiplication of (M, $\lambda_{m,n}$)-summable double sequences and double series.

Theorem 3.3: If $\lim_{m+n\to\infty} a_{m,n} = 0$ and $\{b_{m,n}\}$ is (M, $\lambda_{m,n}$)-summable to B, then $\{c_{m,n}\}$ is (M, $\lambda_{m,n}$)-summable to AB, where,

$$c_{m,n} = \sum_{k,\ell=0}^{m,n} a_{m-k,n-\ell} b_{k,\ell}, \quad m,n = 0,1,2,...$$

and

$$\sum_{m,n=0}^{\infty,\infty} a_{m,n} = A.$$

Proof: We first note that $\lim_{m+n\to\infty} a_{m,n} = 0$ implies that $\sum_{m,n=0}^{\infty,\infty} a_{m,n}$ converges in view of Theorem 1.2. Let $\{t_{m,n}\}$, $\{\tau_{m,n}\}$ be the (M, $\lambda_{m,n}$)-transforms of $\{b_{m,n}\}$, $\{c_{m,n}\}$ respectively,

i.e., $$t_{m,n} = \sum_{k,\ell=0}^{m,n} \lambda_{m-k,n-\ell} b_{k,\ell},$$

$$\tau_{m,n} = \sum_{k,\ell=0}^{m,n} \lambda_{m-k,n-\ell} c_{k,\ell}, \quad m,n = 0,1,2,....$$

We can work out to see that

$$\tau_{m,n} = \sum_{k,\ell=0}^{m,n} a_{m-k,n-\ell}(t_{k,\ell} - B) + B\left(\sum_{k,\ell=0}^{m,n} a_{k,\ell}\right), \quad m,n = 0,1,2,..., \tag{3.2}$$

where, $\lim_{k+\ell\to\infty} t_{k,\ell} = B$, by hypothesis. Since $\lim_{m+n\to\infty} a_{m,n} = 0$ and $\lim_{m+n\to\infty} (t_{m,n} - B) = 0$, using Theorem 3.2, we see that

$$\lim_{m+n\to\infty}\left[\sum_{k,\ell=0}^{m,n} a_{m-k,n-\ell}(t_{k,\ell} - B)\right] = 0,$$

so that, taking limit as m+n $\to \infty$ in (3.2), we have,

$$\lim_{m+n\to\infty} \tau_{m,n} = B\left(\sum_{k,\ell=0}^{\infty,\infty} a_{k,\ell}\right) = AB,$$

i.e., $\{c_{m,n}\}$ is (M, $\lambda_{m,n}$)-summable to AB, completing the proof of the theorem. □

It is now easy to prove the following result on similar lines.

Theorem 3.4: If $\lim_{m+n\to\infty} a_{m,n} = 0$, $\sum_{m,n=0}^{\infty,\infty} b_{m,n}$ is (M, $\lambda_{m,n}$)-summable to B, then $\sum_{m,n=0}^{\infty,\infty} c_{m,n}$ is (M, $\lambda_{m,n}$)-summable to AB, where,

$$c_{m,n} = \sum_{k,\ell=0}^{m,n} a_{m-k,n-\ell} b_{k,\ell}, \quad m,n = 0,1,2,...$$

and

$$\sum_{m,n=0}^{\infty,\infty} a_{m,n} = A.$$

As in the case of the (M, λ_n) method for simple sequences (Natarajan, 2012), using Theorem 3.2 again, we can prove

Theorem 3.5: If $\sum_{m,n=0}^{\infty,\infty} a_{m,n}$ is (M, $\lambda_{m,n}$)-summable to A, $\sum_{m,n=0}^{\infty,\infty} b_{m,n}$ is (M, $\mu_{m,n}$)-summable to B, then $\sum_{m,n=0}^{\infty,\infty} c_{m,n}$ is (M, $\gamma_{m,n}$)-summable to AB, where,

$$c_{m,n} = \sum_{k,\ell=0}^{m,n} a_{m-k,n-\ell} b_{k,\ell},$$

$$\gamma_{m,n} = \sum_{k,\ell=0}^{m,n} \lambda_{m-k,n-\ell} \mu_{k,\ell}, \quad m,n = 0,1,2,....$$

Again as in the case of the Natarajan method (M, λ_n) for simple sequences (Natarajan, 2013), we can prove the following result, using Theorem 3.2.

Theorem 3.6: Let (M, $\lambda_{m,n}$), (M, $\mu_{m,n}$) be regular methods. Then, (M, $\lambda_{m,n}$) (M, $\mu_{m,n}$) is also regular, where, we define, for x = {$x_{m,n}$},

((M, $\lambda_{m,n}$) (M, $\mu_{m,n}$))(x) = (M, $\lambda_{m,n}$) ((M, $\mu_{m,n}$))(x)).

We can prove the following results too.

Theorem 3.7: For given regular methods (M, $\lambda_{m,n}$), (M, $\mu_{m,n}$) and (M, $t_{m,n}$), let $|\lambda_{m,n}| < |\lambda_{0,0}|$, $|t_{m,n}| < |t_{0,0}|$, (m, n) $\neq$ (0, 0), m, n = 0, 1, 2, ... Then

(M, $\lambda_{m,n}$) $\subseteq$ (M, $\mu_{m,n}$)

if and only if

(M, $t_{m,n}$) (M, $\lambda_{m,n}$) $\subseteq$ (M, $t_{m,n}$) (M, $\mu_{m,n}$).

In view of Theorem 3.6 of Natarajan (2014), we can reformulate Theorem 3.7 as follows:

Theorem 3.8: Given the regular methods (M, $\lambda_{m,n}$), (M, $\mu_{m,n}$) and (M, $t_{m,n}$), $|\lambda_{m,n}| < |\lambda_{0,0}|$, $|t_{m,n}| < |t_{0,0}|$, (m, n) $\neq$ (0, 0), m, n = 0, 1, 2, ..., the following statements are equivalent:

(i) (M, $\lambda_{m,n}$) $\subseteq$ (M, $\mu_{m,n}$);

(ii) (M, $t_{m,n}$) (M, $\lambda_{m,n}$) $\subseteq$ (M, $t_{m,n}$) (M, $\mu_{m,n}$);

and

(iii) $\lim_{m+n\to\infty} k_{m,n} = 0$ and $\sum_{m,n=0}^{\infty,\infty} k_{m,n} = 1,$

where,

$$\frac{\mu(x)}{\lambda(x)} = k(x) = \sum_{m,n=0}^{\infty,\infty} k_{m,n} x^m y^n,$$

$$\lambda(x) = \sum_{m,n=0}^{\infty,\infty} \lambda_{m,n} x^m y^n;$$

and

$$\mu(x) = \sum_{m,n=0}^{\infty,\infty} \mu_{m,n} x^m y^n.$$

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