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Research Article

ON HYPER SURFACE OF A FINSLER SPACE WITH AN EXPONENTIAL (α,β)- METRIC OF ORDER M

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ABSTRACT
The purpose of the present paper is to investigate the various kinds of hyper surfaces of a Finsler space with special (α, β)- metric $L=\sum_{r=0}^{m}\frac{1}{r!}\frac{\beta^r}{\alpha^{(r-1)}}$.

Keywords: *Special Finsler Hyper Surface, (α, β)-Metric, Normal Curvature Vector, Second Fundamental Tensor, Hyperplane of First Kind, Hyperplane of Second Kind, Hyperplane of Third Kind*

INTRODUCTION

Let $F^n=(M^n,L)$ be an n-dimensional Finsler space, i.e., a pair consisting of an n-dimensional differentiable manifold M^n equipped with a fundamental function L(x,y). The concept of the (α,β)-metric L(α,β) was introduce by Matsumoto (1991) and has been studied by many authors (Hashiguchi and Ichjyo, 1975; Kikuchi, 1979; Shibata, 1984).

A Finsler metric L (x,y) is called an (α,β)- metric L (α,β) if L is a positively homogeneous function of α and β of degree one, where $\alpha^2=a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form on M^n.

A hypersurface M^{n-1} of the M^n may be represented parametrically by the equations $x^i=x^i(u^\alpha)$, $\alpha=1,2,\dots,n-1$, where u^α are Gaussian coordinates on M^{n-1}. The following notations are also employed (Kitayama, 2002): $B^i_{\alpha\beta}=\partial^2x^i/\partial u^\alpha\partial u^\beta$, $B^i_{0\beta}=v^\alpha B^i_{\alpha\beta}$. If the supporting element y^i at a point (u^α) of M^{n-1} is assumed to be tangential to M^{n-1}, we may then write $y^i=B^i_\alpha(u)v^\alpha$, so that v^α is thought of as the supporting element of M^{n-1}at the point (u^α).

Since the function $\underline{L}(u,v)=L(x(u),y(u,v))$ gives rise to a Finsler metric of M^{n-1}, we get an (n-1)-dimensional Finsler space $F^{n-1}=(M^{n-1},\underline{L}(u,v))$.

In the present paper, we consider an n-dimentional Finsler space $F^n=(M^n,L)$ with (α, β)-metric$L(\alpha,\beta)=L=\sum_{r=0}^{m}\frac{1}{r!}\frac{\beta^r}{\alpha^{(r-1)}}$and the hyper surface of F^n with $b_i(x)=\partial_ib$ being the gradient of a scalar function b(x). We prove the condition for this hyper surface to be a hyperplane of first kind, second kind and third kind.

Preliminaries

Let $F^n=(M^n,L)$ be a special Finsler space with the metric

(2.1) $L(\alpha,\beta)=\sum_{r=0}^{m}\frac{1}{r!}\frac{\beta^r}{\alpha^{(r-1)}}$

The derivatives of the (2.1) with respect to α and β are given by

$$L_\alpha=\sum_{r=0}^{m}\frac{(1-r)}{r!}\frac{\beta^r}{\alpha^r},,L_\beta=\sum_{r=0}^{m}\frac{1}{(r-1)!}\frac{\beta^{(r-1)}}{\alpha^{(r-1)}},$$

$$L_{\alpha\alpha}=\sum_{r=0}^{m}\frac{1}{(r-2)!}\frac{\beta^r}{\alpha^{(r+1)}}\quad,L_{\beta\beta}=\sum_{r=0}^{m}\frac{1}{(r-2)!}\frac{\beta^{(r-2)}}{\alpha^{(r-1)}}\ ,$$

$$L_{\alpha\beta}=-\sum_{r=0}^{m}\frac{1}{(r-2)!}\frac{\beta^{(r-1)}}{\alpha^r}\ ,$$

Where $L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_\beta = \frac{\partial L}{\partial \beta}$, $L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}$, $L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}$, $L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}$ and $L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}$

In the special Finsler space $F^n = (M^n, L)$ the normalized element of support $l_i = \dot{\partial}_i L$ and the angular metric tensor h_{ij} are given by Matsumoto, (1991):

$$l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i ,$$

$$h_{ij} = p a_{ij} + q_0 b_i b_j + q_{-1}\left(b_i Y_j + b_j Y_i\right) + q_{-2} Y_i Y_j ,$$

where $Y_i = a_{ij} y^j$ For the fundamental function (2.1) above scalars are given by

(2.2) $$p = L L_\alpha \alpha^{-1} = \sum_{t=0}^{m} \sum_{r=0}^{m} \frac{(1-t)}{r!\,t!} \frac{\beta^{(r+t)}}{\alpha^{(r+t-1)}} ,$$

$$q_0 = L L_{\beta\beta} = \sum_{t=0}^{m} \sum_{r=0}^{m} \frac{1}{r!(t-2)!} \frac{\beta^{(r+t-2)}}{\alpha^{(r+t-2)}} ,$$

$$q_{-1} = L L_{\alpha\beta} \alpha^{-1} = -\sum_{t=0}^{m} \sum_{r=0}^{m} \frac{1}{r!(t-2)!} \frac{\beta^{(r+t-1)}}{\alpha^{(r+t)}}$$

$$q_{-2} = L \alpha^{-2}(L_{\alpha\alpha} - L_\alpha \alpha^{-1}) = \sum_{t=0}^{m} \sum_{r=0}^{m} \frac{r^2-1}{r!\ t!} \frac{\beta^{(r+t)}}{\alpha^{(r+t+2)}}$$

The Fundamental metric tensor $g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2$ and its reciprocal tensor g^{ij} is given by Matsumoto, (1991)

(2.3) $g_{ij} = p a_{ij} + p_0 b_i b_j + p_{-1}\left(b_i Y_j + b_j Y_i\right) + p_{-2} Y_i Y_j$,

where

(2.4) $$p_0 = q_0 + L_\beta^2 = \sum_{t=0}^{m} \sum_{r=0}^{m} \frac{(r+t-1)}{r!(t-1)!} \frac{\beta^{(r+t-2)}}{\alpha^{(r+t-2)}} ,$$

$$p_{-1} = \mathrm{q}_{-1} + L^{-1} p L_\beta = \sum_{t=0}^{m} \sum_{r=0}^{m} \frac{(2-r-t)}{r!(t-1)!} \frac{\beta^{(r+t-1)}}{\alpha^{(r+t)}} ,$$

$$p_{-2} = q_{-2} + p^2 L^{-2} = \sum_{t=0}^{m} \sum_{r=0}^{m} \frac{(r-1)(r+t)}{r!\ \ t!} \frac{\beta^{(r+t)}}{\alpha^{(r+t+2)}} .$$

The reciprocal tensor g^{ij} of g_{ij} is given by

(2.5) $g^{ij} = p^{-1} a^{ij} - s_0 b^i b^j - s_{-1}\left(b^i y^j + b^j y^i\right) - s_{-2} y^i y^j$,

where $b^i = a^{ij} b_j$, $b^2 = a_{ij} b^i b^j$ and

(2.6) $s_0 = \frac{1}{\tau p}\{p p_0 + (p_0 p_{-2} - p_{-1}^2)\alpha^2\}$,

$s_{-1} = \frac{1}{\tau p}\{p p_{-1} + (p_0 p_{-2} - p_{-1}^2)\beta\}$,

$s_{-2} = \frac{1}{\tau p}\{p p_{-2} + (p_0 p_{-2} - p_{-1}^2) b^2\}$,

$\tau = p(p + p_0 b^2 + p_{-1}\beta) + (p_0 p_{-2} - p_{-1}^2)(\alpha^2 b^2 - \beta^2)$

The hv- torsion tensor $C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}$ is given by Matsumoto, (1991)

(2.7) $$2p C_{ijk} = p_{-1}\left(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j\right) + \gamma_1 m_i m_j m_k$$

where

(2.8) $$\gamma_1 = \frac{\partial p_0}{\partial \beta} - 3 p_{-1} q_0 , \qquad m_i = b_i - \alpha^{-2} \beta Y_i .$$

Here m_i is a non- vanishing covariant vector orthogonal to the element of support y^i.

Let $\left\{ {i \atop jk} \right\}$ be the components of Christoffel symbols of the associated Riemmanian space R^n and ∇_k be the covariant derivative with respect to x^k relative to these Christoffel symbols. Now we define

(2.9) $2E_{ij} = b_{ij} + b_{ji}$, $\qquad 2F_{ij} = b_{ij} - b_{ji}$

where $b_{ij} = \nabla_j b_i$.

Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ be the Cartan connection of F^n. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{{i \atop jk}\}$ of the special Finsler space F^n is given by Matsumoto, (1986).

(2.10)
$$\begin{aligned} D^i_{jk} &= B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} \\ &- C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} \\ &+ \lambda^s \left(C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms} \right), \end{aligned}$$

where

(2.11) $B_k = p_0 b_k + p_{-1} Y_k$, $\quad B^i = g^{ij} B_j$, $\quad F^k_i = g^{kj} F_{ji}$

$$B_{ij} = \frac{1}{2}\left\{ p_{-1}(a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j \right\}, \quad B^k_i = g^{kj} B_{ji},$$
$$A^m_k = B^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k,$$
$$\lambda^m = B^m E_{00} + 2 B_0 F^m_0, \quad B_0 = B_i y^i.$$

where '0' denotes contraction with y^i except for the quantities p_0, q_0 and s_0.

Induced Cartan Connection

Let F^{n-1} be a hyper surface of F^n given by the equations $x^i = x^i(u^\alpha)$

Where α= 1,2,3 …… $(n-1)$.The (n-1) tangent vectors to the hyper surface F^{n-1} are given by $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$.

The element of support y^i of F^n is to be taken tangential to F^{n-1}, that is Matsumoto, (1985),

(3.1) $y^i = B^i_\alpha(u) v^\alpha$

The metric tensor $g_{\alpha\beta}$ and hv-tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta, \qquad C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma$$

At each point (u^α) of F^{n-1}, a unit normal vector $N^i(u,v)$ is defined by
$g_{ij}\{x(u,v), y(u,v)\} B^i_\alpha N^j = 0$, $g_{ij}\{x(u,v), y(u,v)\} N^i N^j = 1$.

Angular metric tensor $h_{\alpha\beta}$ of the hyper surface is such that

(3.2) $h_{\alpha\beta} = h_{ij} B^i_\alpha B^j_\beta$, $\quad h_{ij} B^i_\alpha N^j = 0$, $\quad h_{ij} N^i N^j = 1$

If (B^α_i, N_i) denote the inverse of (B^i_α, N^i), then we have

$$B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta, \quad B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^\alpha_i N^i = 0, \quad B^i_\alpha N_i = 0,$$
$$N_i = g_{ij} N^j, \quad B^K_i = g^{kj} B_{ji}, \quad B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j.$$

The induced connection $IC\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ of F^{n-1} induced from the Cartan's connection $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^{*i}_{jk})$ is given by Matsumoto, (1985)

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i \left(B^i_{\beta\gamma} + \Gamma^{*i}_{jk} B^j_\beta B^k_\gamma \right) + M^\alpha_\beta H_\gamma$$

$$G^\alpha_\beta = B^\alpha_i \left(B^i_{0\beta} + \Gamma^{*i}_{0j} B^j_\beta \right), C^\alpha_{\beta\gamma} = B^\alpha_i C^i_{jk} B^j_\beta B^k_\gamma,$$

where $\quad M_{\beta\gamma} = N_i C^i_{jk} B^j_\beta B^k_\gamma$, $\quad M^\alpha_\beta = g^{\alpha\gamma} M_{\beta\gamma}$, $\quad H_\beta = N_i \left(B^i_{0\beta} + \Gamma^{*i}_{0j} B^j_\beta \right)$,

and $B^i_{\beta\gamma} = \frac{\partial B^i_\beta}{\partial u^\gamma}$, $\qquad B^i_{0\beta} = B^i_{\alpha\beta} v^\alpha$.

The quantities $M_{\beta\gamma}$ and H_β are called the second fundamental v- vector and normal curvature vector respectively (Matsumoto, 1985) .The second fundamental h-tensor $H_{\beta\gamma}$ is defined as Matsumoto, (1985).

(3.3) $$H_{\beta\gamma} = N_i \left(B^i_{\beta\gamma} + \Gamma^{*i}_{jk} B^j_\beta B^k_\gamma \right) + M_\beta H_\gamma$$

where

(3.4) $M_\beta = N_i C^i_{jk} B^j_\beta N^k$

The relative h- and v- covariant derivatives of projection factor B^i_α with respect to $IC\Gamma$ are given by

(3.5) $$B^i_{\alpha|\beta} = H_{\alpha\beta} N^i, \qquad B^i_\alpha|_\beta = M_{\alpha\beta} N^i$$

The equation (3.3) shows that $H_{\beta\gamma}$ is generally not symmetric and

(3.6) $$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta$$

The above equations yield

(3.7) $H_{0\gamma} = H_\gamma$, $H_{\gamma 0} = H_\gamma + M_\gamma H_0$

We shall use following lemmas which are due to Matsumoto (1985):

Lemma 1: The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if the normal curvature vector H_β vanishes.

Lemma 2: A hyper surface F^{n-1} is a hyperplane of the first kind with respect to connection $C\Gamma$ if and only if $H_\alpha = 0$.

Lemma 3: A hyper surface F^{n-1} is a hyperplane of the second kind with respect to connection $C\Gamma$ if and only if $H_\alpha = 0$ and $Q_{\alpha\beta} = 0$ where $Q_{\alpha\beta} = C_{ijk|0} B^i_\alpha B^j_\beta N^k$ and then $H_{\alpha\beta} = 0$

Lemma 4: A hyper surface F^{n-1} is a hyperplane of the third kind with respect to connection $C\Gamma$ if and only if $H_\alpha = 0 H_{\alpha\beta} = 0\ and\ M_{\alpha\beta} = 0$.

Hypersurface $F^{n-1}(c)$ of the Special Finsler Space

Let us consider a Finsler space with the metric $L = L = \sum_{r=0}^{m} \frac{1}{r!} \frac{\beta^r}{\alpha^{(r-1)}}$, with a gradient $b_i(x) = \frac{\partial b}{\partial x^i}$ for ascalar function b(x) and a hypersurface $F^{n-1}(c)$ given by the equation b(x) = c (constant) (Lee *et al.*, 2001).

From the parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get $\frac{\partial b(x)}{\partial u^\alpha} = 0 = b_i B^i_\alpha$,

So, that $b_i(x)$ are regarded as covariant components of a normal vector field of hypersurface $F^{n-1}(c)$. Therefore, along the $F^{n-1}(c)$, we have

(4.1) $b_i B^i_\alpha = 0$ and $b_i y^i = 0$ i.e. $\beta = 0$

The induced metric L(u,v) of $F^{n-1}(c)$ is given by

(4.2) $$L(u,v) = a_{\alpha\beta} v^\alpha v^\beta \ , \ a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta$$

At a point of $F^{n-1}(c)$, from equations (2.2), (2.3) and (2.5) we get

(4.3) $p = 1 \qquad q_0 = 1\ ,\ q_{-1} = 0\ ,\ q_{-2} = -\alpha^{-2}$

$p_0 = 2 \qquad p_{-1} = \alpha^{-1} p_{-2} = 0 \qquad \tau = (1 + b^2),$

$s_0 = \frac{1}{(1+b^2)} s_{-1} = \frac{1}{\alpha(1+b^2)} s_{-2} = \frac{1}{\alpha^2(1+b^2)}$.

Therefore, from (4.2) we get,

(4.4) $$g^{ij} = a^{ij} - \frac{1}{(1+b^2)} b^i b^j - \frac{1}{\alpha(1+b^2)}\left(b^i y^j + b^j y^i\right) + \frac{1}{\alpha^2(1+b^2)} y^i y^j$$

Thus, along $F^{n-1}(c)$, (4.4) and (4.1) lead to

$$g^{ij} b_i b_j = \frac{b^2}{(1+b^2)}$$

Therefore, we get

(4.5) $b_i\big(x(u)\big) = \sqrt{\frac{b^2}{(1+b^2)}} N_i$, $b^2 = a^{ij} b_i b_j$

where b is the length of the vector b^i.

Again from (4.4) and (4.5), we get

(4.6) $$b^i = a^{ij} b_j = \sqrt{b^2(1+b^2)} N^i + \frac{b^2}{\alpha} y^i$$

Thus, we have

Theorem 4.1 In a special Finsler hypersurface $F^{n-1}(c)$, the induced metric is a Riemannian metric given by (4.2) and the scalar function $b(x)$ is given by (4.5) and (4.6)

The angular metric tensor h_{ij} and metric tensor g_{ij} of F^n are given by

(4.7) $h_{ij} = a_{ij} + b_i b_j - \frac{1}{\alpha^2} Y_i Y_j$,

$g_{ij} = a_{ij} + 2b_i b_j + \frac{1}{\alpha}\left(b_i Y_j + b_j Y_i\right)$.

From equations (4.1), (4.7) and (3.2) it follows that if $h^{(a)}_{\alpha\beta}$ denotes the angular metric tensor corresponding to the Riemannian metric tensor $a_{ij}(x)$, then we have along $F^{n-1}(c)$, $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$

Thus, along $F^{n-1}(c)$, From (2.3) we get, $\frac{\partial p_0}{\partial \beta} = \frac{4}{\alpha}$ and therefore, (2.8) give

$\gamma_1 = \frac{1}{\alpha}$, $m_i = b_i$,

At the points of $F^{n-1}(c)$, the hv-torsion tensor becomes

(4.8) $$C_{ijk} = \frac{1}{2\alpha}\left(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j\right) + \frac{1}{2\alpha} b_i b_j b_k$$

Therefore, from (3.2),(3.3) ,(3.5),(4.1) and (4.8), we have

(4.9) $$M_{\alpha\beta} = \frac{1}{2\alpha}\sqrt{\frac{b^2}{(1+b^2)}}\, h_{\alpha\beta}$$

and $M_\alpha = 0$.

Thus, from equation (3.6) it follows that $H_{\alpha\beta}$ is symmetric. Hence we have

Theorem 4.2: The second fundamental v-tensor of the special Finsler hyper surface $F^{n-1}(c)$ is given by (4.9) the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.

From $b_i B^i_\alpha = 0$, we have

$$b_{\alpha|\beta} B^i_\alpha + b_i B^i_{\alpha|\beta} = 0$$

Therefore, from (3.5) and using $b_{i|\beta} = b_{i|j} B^j_\beta + b_i|_j N^j H_\beta$ (Matsumoto, 1985) we have

(4.10) $$b_{i|j} B^i_\alpha B^j_\beta + b_i|_j B^i_\alpha N^j H_\beta + b_i H_{\alpha\beta} N^i = 0 .$$

Since $b_i|_j = -b_h C^h_{ij}$ and $M_\alpha = 0$, therefore

$$b_i|_j B^i_\beta N^j = 0$$

Then, from equation (4.10) we have

(4.11) $$\sqrt{\frac{b^2}{(1+b^2)}}\, H_{\alpha\beta} + b_{i|j} B^i_\alpha B^j_\beta = 0.$$

Now contracting (4.11) with v^β and using (3.1)and(3.6)we get

(4.12) $$\sqrt{\frac{b^2}{(1+b^2)}}\, H_\alpha + b_{i|j} B^i_\alpha y^j = 0 .$$

Again contracting equation (4.12) by v^α and using (3.1) we have

(4.13) $$\sqrt{\frac{b^2}{(1+b^2)}}\, H_0 + b_{i|j} y^i y^j = 0 .$$

From lemma (3.1) and (3.2), it is clear that the hypersurface $F^{n-1}(c)$ is a hyperplane of first kind if and only if $H_0 = 0$. Thus, from (4.13) it is obvious that $F^{n-1}(c)$ is a hyperplane of first kind if and only if $b_{i|j} y^i y^j = 0$. This $b_{i|j}$ being the covariant derivative with respect to $C\Gamma$ of F^n depends on y^i, but $b_{ij} = \nabla_j b_i$ is the covariant derivative with respect to Riemannian connection $\{^{\,i}_{jk}\}$ constructed from $a_{ij}(x)$. Hence, b_{ij} does not depend on y^i. We shall consider the difference $b_{i|j} - b_{ij}$. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^{\,i}_{jk}\}$ is given by (2.10). Since b_i is a gradient vector, from (2.9) we have

$$E_{ij} = b_{ij} \quad , \quad F_{ij} = 0 \quad and \quad F^i_j = 0 .$$

Thus, (2.10) reduces to

(4.14) $$D^i_{jk} = B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}) .$$

From (2.11) and (4.3) it follows that at $F^{n-1}(c)$, we have

(4.15) $$B_i = 2b_i + \alpha^{-1} Y_i \ , B^i = \frac{1}{(1+b^2)} b^i + \frac{1}{\alpha(1+b^2)} y^i ,$$

$\lambda^m = B^m b_{00}\, , B_{ij} = \frac{1}{2\alpha}(a_{ij} - \alpha^{-2} Y_i Y_j\,) + \frac{2}{\alpha} b_i b_j$,

$B^i_j = \frac{1}{2\alpha}(\delta^i_j - \alpha^{-2} y^i Y_j\,) + \frac{3}{2\alpha(1+b^2)} b^i b_j - \frac{(1+4b^2)}{2\alpha^2(1+b^2)} y^i \mathrm{b_j}$.

$$A^m_k = B^m_k b_{00} + B^m b_{k0}\ .$$

From (4.15) we have $B^i_0 = 0\, , B_{i0} = 0$ which leads to $A^m_0 = B^m b_{00}$.
Now contracting (4.14) by y^k we get

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - B^m C^i_{jm} b_{00}.$$

Again contracting the above equation with respect to y^j, we have

$$D^i_{00} = B^i b_{00} = \{ \frac{1}{(1+b^2)} b^i + \frac{1}{\alpha(1+b^2)} \mathrm{y^i} \}\ \ b_{00}$$

Paying attention to (4.1) along $F^{n-1}(c)$, we get

(4.16) $\quad b_i D^i_{j0} = \frac{b^2}{(1+b^2)} b_{j0} + \frac{(1+4b^2)}{2\alpha(1+b^2)} \mathrm{b_j} b_{00} - \frac{1}{(1+b^2)} b^m \mathrm{b_i} C^i_{jm} b_{00}$

Now we contract (4.16) by y^j, we have

(4.17) $\quad b_i D^i_{00} = \frac{b^2}{(1+b^2)} b_{00}.$

From (3.3),(4.5),(4.9) and $M_\alpha = 0$,we have

$$b^m \mathrm{b_i} C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0$$

Thus, the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$, after use of (4.17), gives

$$b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = \frac{1}{(1+b^2)} b_{00}\ .$$

Consequently (4.12) and (4.13) may be written as

(4.18) $\sqrt{\frac{b^2}{(1+b^2)}} H_\alpha + \frac{1}{(1+b^2)} b_{i0} B^i_\alpha = 0$

$$\sqrt{\frac{b^2}{(1+b^2)}} H_0 + \frac{1}{(1+b^2)} b_{00} = 0$$

respectively. We see that the condition $H_0 = 0$ is equivalent to $b_{00} = 0$,where $\mathrm{b_{ij}}$ does not depend on $\mathrm{y^i}$. Using the fact that $\beta = b_i y^i = 0$ on the $F^{n-1}(c)$, the condition $b_{00} = 0$ can be written as $b_{ij} y^i y^j = b_i y^i c_j y^j$ for some $c_j(x)$. Thus, we can write

(4.19) $\quad 2\mathrm{b_{ij}} = \mathrm{b_i c_j} + \mathrm{b_j c_i}$

Now from (4.1) and (4.19) we get

$$\mathrm{b_{00}} = 0\, ,\ \ \mathrm{b_{ij} B^i_\alpha B^j_\beta} = 0\, b_{ij} B^i_\alpha y^j = 0\ .$$

Hence, from (4.18) we get $H_\alpha = 0$, Again from (4.19) and (4.15) we get

$b_{i0} b^i = \frac{c_0 b^2}{2}$, $\lambda^m = 0$, $A^i_j B^j_\beta = 0$ and$B_{ij} B^i_\alpha B^j_\beta = \frac{1}{2\alpha} h_{\alpha\beta}$

Now we use equations (3.3),(4.4), (4.6),(4.9) and (4.14) to get

(4.20)$b_r D^r_{ij} B^i_\alpha B^j_\beta = \frac{-C_0 b^2}{4\alpha(1+b^2)^2} h_{\alpha\beta}$

Using (4.20) the equation (4.11) becomes

(4.21) $\quad \sqrt{\frac{b^2}{1+b^2}} H_{\alpha\beta} + \frac{b^2}{4\alpha(1+b^2)^2} h_{\alpha\beta} = 0$

Hence, the hyper surface $F^{n-1}(c)$ is umbilical.

Theorem 4.3: The necessary and sufficient condition for $F^{n-1}(c)$ to be a hyperplane of first kind is that (4.21) holds good. Hence, the hyper surface $F^{n-1}(c)$ is umbilical. From (4.21) we see that the second fundamental tensor of $F^{n-1}(c)$ is proportional to its angular metric tensor.

Now from lemma (3.3), $F^{n-1}(c)$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $Q_{\alpha\beta} = 0$ which implies that $H_{\alpha\beta} = 0$. Thus, from (4.21) we get

$$c_0 = c_i(x) y^i = 0$$

Therefore, there exists a function $\psi(x)$ such that

$$c_i(x) = \psi(x) b_i(x)$$

Thus, equation (4.19) is written as

$$2b_{ij} = b_i(x)\psi(x)b_j(x) + b_j(x)\psi(x)b_i(x)$$

or

(4.22) $b_{ij} = \psi(x)\ b_i b_j$

Hence, we have

Theorem 4.4: The necessary and sufficient condition for a hypersurface $F^{n-1}(c)$ to be a hyperplane of second kind is that (4.22) holds good.

Next, lemma (3.4) together with (4.9) and $M_\alpha = 0$ shows that $F^{n-1}(c)$ can never become a hyperplane of the third kind. Hence, we have

Theorem 4.5**:** The hyper surface $F^{n-1}(c)$ can never become a hyperplane of the third kind.

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