

QUANTUM THERMODYNAMICS OF A STRONGLY INTERACTING BOSE-FERMI MIXTURE IN A THREE-DIMENSIONAL ANHARMONIC POTENTIAL

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ABSTRACT

Thermodynamic properties of Bose-Fermi mixture at ultra-low temperatures were studied using perturbation theory in three Cartesian coordinates. The Hamiltonian for the mixture was developed in terms of boson-boson, boson-fermion and fermion-fermion interactions. In both first and second excited states the mixture had about 150 joules of energy at about 40 kelvins. In both first and third excited states, the mixture had energy value of about 200 joules at a temperature of about 48 kelvins. The mixture had energy value of about 240 joules at about 58 kelvins in both the second and the third excited states. The specific heat capacity versus temperature curves showed sharp turning points with peak values of specific heat capacity being 11.58J/mol/K at about 6.9 kelvins for first excited state, 7.88J/mol/K at about 13.7 kelvins for second excited state and 6.758J/mol/K at about 20.9 kelvins for third excited state. Entropy of the BF mixture in an excited state was found to increase with temperature and became constant at higher temperatures.

Keywords: Anharmonic Potential, Bose-Fermi Mixture, Perturbed Hamiltonian, Unperturbed Hamiltonian, Strong Interaction

INTRODUCTION

Quantum thermodynamics addresses the emergence of thermodynamics from quantum mechanics (Kosloff, 2013). It deals with microscopic properties, based on the idea that every particle behaves as a wave and vice versa. Fermions are identical particles which obey the Pauli Exclusion Principle which states that this type of particle must be in a state that is antisymmetric with respect to particle exchange (Romero-Rochin, 2011). A fermion has a half integer spin angular momentum. Bosons are integer spin particles which are governed by Bose-Einstein (BE) statistics (Aversa, 2011). It has been observed that fermions in superconductors undergo Bose-Einstein condensation through interactions with the bosons at temperatures near absolute zero (Mathew, 2013).

High-precision measurements on the thermodynamics of a strongly interacting Fermion gas across the superfluid transition were done (Bakr *et al.*, 2013). The onset of superfluidity was observed in the compressibility, the chemical potential, the entropy and the heat capacity. The ground state properties of Heavy isotopes of ¹⁷¹Yb and ¹⁷²Yb were shown to be degenerate with anisotropic interactions (Haiping, 2015). A quantum-degenerate BF mixture of ²³Na and ⁴⁰K with widely tunable interactions was created (Park, 2012). Over thirty Feshbach resonances were identified, including p-wave multiple resonance. A mixture of interacting atomic bosons and atomic fermions have been described by the Hamiltonian in which all the interaction potentials are approximated by contact interactions and parameterized by s-wave scattering length (Malatsetxebarria *et al.*, 2013). A Hamiltonian for a strongly repulsive one-dimensional Bose-Fermi mixture in a double well potential was determined and used to calculate the reduced single-particle density matrix, momentum distribution, natural orbitals and their occupancies resulting in a smooth momentum distribution for even mixtures and distinct modulations in the momentum distribution of odd mixtures (Lelas *et al.*, 2009).

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Anharmonic oscillation is a deviation of a system from harmonic oscillation (Jafarpour & Afshar, 2008). The ground state properties of trapped atomic BF mixture at near absolute zero temperature have been studied using second quantization techniques (Sakwa *et al.*, 2013).

A grand canonical binary system of $^4\text{He}-^3\text{He}$ was used. The results for variation of specific heat versus temperature showed a peak or turning point in the vicinity of $T = 0.5\text{K}$ on a nearly Gaussian curve. The variation of entropy with temperature of the binary mixture was found to be a curve with a gently decreasing slope, nearly saturating at 1.3K . It is assumed that quartic and sextic anharmonic oscillator potentials have similar properties as those of the harmonic oscillator potential in the derivation of the eigen states of different particles (Joshua *et al.*, 2008).

This is because the potentials are even functions.

The influence of boson-boson (bb) and boson-fermion (bf) interactions on the stability of a binary mixture of bosonic and fermionic atoms were investigated (Roth & Feldmeier, 2013). Boson-Boson and Boson-Fermion interactions were found to have very different effects on the collapse. Massive bosonic or fermionic particles with only two types of contact interaction potentials allow for either bosons with s-wave interactions or fermions with p-wave interactions (Dominik & Michael, 2010). The s-wave interactions are dominant in boson-boson and boson-fermion systems while p-wave interactions are dominant in fermion-fermion systems.

Mixtures of fermionic ^{40}K and bosonic ^{87}Rb quantum gases in a three-dimensional optical lattice have been studied (Gunter *et al.*, 2008). It is observed that an increasing admixture of the fermionic species diminishes the phase coherence of the bosonic atoms as measured by studying both the visibility of the matter wave interference pattern and the coherence length of the bosons. A triply quantum degenerate mixture of bosonic ^{41}K and two fermionic species ^{40}K and ^6Li has been created (Wu *et al.*, 2011). The boson is shown to be an efficient coolant for the two fermions, spurring hopes for the observation of fermionic superfluids with imbalanced masses. Bose-Fermi mixtures have lower critical temperature due to the presence of antisymmetric and mixed symmetry states (Mirza, 2007). The dependence of the critical temperature shift on the Bose component of a degenerate Bose-Fermi gas mixture in a harmonic trap has been obtained on the basis of the effective Hamiltonian of the Bose subsystem (Belemuk & Ryzhov, 2008).

For calculations it is often more practical to consider the system being in contact with a large reservoir, which allows for the exchange of particles and energy (Will, 2013).

The low-temperature thermodynamic variables of entropy, internal energy and specific heat capacity of a liquid $^4\text{He}-^3\text{He}$ binary system have been studied (Ayodo *et al.*, 2010). A micro-canonical ensemble model with pair interactions has been used to obtain entropy, internal energy and specific heat capacity in terms of the partition function.

Anharmonic perturbation of neutron-proton (np) pairs by the unpaired neutrons in heavy finite nuclei has been studied (Khanna *et al.*, 2010). Heavy nuclei of ^{161}Dy and ^{163}Dy were used. Second quantization with ladder operator techniques was used to calculate the total energy (E_n), the specific heat (C) and entropy (S) in one dimension. The results showed that the nucleus ^{163}Dy has larger values of specific heat capacity and entropy than the nucleus ^{161}Dy at any temperature. The nucleus ^{163}Dy has two more neutrons than ^{161}Dy which contribute more perturbation energy to the core of the nucleus. The thermodynamic properties of finite heavy mass nuclei of ^{64}Zn , ^{101}Ru and ^{150}Sm with the number of neutrons greater than the number of protons have been investigated (Ndinya & Okello, 2014). The core of the nucleus contains the neutron-proton pairs that interact harmonically. The excess neutrons reside on the surface of the nucleus and they introduce the anharmonic effect. The plot of specific heat capacity versus temperature at occupation number $n_x = 1$ depicted a nearly Gaussian curve with peak value at around $T = 0.5\text{K}$.

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Theoretical Derivations

Energy eigenvalue of unperturbed Hamiltonian operator was first derived for the Bose-Fermi mixture in the energy state n in three dimensional Cartesian coordinates. The oscillator was considered to be isotropic.

Hamiltonian of a System

For a perturbed boson-boson system we have

$$H_{1,2}(x) = -\frac{\hbar^2}{2m_{1,2}} \frac{d^2}{dx^2} + \frac{m_{1,2}\omega^2 x^2}{2} + \beta x^3 + \gamma x^4 \quad (2.1)$$

Where, βx^3 and γx^4 are cubic and quartic anharmonic perturbation terms respectively and the subscript 1,2 refers to a boson-boson or boson-fermion or fermion-fermion pair. Equation (2.1) can be generalized to three Cartesian coordinates to obtain total Hamiltonian for boson-boson, boson-fermion and fermion-fermion systems. The sum of the first and second terms on the right of equation (2.1) represents the unperturbed Hamiltonian.

Eigenvalue of Unperturbed Hamiltonian

The general expression for eigenvalue of unperturbed Hamiltonian operator of a system in x - coordinates is

$$E_{1,2}^0(x) = -\frac{\hbar^2}{2m_{1,2}} \left\langle \psi_n^0(x) \left| \frac{d^2}{dx^2} \right| \psi_n^0(x) \right\rangle + \frac{m_{1,2}\omega^2}{2} \left\langle \psi_n^0(x) \left| x^2 \right| \psi_n^0(x) \right\rangle \quad (2.2)$$

The wave function is a solution to the Hermite equation. From equation (2.2) we can write the eigenvalue of kinetic energy operator of a binary system as

$$-\frac{\hbar^2}{2m_{\eta\lambda}} \left(\frac{d^2}{dx^2} \right)_{n,n} = -\frac{\hbar^2}{2m_{\eta\lambda}} N_{nx}^2 \int_{-\infty}^{\infty} H_n(x) e^{-\frac{\alpha}{2}x^2} \frac{d^2}{dx^2} H_n(x) e^{-\frac{\alpha}{2}x^2} dx \quad (2.3)$$

This gives

$$\frac{d^2}{dx^2} H_n(x) e^{-\frac{\alpha}{2}x^2} = 4n_x(n_x - 1)H_{n-2}(x)e^{-\frac{\alpha}{2}x^2} - 4n_x H_{n-1}(x)\alpha x e^{-\frac{\alpha}{2}x^2} - H_n(x)\alpha e^{-\frac{\alpha}{2}x^2} + H_n(x)\alpha^2 x^2 e^{-\frac{\alpha}{2}x^2} \quad (2.4)$$

Substituting the terms on the right of equation (2.4) in equation (2.3) we obtain the first integral as

$$I_1 = \frac{-\hbar^2}{2m_{bb}} \cdot N_{nx}^2 \cdot 4n_x(n_x - 1) \int_{-\infty}^{\infty} H_{n_x}(x) H_{n_x-2}(x) e^{-\alpha x^2} dx = 0 \quad (2.5)$$

Since, $H_{n_x}(x)e^{-\frac{\alpha}{2}x^2}$ and $H_{n_x-2}(x)e^{-\frac{\alpha}{2}x^2}$ are orthogonal functions. Similarly, we can write the second integral as

$$I_2 = \frac{-\hbar^2}{2m_{bb}} \cdot N_{nx}^2 \cdot (-4n_x\alpha) \cdot \int_{-\infty}^{\infty} H_{n_x}(x) H_{n_x-1}(x) x e^{-\alpha x^2} dx \quad (2.6)$$

Hence,

$$\frac{\hbar^2}{2m_{bb}} \cdot \frac{N_{nx}}{N_{n_x-1}} \cdot (4n_x\alpha) \cdot \int_{-\infty}^{\infty} N_{nx} N_{n_x-1} H_{n_x}(x) H_{n_x-1}(x) x e^{-\alpha x^2} dx = \frac{\hbar^2}{2m_{bb}} \cdot \frac{N_{nx}}{N_{n_x-1}} \cdot (4n_x\alpha) \cdot (x)_{n,n-1} \quad (2.7)$$

The identity for matrix element is

$$(x)_{i,j} = \sqrt{\frac{j+1}{2\alpha}} \delta_{j,i-1} + \sqrt{\frac{j}{2\alpha}} \delta_{j,i+1} \quad (2.8)$$

Substituting for $\frac{N_{nx}}{N_{n_x-1}} = \frac{1}{\sqrt{2n_x}}$ in equation (2.7) we now obtain

$$I_2 = \left(\frac{\hbar^2}{2m_{\eta\lambda}} \cdot 2n_x\alpha \right) \alpha^{-\frac{1}{2}} \quad (2.9)$$

The value in brackets of equation (2.9) represents part of kinetic energy, K_2 expressed as

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$$K_2 = \frac{\hbar^2}{2m_{\eta\lambda}} \cdot 2n_x \alpha \quad (2.10)$$

The third integral is obtained by substituting $\frac{d^2}{dx^2} H_n(x)$ in equation (2.3) with the third term on the right of equation (2.4) to obtain

$$I_3 = \frac{-\hbar^2}{2m_{1,2}} \cdot N_{n_x}^2 \cdot (-\alpha) \int_{-\infty}^{\infty} H_{n_x}(x) H_{n_x}(x) e^{-\alpha x^2} dx = \frac{\hbar^2}{2m_{1,2}} \alpha \quad (2.11)$$

The fourth integral is obtained by substituting $\frac{d^2}{dx^2} H_n(x)$ in equation (2.3) with the fourth term on the right of equation (2.4) to get

$$I_4 = \frac{-\hbar^2}{2m_{1,2}} \cdot N_{n_x}^2 \cdot \alpha^2 \int_{-\infty}^{\infty} H_{n_x}(x) H_{n_x}(x) x^2 e^{-\alpha x^2} dx = \frac{-\hbar^2}{2m_{1,2}} \alpha \left(n_x + \frac{1}{2}\right) \quad (2.12)$$

The sum of the integrals I_3 and I_4 and the kinetic energy K_2 give the kinetic energy eigenvalues for boson-boson (bb), boson-fermion (bf) and fermion-fermion (ff) systems as

$$-\frac{\hbar^2}{2m_{bb}} \left(\frac{d^2}{dx^2} \right)_{n,n} = -\frac{\hbar^2}{2m_{bf}} \left(\frac{d^2}{dx^2} \right)_{n,n} = -\frac{\hbar^2}{2m_{ff}} \left(\frac{d^2}{dx^2} \right)_{n,n} = \frac{\hbar\omega}{2} \left(n_x + \frac{1}{2} \right) \quad (2.13)$$

Hence the total energy eigenvalue of unperturbed system in three Cartesian coordinates is

$$E_n^0 = 3\hbar\omega \left(n_x + n_y + n_z + \frac{3}{2} \right) \quad (2.14)$$

Eigenvalue of Perturbed Hamiltonian

Cubic perturbation

The total first order energy correction to cubic perturbation operator is

$$\beta((x^3)_{n,n} + (y^3)_{n,n} + (z^3)_{n,n}) = 0 \quad (2.15)$$

Since the integrand is odd and therefore not symmetric in the interval $(-\infty, \infty)$ of x , y and z . Applying the second order time independent perturbation theory, the energy eigenvalue for the a system in x -coordinate reads (Harrison, 2008),

$$E_n^2(x) = \frac{\beta^2}{\hbar\omega} \sum_{m \neq n} \frac{(x^3)_{n,m}^2}{n-m} = \frac{\beta^2}{\hbar\omega} \left[\frac{(x^3)_{n,n+2}^2}{-3} + \frac{(x^3)_{n,n+1}^2}{-1} + \frac{(x^3)_{n,n-1}^2}{1} + \frac{(x^3)_{n,n-3}^2}{3} \right] \quad (2.16)$$

which simplifies to

$$E_n^2(cubic) = -\frac{30\beta^2}{8\hbar\omega} \left[\left(\frac{\hbar}{m_{bb}\omega} \right)^3 + \left(\frac{\hbar}{m_{bf}\omega} \right)^3 \right] \left(n_x^2 + n_y^2 + n_z^2 + n_x + n_y + n_z + \frac{11}{10} \right) \quad (2.17)$$

Quartic perturbation

The first order correction to quartic perturbation in the x -coordinate is

$$E_n^1(x, quartic) = \frac{3\gamma}{4} \left[\left(\frac{\hbar}{m_{bb}\omega} \right)^2 + \left(\frac{\hbar}{m_{bf}\omega} \right)^2 \right] (2n_x^2 + 2n_x + 1) \quad (2.18)$$

Therefore, in three Cartesian coordinates we can write the total first order correction to perturbation of the mixture as

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$$E_n^1 = \frac{3\gamma}{4} \left[\left(\frac{\hbar}{m_{bb}\omega} \right)^2 + \left(\frac{\hbar}{m_{bf}\omega} \right)^2 \right] (2n_x^2 + 2n_y^2 + 2n_z^2 + 2n_x + 2n_y + 2n_z + 3) \quad (2.19)$$

The second order energy correction is

$$E_n^2(x, \text{quartic}) = \frac{\gamma^2}{\hbar\omega} \left[\frac{(x^4)_{n,n+4}^2}{-4} + \frac{(x^4)_{n,n-4}^2}{4} + \frac{(x^4)_{n,n+2}^2}{-2} + \frac{(x^4)_{n,n-2}^2}{2} \right] \quad (2.20)$$

Solving equation (2.20) we obtain in three dimensions,

$$E_n^2(\text{quartic}) = -\frac{\gamma^2}{8\hbar\omega} \left[\left(\frac{\hbar}{m_{bb}\omega} \right)^4 + \left(\frac{\hbar}{m_{bf}\omega} \right)^4 \right] \times \left(\begin{matrix} 34n_x^3 + 34n_y^3 + 34n_z^3 + 51n_x^2 + 51n_y^2 + \\ 51n_z^2 + 59n_x + 59n_y + 59n_z + 63 \end{matrix} \right)$$

Expectation Value of Energy

The total energy of the binary mixture is the sum,

$$E_n = E_n^0 + E_n^1 + E_n^2 \quad (2.21)$$

Substituting for E_n^0 , E_n^1 and E_n^2 in equation (2.21) we obtain,

$$\begin{aligned} E_n = & 3\hbar\omega \left(n_x + n_y + n_z + \frac{3}{2} \right) + \frac{3\gamma}{4} \left[\left(\frac{\hbar}{m_{bb}\omega} \right)^2 + \left(\frac{\hbar}{m_{bf}\omega} \right)^2 \right] \left(\begin{matrix} 2n_x^2 + 2n_y^2 + 2n_z^2 + 2n_x \\ + 2n_y + 2n_z + 3 \end{matrix} \right) - \\ & \frac{30\beta^2}{8\hbar\omega} \left[\left(\frac{\hbar}{m_{bb}\omega} \right)^3 + \left(\frac{\hbar}{m_{bf}\omega} \right)^3 \right] \left(\begin{matrix} n_x^2 + n_y^2 + n_z^2 + n_x + n_y + n_z + \frac{11}{10} \end{matrix} \right) - \\ & \frac{\gamma^2}{8\hbar\omega} \left[\left(\frac{\hbar}{m_{bb}\omega} \right)^4 + \left(\frac{\hbar}{m_{bf}\omega} \right)^4 \right] \times \left(\begin{matrix} 34n_x^3 + 34n_y^3 + 34n_z^3 + 51n_x^2 + 51n_y^2 + \\ 51n_z^2 + 59n_x + 59n_y + 59n_z + 63 \end{matrix} \right) \end{aligned} \quad (2.22)$$

The total internal energy of the Bose-Fermi mixture is (Sakwa et al., 2013)

$$\langle E_{neff} \rangle = \rho_n (E_n - \mu) \quad (2.23)$$

Where, μ is the fermion chemical potential and ρ_n is the probability density operator for the single mode or excitation in three Cartesian coordinates given by

$$\rho_n = e^{-\frac{3\hbar\omega(n_x+n_y+n_z)}{kT}} \quad (2.24)$$

Where, $\omega = \omega_x = \omega_y = \omega_z$ for an isotropic oscillator and T is thermodynamic temperature in Kelvins.

Thus, $\omega_x + \omega_y + \omega_z = 3\omega$.

Heat Capacity

Heat capacity at constant volume is given by (Laud, 2012)

$$C_V = (E_n - \mu) \frac{\partial \rho_n}{\partial T} \quad (2.25)$$

Hence by substituting for ρ_n in equation (2.25) and differentiating we obtain

$$C_V(T) = (E_n - \mu) \frac{3\hbar\omega(n_x + n_y + n_z)}{kT^2} e^{-\frac{3\hbar\omega(n_x+n_y+n_z)}{kT}} \quad (2.26)$$

Entropy

Entropy S is given by (Laud, 2012)

$$S(T) = \int C_V \frac{dT}{T} \quad (2.27)$$

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Substituting for C_V in equation (2.26) gives,

$$S(T) = (E_n - \mu) \left(\frac{1}{T} + \frac{k}{3\hbar\omega(n_x + n_y + n_z)} \right) e^{-\frac{3\hbar\omega(n_x + n_y + n_z)}{kT}} \quad (2.28)$$

Parameters for Data Analysis

The parameters βx^3 and γx^4 in equation (2.1) have dimensions of energy, then β and γ can be expressed in terms of the scattering length a_0 as:

$$\beta = \frac{\hbar\omega}{a_0^3}, \gamma = \frac{\hbar\omega}{a_0^4} \quad (2.29)$$

The scattering length is given as: $a_0 = 1.3 \times 10^{-13} A^{\frac{1}{3}} \text{cm}$ (Sakwa et al., 2013), where A is the mass number of a boson.

The parameter μ_f is the fermion chemical potential given by:

$$\mu_f = \frac{\pi^2 \hbar^2}{2m_f} \left(\frac{3N_A}{\pi V} \right)^{\frac{2}{3}} \quad (2.30)$$

Where, m_f is the molar mass of the fermion and N_A is the number of the particles in one mole, known as the Avogadro's number, whose value is $N_A = 6.025 \times 10^{23} \text{mol}^{-1}$. Atomic mass of ^4He is $A = 4$, atomic mass of ^3H is $A = 3$, boson-boson reduced molar mass is $1.86 \times 10^{-3} \text{Kgmol}^{-1}$, boson-fermion reduced molar mass is $1.63 \times 10^{-3} \text{Kgmol}^{-1}$, fermion chemical potential is $\mu_f = 5.0944 \times 10^{-46} \text{J}$.

Therefore, $m_{bb} = \frac{1.86 \times 10^{-3}}{6.025 \times 10^{23}} = 3.087 \times 10^{-27} \text{Kg}$, $m_{bf} = 2.705 \times 10^{-27} \text{Kg}$.

RESULTS AND DISCUSSION

Equations (2.23), (2.26) and (2.28) were used to obtain graphs of variation of energy, specific heat capacity and entropy with temperature respectively for the mixture in the first, second and third excited states as shown in figures 1, 2 and 3 respectively.

Energy of the Binary $^4\text{H} - ^3\text{H}$ System

Figure 1 shows the variation of internal energy, $\langle E_{eff} \rangle$ with temperature for $n=1$, $n=2$ and $n=3$.

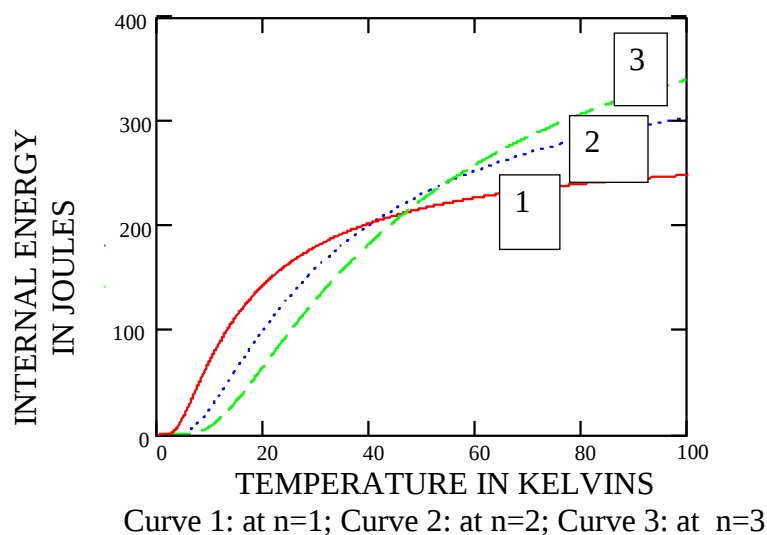


Figure 1: Graph of Variation of Internal Energy Versus Temperature

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Figure 1 shows that for an excited state the internal energy of an interacting $^4\text{He}-^3\text{He}$ mixture in a three dimensional anharmonic potential increases with increase in temperature then becomes constant at higher temperatures. This result is in agreement with the results obtained by Sakwa *et al.*, (2013) for a $^4\text{He}-^3\text{He}$ mixture and Khanna *et al.*, (2010) for ^{163}Dy nuclei, both in a one dimensional anharmonic potential. A temperature rise is accompanied by an increase in the kinetic energy of particles up to some temperature where no more heat is absorbed. Hence, beyond some higher temperature, the strength of repulsive interactions is just enough to resist further absorption of energy. At lower temperatures near absolute zero internal energy of the BF mixture at lower excited state is higher than that at higher excited state and at higher temperatures internal energy at higher excited state is higher than at lower excited state as shown in figure 1. At higher temperatures, more particles with greater kinetic energy occupy higher excited state. At lower temperatures, more particles occupy lower occupation state. At temperatures very close to absolute zero temperature all the curves of figure 3 show that internal energy at excited states are zero. Hence, all the particles at absolute zero condense into the ground state. It is also noted that in the first, second and third excited states the mixture has internal energy of about 150 joules at 40 Kelvins, 200 joules at 48 Kelvins and 240 joules at 58 Kelvins respectively. At temperatures less than 40 Kelvins the mixture has less internal energy at higher excited states. At temperatures higher than 58 Kelvins the mixture has more internal energy at higher excited states. Near absolute zero temperature, the internal energy of the mixture in an excited state is zero.

Specific Heat Capacity of $^4\text{He}-^3\text{He}$ System

Figure 2 shows the variation of specific heat capacity ($C_v(T)$) with temperature for $n=1$, $n=2$ and $n=3$.

The curves show peaks in specific heat capacity at different temperatures. At higher excited state, repulsive interactions in the mixture are stronger, increasing rate of cooling. Hence, the mixture condenses easily at higher temperature. The mixture at higher transition temperature requires less energy to be absorbed to raise temperature by one Kelvins since attractive interactions are weaker at higher excited states. At lower transition temperature attractive interactions are much stronger, requiring more energy to be absorbed per Kelvins rise in temperature. Thus, an additional second order correction to quartic perturbation lowers peak value of specific heat capacity and raises the transition temperature in the present case. An additional second order correction to quartic perturbation increases rate of cooling by further increasing strength of repulsive interactions.

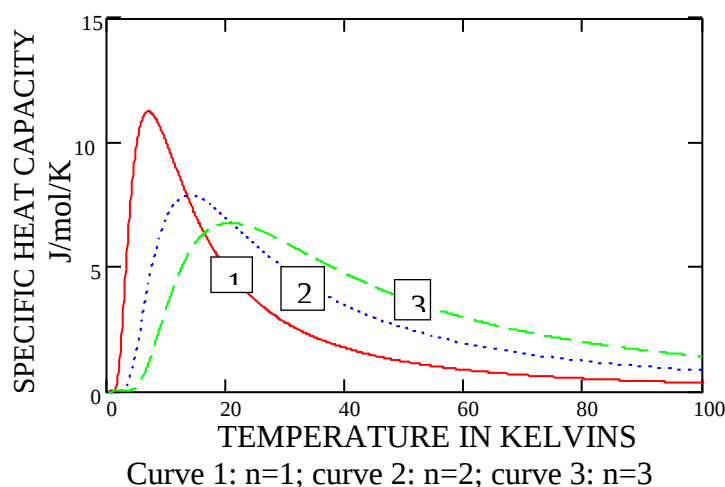


Figure 2: Graph of Variation of Specific Heat Capacity at Constant Volume Versus Temperature

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The three Cartesian coordinates contribute to a larger second order correction to perturbation. In figure 2 the peak values of specific heat capacity of the mixture in the first, second and third excited states are about 11.58J/mol/K at 6.9 kelvins, 7.88J/mol/K at 13.7 Kelvins and 6.75J/mol/K at 20.9 kelvins respectively. The turning point in the curve is sharpest at first excited state. At temperatures higher than 20.9 kelvins specific heat capacity of the mixture has a highest value in the third excited state.

Entropy of $^4\text{He}-^3\text{He}$ System

Figure 3 shows the variation of entropy, $S(T)$ with temperature for $n=1$, $n=2$ and $n=3$.

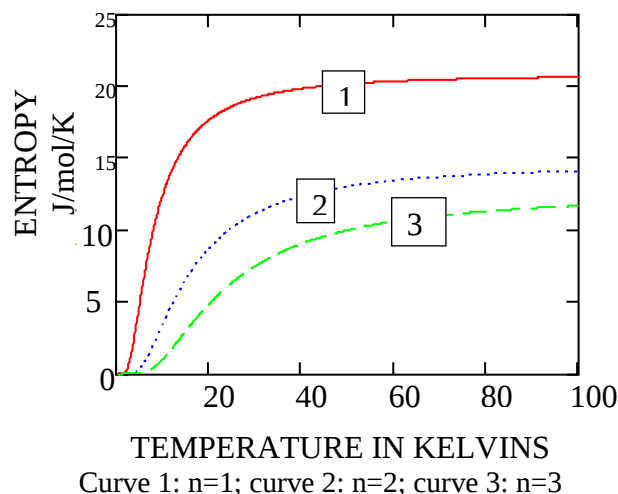


Figure 3: Graph of Entropy Versus Temperature

It can be observed from figure 3 that at temperatures higher than absolute zero, entropy at lower excited states is higher than entropy at higher excited states. At higher occupation states stronger repulsive interactions result in more ordered fermionic spins which reduce dispersion of energy from surroundings into the system at constant temperature. Entropy at an excited state is zero at temperatures closer to absolute zero since there are no particles in higher occupation states at these temperatures. This is due to BE condensation resulting in an infinite number of particles entering the ground state. It is observed that entropy at an excited state increases with temperature, then levels off at higher temperatures. This is similar to results of Khanna *et al.*, (2010) and Sakwa *et al.*, (2013). Increase in temperature provides more kinetic energy, causing increase in thermal entropy.

Conclusion

In the article the $^4\text{He}-^3\text{He}$ mixture is assumed to be strongly interacting in pairs. The three-dimensional Hamiltonian of the mixture consists of the harmonic part, with anharmonic parts being introduced by unpaired proton in ^3He nucleus. A temperature rise is accompanied by an increase in the kinetic energy of particles up to some temperature where no more heat is absorbed. This result is in agreement with the results obtained by Sakwa *et al.*, (2013) for a $^4\text{He}-^3\text{He}$ mixture and Khanna *et al.*, (2010) for ^{163}Dy nuclei. Energy and particle equilibrium between higher states is achieved at higher temperatures than between lower excited states as shown in figure 1. The binary mixture in a higher excited state condenses more easily at higher transition temperature as shown in figure 2 due to presence of stronger repulsive interactions that increase the rate of cooling. The mixture at higher excited state has a smaller peak value of specific heat capacity at transition temperature due to presence of stronger repulsive interactions that reduce rate of absorption of heat energy. An additional second order correction to quartic perturbation

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increases rate of cooling by further increasing strength of repulsive interactions. The three Cartesian coordinates contribute to a larger second order correction to perturbation, which in turn raises transition temperature and lowers the peak value of specific heat capacity at the transition temperature. This is supported by comparison with results of Sakwa *et al.*, (2013) in which the peak value of specific heat capacity at first excited state is higher and transition temperature is lower due to only the first order correction to quartic perturbation and the second order correction to cubic perturbation in one-dimensional coordinate.

Entropy at a particular temperature decreases with increase in the occupation number since at higher states the spins of particles tend to align more in the same direction.

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