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Research Article

ON THE ABSOLUTE NEVANLINNA SUMMABILITY TO THE SERIES RELATED TO WALSH-FOURIER SERIES

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ABSTRACT
The object of the present paper is to study the absolute Nevanlinna summability of the factored Walsh-Fourier series with factor of the type n^{α} , 0≤ $\alpha < 1$, which includes absolute Nevanlinna summability of the Walsh-Fourier series ($\alpha = 0$).

Keywords: *Dyadic Rational, Dyadic Derivative, Walsh-Fourier Series, Trigonometric Fourier Series, Absolute Nevanlinna Summability, Cesa`ro Summability*

INTRODUCTION

1.1 In a paper [3] published in 1921, Nevanlinna (1921) suggested and discussed an interesting method, called N_q-method. Ray and Samal (1980) studied absolute N_q -summability of Trigonometric Fourier and its conjugate series. Later Samal (1986) extended N_q-method to N_{q_α} -method $(0 \le \alpha < 1)$ and studied absolute N_{q_α}–summability of factored Trigonometric Fourier series and factored conjugate series with factor of the type n^{α}(0< $\alpha < 1$). The object of the present paper is to study the analogous result of Samal's result.

1.2 The N_q-Method

Let F (w) be a function continuous parameter w defined for all w > 0 and let it be desired to consider the "generalized limit" of F(w) as w→ ∞ . The N_q-method consists in forming the N_q -transform or mean

$$N_q F(w) \equiv \int_0^1 q(t)F(wt)\,dt$$

and then considering the

$$\lim_{w\to\infty} N_q F(w)$$

where, the class of functions q is such that

q(t) is non-negative and monotonic increasing for 0< $t < 1$, (1)

$\int_0^1 q(t)\,dt = 1$ (2)

and $\lim_{\delta\to 0}\int_0^{1-\delta} q(t)\log\frac{1}{1-t}\,dt$ exists (3)

When $\lim_{w\to\infty} N_q F(w)$ exists, it is called the N_q-limit of F(w).

By a lemma due to Moursund (1832)

$\int_0^1 \frac{Q(t)}{t}\,dt$ exists if and only if $\lim_{\delta\to 0}\int_0^{1-\delta} q(t)\log\frac{1}{1-t}\,dt$ exists,

Where, Q (t) =$\int_{1-t}^1 q(u)\,du$.

So the condition (3) can be replaced by $\int_0^1 \frac{Q(t)}{t}\,dt$ exists.

The N_{q_α} -Method $(\mathbf{0} \le \boldsymbol{\alpha} < 1)$ ***(Samal, 1986).***

Let the class of functions q_α be such that

q_α(t) is non-negative and monotonic increasing for 0< $t < 1$,

$\int_0^1 q_\alpha(t)\,dt = 1$

and $\int_0^1 \frac{Q_\alpha(t)}{t^{\alpha+1}}\,dt$ exists, where $Q_\alpha(t) = \int_{1-t}^1 q_\alpha(u)\,du$.

The series ΣU_n is summable by N_{q_α}-method to S if $\lim_{w\to\infty}\sum_{n\le w} U_n\ Q_\alpha(1-\frac{n}{w}) = S$

Again the series ΣU_n summable $|N_{q_\alpha}|$ (or absolute N_{q_α}-summable) if
$\int_A^\infty \frac{dw}{w^2}\left|\sum_{n\le w} nU_n q_\alpha\left(\frac{n}{w}\right)\right| < \infty$, where, A is a positive constant.
For $\alpha = 0$ the N_{q_α}-method reduces to N_q-method.

1.3

The Rademacher functions are defined by

$$r_0(x) = \begin{cases} 1, & if\ x \in [0,1/2) \\ -1, & if\ x \in [1/2,1) \end{cases}$$

$r_0(x+1) = r_0(x)$, $\quad r_n(x) = r_0(2^n x)$, $\quad (n = 1,2,3, \dots\dots\dots\dots\dots\dots.)$
Walsh functions are given by
$w_0(x) = 1$
and $w_n(x) = r_{n_1}(x) r_{n_2}(x) \dots\dots\dots\dots\dots\dots\dots\dots\dots.. r_{n_k}(x)$,
for n= $2^{n_1} + 2^{n_2} + \dots\dots\dots\dots\dots + 2^{n_k}$, where integers n_i are uniquely determined with $n_{i+1} < n_i$.
Any $x \in [0,1]$ can be written in the form

$$x = \sum_{k=0}^\infty x_k 2^{-(k+1)}\ ; \tag{4}$$

Where, each $x_k = 0$ or 1. If x is not dyadic rational, it can be uniquely expressed in the form (4). We call it dyadic expansion of x. For dyadic rationals, there are two expressions of this form, one which terminates with 0's and other which terminates with 1's. By dyadic expansion of dyadic rational, we shall mean the one which terminates with 0's.

Dyadic Derivative

For each function f defined on $[0,1)$ and non-negative integer n, let
$d_n f(x) = \sum_{j=0}^{n-1} 2^{j-1}\{f(x) - f(x \dotplus 2^{-j-1})\}$ for $x \in 0,1)$.
Then, f is said to be dyadically differentiable at x if $f^{[1]}(x) = \lim_{n\to\infty} d_n f(x)$ exists and finite. $f^{[1]}(x)$ is called dyadic derivative of f at x.

1.4.

Let $f(x)$ be a periodic function with period 1 and Lebesgue integrable over (0, 1). Then, the Walsh-Fourier series of f is
$f(x) \sim \sum_0^\infty c_k\, w_k(x) \equiv \sum_o^\infty A_k(x)$, where, $c_k = \int_0^1 f(u)\, w_k(u)\, du$.
We have

$$w_k(x)\, w_k(y) = w_k(x \dotplus y)\ \text{, where } x \dotplus y = \sum_{k=0}^\infty |x_k - y_k|\, 2^{-(k+1)}\ ,$$

if $x = \sum_{k=o}^\infty x_k\, 2^{-(k+1)}$ and $y = \sum_{k=0}^\infty y_k\, 2^{-(k+1)}$ are dyadic expansion x and y respectively.
By a result due to Fine [1]

$$A_k(x) = \int_0^1 f(u)\, w_k(u) w_k(x)\, du$$
$$= \int_0^1 f(u)\, w_{k(x+u)}\, du$$
$$= \int_0^1 f(x \dotplus u)\, w_k(u)\, du$$

Purpose of the Present Work

The absolute Nevanlinna summability of the Trigonometric Fourier series, Conjugate series and their factored series has been studied in the following problems.

Theorem A (Ray and Samal, 1980)

$\phi(s) \in BV\ (0,\pi) \Rightarrow \sum_{n=0}^\infty A_n(x) \in |N_q|$

Theorem B (Ray and Samal 1980)

$\psi(s) \in BV\ (0,\pi)$ *and* $\frac{\psi(s)}{s} \in L(0,\pi) \Rightarrow \sum_{n=1}^\infty B_n(x) \in |N_q|$

Theorem C (Samal, 1986)

If $0 < \alpha < 1$ and $\int_0^\pi u^{-\alpha}\, |d\phi(u)| < \infty$,
then $\sum_{n=1}^\infty n^\alpha A_n(u)$ *is* $|N_{q_\alpha}|$-summable at $u = x$.

Theorem D (Samal, 1986)

If $0 < \alpha < 1$, (i) $\psi(+0) = 0$ and (ii) $\int_0^{\pi} u^{-\alpha} |d\psi(u)| < \infty$,
then $\sum_{n=1}^{\infty} n^{\alpha} B_n(u)$ *is* $|N_{q_\alpha}|$-summable at $u = x$.

In the year 2000, in a paper Sahoo (2000) we have studied the following problem on absolute cesàro summability of factored Walsh-Fourier series.

Theorem E (Sahoo, 2000)

Let $\phi(t) = f(x \dot{+} t) - S$, where S is function of x. If $\phi(t)$ is strongly differentiable in X and $\int_0^1 \frac{|\phi^{[1]}(t)|}{t^{\alpha}} dt < \infty$, $0 < \alpha < 1$, then the series $\sum_{n=1}^{\infty} n^{\alpha} A_n(x)$ is summable $|C, \beta|$, $\beta > \alpha$.

The object of the following paper is to prove the following theorem on Walsh- Fourier series.

Theorem

Let $\phi(t) = f(x \dot{+} t) - S$, where S is function of x and $0 \leq \alpha < 1$.

If $\int_0^1 u^{-\alpha} |d\phi(u)| < \infty$, then the series $\sum_{n=1}^{\infty} n^{\alpha} A_n(x)$ is summable $|N_{q_\alpha}|$.

For $\alpha = 0$, the theorem becomes

If $\phi(u) \in BV\ (0,1)$ *then* $\sum_{n=1}^{\infty} A_n(x)$ *is summable* $|N_q|$

If we take $q_\alpha(t) = \begin{cases} (\alpha+\delta)(1-t)^{\alpha+\delta-1}, & 0 \leq \alpha < 1 \text{ and } 0 < \alpha+\delta \leq 1 \text{ for } 0 < t < 1 \\ 0, & \text{for } t \geq 1 \end{cases}$

Then, N_{q_α}-method reduces to Ces'aro method (C, $\alpha + \delta$).

So by taking $q_\alpha(t)$ as above in our theorem we obtain the following corollary.

Corollary

If $0 \leq \alpha < 1$ and $\int_0^1 u^{-\alpha} |d\phi(u)| < \infty$, then the series is $\sum_{n=1}^{\infty} n^{\alpha} A_n(x)$ is summable $|C, \beta|$, $\beta > \alpha$.

Notations and Lemmas

3.1.

We need the following notations.

$[w]$= N

$D_n(u) = \sum_{k=0}^{n-1} w_k(u)$

$J_k(u) = \int_0^u w_k(x)\, dx$.

3.2.

We need the following lemmas for the proof of our theorem.

Lemma 1 (Samal, 1986)

For $0 < \alpha < 1$, $\int_0^1 \frac{Q_\alpha(t)}{t^{\alpha+1}} dt$ exists if and only if $\int_0^1 \frac{q_\alpha(t)}{(1-t)^{\alpha}} dt$ exists.

Lemma 2 (Samal, 1986)

For $0 \leq \alpha < 1$, $\sum_{k=1}^{\infty} \frac{Q_\alpha\left(\frac{1}{k}\right)}{k^{1-\alpha}}$ is convergent.

Lemma 3 (Fine, 1949)

$J_k(u) = w_{k'}(u) J_{2^n}(u)$, *where* $k \geq 1, k = 2^n + k', 0 \leq k' < 2^n$, $n = 0,1,2,3, \ldots\ldots\ldots\ldots\ldots$

and $J_{2^n}(u) = 2^{-(n+2)} \{1 - \sum_{r=1}^{\infty} 2^{-r} w_{2^r+1}(2^n u)\}$, $n = 0, 1, 2, 3 \ldots\ldots\ldots\ldots\ldots$

Lemma 4 ((Schip et al., 1990) page 35, Theorem 10)

For $0 < u < 1$, $|D_n(u)| < min\left(n, \frac{2}{u}\right)$

Lemma 5

$$D_{2^n}(u) = \begin{cases} 2^n, if\ u \in \left[0, \frac{1}{2^n}\right) \\ 0, if\ u \in \left[\frac{1}{2^n}, 1\right) \end{cases}$$

The proof of this lemma is similar to Paley's Lemma([7], *page* 7).

Lemma 6

For $0 < u < 1$ *and* $0 < wu < 1$,

$\sum_{k\le w-1} k^{\alpha+1}\ q_\alpha\left(\frac{k}{w}\right) J_k(u) = O(uw^{\alpha+2})$.

Proof.

$\sum_{k\le w-1} k^{\alpha+1}\ q_\alpha\left(\frac{k}{w}\right) J_k(u)$

$\le uw^{\alpha+1}\ \sum_{k\le w-1} q_\alpha\left(\frac{k}{w}\right)$

$< uw^{\alpha+2}$.

Lemma 7

For $k = 2^n + k^{'}, 0 \le k^{'} < 2^n and\ 0 < u < 1$,

$\sum_{k=1}^{p} 2^{(n+2)} J_k(u) = O\left(\frac{1}{u}\right)$,

for any positive integer p .

Proof.

Let $p = 2^q + p^{'}, 0 \le p' < 2^q$ for some positive integer q and $\frac{1}{2^m} \le u < \frac{1}{2^{m-1}}$.

Then

$\left|\sum_{k=1}^{p} 2^{(n+2)} J_k(u)\right|$

$= \left|\sum_{r=0}^{q-1}\sum_{k^{'}=0}^{2^r-1} 2^{r+2}\ J_{2^r+k'}(u) + \sum_{k^{'}=0}^{p'} 2^{q+2} J_{2^q+k'}(u)\right|$

$\le \left|\sum_{r=0}^{q-1}\sum_{k^{'}=0}^{2^r-1}\{1-\sum_{i=1}^{\infty} 2^{-i} w_{2^i+1}(2^r u)\} w_{k'}(u)\right|$

$+\left|\sum_{k^{'}=0}^{p^{'}}\{1-\sum_{i=1}^{\infty} 2^{-i} w_{2^i+1}(2^q u)\} w_{k'}(u)\right|\ by\ \ Lemma\ 3$

$= \left|\sum_{r=0}^{q-1} D_{2^r}(u)\{1-\sum_{i=1}^{\infty} 2^{-i} w_{2^i+1}(2^r u)\}\right|$

$+\left|D_{p'}(u)\{1-\sum_{i=1}^{\infty} 2^{-i} w_{2^i+1}(2^q u)\}\right|$

$\le \sum_{r=0}^{q-1} 2|D_{2^r}(u)| + 2\left|D_{p'}(u)\right|$

$\le \sum_{r=0}^{m-1} 2^{r+1}\ + O\left(\frac{1}{u}\right),\ by\ Lemma\ 5\ and\ Lemma 4.$

$= 2^{m+1} - 2\ + O\left(\frac{1}{u}\right)$

$= O\left(\frac{1}{u}\right)$

Lemma 8

For $0 < u \le 1$,

$\sum_{k=1}^{2^n-1} kJ_k(u) = O\left(\frac{1}{u}\right)$

Proof.

For $0 < u \le 2^{-n}$ and $1 \le k \le 2^n - 1$

$J_k(u) = u$ and therefore, it is easy to show that $\sum_{k=1}^{2^n-1} kJ_k(u) = O\left(\frac{1}{u}\right)$ for $0 < u \le 2^{-n}$.

For $u > 2^{-n}$, suppose $\frac{1}{2^m} < u \le \frac{1}{2^{m-1}}$.

Then $\left|\sum_{k=1}^{2^n-1} kJ_k\ (u)\ \right| = \left|\int_0^u \left(\sum_{k=1}^{2^n-1} kw_k(t)\right) dt\right|$

$=\left|\int_0^u D_{2^n}^{[1]}(t)\ dt\right|$ as $w_k^{[1]}(t) = kw_k(t)$

$=\left|\int_0^u \left[\sum_{j=0}^{\infty} 2^{j-1}\{D_{2^n}(t) - D_{2^n}\left(t \dotplus 2^{-j-1}\right)\}\right] dt\right|$

$\le \sum_{j=0}^{\infty} 2^{j-1}\left|\int_0^u \{D_{2^n}(t) - D_{2^n}\left(t \dotplus 2^{-j-1}\right)\} dt\right|$

$= \sum_{j=0}^{\infty} 2^{j-1}\left|\int_0^u D_{2^n}(t)\, dt - \int_0^u D_{2^n}\left(t \dotplus 2^{-j-1}\right) dt\right|$

$=\sum_{j=0}^{m-2} 2^{j-1}\ + 2^{m-2}\left|1 - \int_{2^{-m}}^{u} D_{2^n}(t \dotplus 2^{-m})\ dt\right|\ + \sum_{j=m}^{n-1} 2^{j-1}\left|1 - 2^n\int_{2^{-j-1}}^{2^{-j-1}+2^n} dt\right|$

$+\sum_{j=n}^{\infty} 2^{j-1}\left|1 - 2^n\int_0^{2^{-n}} dt\right|$ by Lemma 5.

(For $n = m$, the sum $\sum_{j=m}^{n-1} 2^{j-1} \left|1 - 2^n \int_{2^{-j-1}}^{2^{-j-1}+2^n} dt\right|$ vanishes.)
$< \sum_{j=0}^{m-1} 2^{j-1}$
$= 2^{m-1} - \frac{1}{2}$
$=O\left(\frac{1}{u}\right)$
So, for $0 < u \leq 1$, $\sum_{k=1}^{2^n-1} kJ_k(u) = O\left(\frac{1}{u}\right)$.

Lemma 9

For $0 < u \leq 1$ and positive integer p,
$\sum_{k=1}^{p} kJ_k(u) = O\left(\frac{1}{u}\right)$.
Proof.
Let p=$2^n + p', where\ 0 \leq p' < 2^n$.
Then
$\sum_{k=1}^{p} kJ_k(u) = \sum_{k=1}^{2^n-1} kJ_k(u) + \sum_{k=2^n}^{p} kJ_k(u)$
$=O\left(\frac{1}{u}\right) + \sum_{k=0}^{p'} (2^n + k) J_{(2^n+k)}(u)$, by Lemma 8.
$=O\left(\frac{1}{u}\right) + \left(\frac{2^n+p'}{2^{n+2}}\right) \max_{0\leq L,L'\leq p'} \left|\sum_{L}^{L'} (2^{n+2}) J_{(2^n+k)}(u)\right|$
$=O\left(\frac{1}{u}\right)$, by Lemma 7.

Lemma 10

For $0 < u \leq 1\ and\ wu > 1$,
$\sum_{k\leq w-\frac{1}{u}} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u) = O\left(\frac{w^\alpha q_\alpha\left(1-\frac{1}{wu}\right)}{u}\right)$
Proof.
$\left|\sum_{k\leq w-\frac{1}{u}} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u)\right|$
$\leq \left(w - \frac{1}{u}\right)^\alpha q_\alpha\left(1 - \frac{1}{wu}\right) \max_{1\leq L,L'\leq w-\frac{1}{u}} \left|\sum_{k=L}^{L'} kJ_k(u)\right|$
$= O\left(\frac{w^\alpha q_\alpha\left(1-\frac{1}{wu}\right)}{u}\right)$, by Lemma 9.

Lemma 11

For $0 < u \leq 1\ and\ wu > 1$,
$\sum_{k=\left[w-\frac{1}{u}\right]+1}^{[w-1]} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u) = O\left(w^{\alpha+1} Q_\alpha\left(\frac{1}{wu}\right)\right)$
Proof.
$.\left|\sum_{k=\left[w-\frac{1}{u}\right]+1}^{[w-1]} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u)\right|$

$\leq \int_{w-\frac{1}{u}}^{w} x^\alpha q_\alpha\left(\frac{x}{w}\right) dx$ Since $J_k(\mathrm{u}) < \frac{1}{\mathrm{k}}$

$=\int_{1-\frac{1}{wu}}^{1} w^{\alpha+1} t^\alpha q_\alpha(t)\, dt$
$< w^{\alpha+1} Q_\alpha\left(\frac{1}{wu}\right)$.

Proof of the Theorem

For $k \geq 1$, we have

$A_k(x) = \int_0^1 \phi(u) w_k(u)\, du$
$= [\phi(u) J_k(u)]_{u=0}^{1} - \int_0^1 J_k(u)\, d\phi(u)$
$= -\int_0^1 J_k(u)\, d\phi(u)$, $as\ \phi(+0)\ and\ \phi(1)\ are\ finite\ ;\ J_k(0) = J_k(1) = 0.$

The series $\sum_{k=1}^{\infty} k^\alpha A_k(x)$ is summable $|N_{q_\alpha}|, (0 \le \alpha < 1)$ if

$\int_1^\infty \frac{dw}{w^2} \left| \sum_{k \le w} k^{\alpha+1} A_k(x) q_\alpha\left(\frac{k}{w}\right) \right| < \infty.$

Now $\int_1^\infty \frac{dw}{w^2} \left| \sum_{k \le w} k^{\alpha+1} A_k(x) q_\alpha\left(\frac{k}{w}\right) \right|$
$= \int_1^\infty \frac{dw}{w^2} \left| \sum_{k \le w} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) \int_0^1 J_k(u)\, d\phi(u) \right|$
$\le \int_0^1 |d\phi(u)| \int_1^\infty \frac{dw}{w^2} \left| \sum_{k \le w} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u) \right|$. …………….. (5)

$\int_1^\infty \frac{dw}{w^2} \left| \sum_{k \le w} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u) \right|$
$= \int_1^\infty \frac{dw}{w^2} \left| \sum_{k \le w-1} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u) \right| + \int_1^\infty \frac{dw}{w^2} \left| N^{\alpha+1} q_\alpha\left(\frac{N}{w}\right) J_N(u) \right|$ where $N = [w]$ …… (6)

$\int_1^\infty \frac{dw}{w^2} \left| \sum_{k \le w-1} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u) \right|$
$= \int_1^{\frac{1}{u}} \frac{dw}{w^2} \left| \sum_{k \le w-1} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u) \right| + \int_{\frac{1}{u}}^\infty \frac{dw}{w^2} \left| \sum_{k \le w-1} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u) \right|$ ….. (7)

By the use of Lemma 6,

$\int_1^{\frac{1}{u}} \frac{dw}{w^2} \left| \sum_{k \le w-1} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u) \right|$
$= O\left(u \int_1^{\frac{1}{u}} w^\alpha\, dw\right)$
$= O(u^{-\alpha})$ …………… (8)

Using Lemma 10 and Lemma 11; for $0 < u \le 1\ and\ wu > 1$, we have

$\sum_{k \le w-1} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u)$
$= \sum_{k \le w - \frac{1}{u}} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u) + \sum_{k=\left[w-\frac{1}{u}\right]+1}^{[w-1]} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u)$
$= O\left(\frac{w^\alpha q_\alpha\left(1-\frac{1}{wu}\right)}{u}\right) + O\left(w^{\alpha+1} Q_\alpha\left(\frac{1}{wu}\right)\right)$

So,

$\int_{\frac{1}{u}}^\infty \frac{dw}{w^2} \left| \sum_{k \le w-1} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u) \right|$
$= \int_{\frac{1}{u}}^\infty O\left(\frac{w^\alpha q_\alpha\left(1-\frac{1}{wu}\right)}{u}\right) \frac{dw}{w^2} + \int_{\frac{1}{u}}^\infty O\left(w^{\alpha+1} Q_\alpha\left(\frac{1}{wu}\right)\right) \frac{dw}{w^2}$
$= O\left(u^{-\alpha} \int_0^1 \frac{q_\alpha(t)}{(1-t)^\alpha}\, dt\right) + O\left(u^{-\alpha} \int_0^1 \frac{Q_\alpha\{t\}}{t^{\alpha+1}}\, dt\right)$
$= O(u^{-\alpha})$ ……… (9)

Now (7), (8) and (9) together imply

$\int_1^\infty \frac{dw}{w^2} \left| \sum_{k \le w-1} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u) \right| = O(u^{-\alpha})$ ….. (10)

Also we have

$\int_1^\infty \frac{dw}{w^2} \left| N^{\alpha+1} q_\alpha\left(\frac{N}{w}\right) J_N(u) \right|$
$\le \int_1^\infty N^\alpha q_\alpha\left(\frac{N}{w}\right) \frac{dw}{w^2}$
$= \sum_{k=1}^\infty \int_k^{k+1} N^\alpha q_\alpha\left(\frac{N}{w}\right) \frac{dw}{w^2}$
$= \sum_{k=1}^\infty \int_k^{k+1} k^\alpha q_\alpha\left(\frac{k}{w}\right) \frac{dw}{w^2}$ as $k < w < k+1$

$= \sum_{k=1}^{\infty} \frac{1}{k^{1-\alpha}} \int_{\frac{1}{k+1}}^{\frac{1}{k}} q_\alpha(u)\, du$

$= \sum_{k=1}^{\infty} \frac{Q_\alpha\left(\frac{1}{k+1}\right)}{k^{1-\alpha}}$

$< \sum_{k=1}^{\infty} \frac{Q_\alpha\left(\frac{1}{k}\right)}{k^{1-\alpha}}$

$= O(1)$ by Lemma 2. ….. (11)

By use of (10) and (11) in (6)

$\int_1^\infty \frac{dw}{w^2} \left| \sum_{k\le w} k^{\alpha+1} q_\alpha\left(\frac{k}{w}\right) J_k(u) \right| = O(u^{-\alpha})$

So from (5) we have

$\int_1^\infty \frac{dw}{w^2} \left| \sum_{k\le w} k^{\alpha+1} A_k(x) q_\alpha\left(\frac{k}{w}\right) \right|$

$= \int_0^1 O(u^{-\alpha}) |d\phi(u)|$

$= O\left(\int_0^1 \frac{|d\phi(u)|}{u^\alpha} \right)$

$= O(1)$.

Hence, $\sum_{k=1}^{\infty} k^\alpha A_k(x)$ is summable $\left|N_{q_\alpha}\right|$.

This completes the proof of our theorem.

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