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Research Article

A NOTE ON NEW TYPE OF PARANORMED SEQUENCE SPACE

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ABSTRACT
The Riesz sequence space $r^q(u,p)$ of non-absolute type, recently studied by Neyaz and Hamid (16) and the classes $(r^q(u,p):\ell_\infty)$, $(r^q(u,p):c)$ and $(r^q(u,p):c_0)$ of infinite matrices. This paper is devoted to characterize the classes $(r^q(u,p):bs)$, $(r^q(u,p):cs)$ and $(r^q(u,p):cs_0)$ of of infinite matrices and characterize a basic theorem where bs, cs and cs_0 denote respectively the space of all bounded series, the convergent series and the series converging to zero. ***AMS Mathematical Subject Classification 2010*:** 46A45; 46B25; 40C05.

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INTRODUCTION

Preliminaries, Background and Notation

A sequence space is defined to be a linear space with real or complex sequences. Throughout the paper $\mathbb{N},\mathbb{R}$ and $\mathbb{C}$ denotes the set of non-negative integers, the set of real numbers and the set of complex numbers respectively.

Let ω denote the space of all sequences (real or complex); ℓ_∞ and c respectively de- notes the space of all bounded sequences, the space of convergent sequences. Also by bs, cs, ℓ_1 and ℓ_p we denote the spaces of all bounded, con- vergent, absolutely and p-absolutely convergent series, respectively. A linear Topological space X over the field of real numbers $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $h: X \to \mathbb{R}$ such that

$h(\theta)=0, h(x)= h(-x)$ and scalar multiplication is continuous, that is, $|\alpha_n-\alpha|\to 0$ and $h(x_n-x)\to 0$ imply $h(\alpha_n x_n-\alpha x)\to 0$ for all α's in $\mathbb{R}$ and $x's$ in X, where θ is a zero vector in the linear space X. Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup_k p_k = H$ and $M = max\{1,H\}$. Then, the linear spaces $\ell(p)$ and $\ell_\infty(p)$ were defined by Maddox (1967) (see, also Nakano (Nakano, 1951) etc) as follows:

$$\ell(p)=\left\{x=(x_k)\in\omega: \sum_k |x_k|^{p_k} < \infty\right\} \text{with } 0<p_k\le H<\infty$$

and

$$\ell_\infty(p)=\left\{x=(x_k)\in\omega: \sup_k |x_k|^{p_k}<\infty\right\},$$

which are complete spaces paranormed by

$$g_1(x)=\left(\sum_k |x_k|^{p_k}\right)^{\frac{1}{M}} \text{ and } g_2(x)= \sup_k |x_k|^{\frac{p_k}{M}} \text{ iff } \inf p_k>0.$$

We shell assume throughout the text that $p_k^{-1}+(p_k')^{-1}=1$ provi- ded $1<\inf p_k\le H<\infty$ and we denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$, where $\mathbb{N}=\{0,1,2,\dots\}$.

For the sequence space X and Y, define the set

$$S(X,Y)=\{z=(z_k)\in\omega: xz=(x_k z_k)\in Y \forall x\in X\}. \quad (1)$$

With the notation of (1), the α-,β- and γ- duals of a sequence space X, which are respectively denoted by X^α, X^β and X^γ are defined by

$X^\alpha = S(X:\ell_1)$, $X^\beta = S(X:cs)$ and $X^\gamma = S(X:bs)$.

If a sequence space X paranormed by h contains a sequence (b_n) with the property that for every $x \in X$ there is a unique sequence of scalars (α_n) such that

$$\lim_n h\left(x - \sum_{k=0}^{n} \alpha_k b_k \right) = 0,$$

Then (b_n) is called a Schauder basis (or briefly basis) for X. The series $\sum_k \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum \alpha_k b_k$.

Let X and Y be two nonempty subsets of ω. Let $A = (a_{nk}), (n, k \in \mathbb{N})$ be an infinite matrix of real or complex numbers. We write $(Ax)_n = A_n(x) = \sum_k a_{nk} x_k$ .Then, $Ax = \{A_n(x)\}$ is called the A-transform of x, when- ever $A_n(x) = \sum_k a_{nk} x_k < \infty$ for all n. We write $\lim_n Ax = \lim_n A_n(x)$. If $x \in X$ implies $Ax \in Y$, we say that A defines a matrix transformations from X into Y, denoted by $A: X \to Y$.By $(X: Y)$, we mean the class of all matrices A such that $A: X \to Y$. The matrix domain X_A of an infinite matrix A in a sequence space X is defined as

$$X_A = \{x = (x_k) \in \omega: Ax \in X\}. \qquad (2)$$

Let (q_k) be a sequence of positive numbers and let us write,

$$Q_n = \sum_{k=0}^{n} q_k \text{, for } n \in \mathbb{N}.$$

Then the matrix $R^q = (r_{nk}^q)$ of the Rieszmean (R, q_n) is given by

$$r_{nk}^q = \begin{cases} \dfrac{q_k}{Q_n} & ; \ 0 \le k \le n, \\ 0 & ; \ k > n. \end{cases}$$

The Riesz mean (R, q_n) is regular if and only if $Q_n \to \infty$ as $n \to \infty$ (see, (Petersen, 1966)).

The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors viz., (see,(Altay and Başar, 2002; Altay *et al.*, 2006; Başar and Altay, 2002; Başar *et al.*, 2008; Choudhary and Mishra, 1993; Gross Erdmann, 1993; Lascarides and Maddox, 1970; Maddox, 1967; Maddox, 1968; Maddox, 1988; Mursaleen, 1983; Mursaleen *et al.*, 2006; Nakano, 1951; Ng and Lee, 1978; Petersen, 1966; Sheikh and Ganie, 2012; Wilansky, 1984)) characterize the classes $(r^q(u,p): bs)$, $(r^q(u,p): cs)$ and $(r^q(u,p): cs_0)$ of infinite matrices and characterize a basic theorem where bs, cs and cs_0 denote respectively the space of all bounded series, the convergent series and the series converging to zero.

We define the Reisz sequence space $r^q(u,p)$ (see (Sheikh and Ganie, 2012)) as the set of all sequences such that R_u^q- transform of it is in the space $\ell(p)$, that is,

$$r^q(u,p) = \left\{ x = (x_k) \in \omega : \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j x_j \right|^{p_k} < \infty \right\}; \left(0 < p_k \le H < \infty\right).$$

In the case $(u_k) = e = (1, 1, 1, \dots)$, the sequence spaces $r^q(u,p)$ reduces to $r^q(p)$, introduced by Altay and Baŝar (see (Altay and Başar, 2002)). For $(u_k) = e = (1, 1, 1, \dots)$ and $q_n = 1$, this reduces to Cesāro sequence space X_p of non-absolute type, introduced by Ng and Lee (see (Ng and Lee, 1978)), consisting of sequences whose arithmetic means are in ℓ_p , where $1 \le p < \infty$.

With the notation of (2) that, $r^q(u,p) = \{\ell(p)\}_{R_u^q}$.

Define the sequence $y = (y_k)$, which will be used, by the R^q-transform of a sequence $x = (x_k)$,*i.e.*,

$$y_k = \frac{1}{Q_k}\sum_{j=0}^{k} u_j q_j x_j \;\; ; \;\; \text{for } k \in \mathbb{N}. \tag{3}$$

Now, we begin with the following theorem which is essential in the text.

Lemma 2.1: $r^q(u,p)$ is a complete linear metric space paranormed by g defined by

$$g(x) = \left(\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k} u_j q_j x_j\right|^{p_k}\right)^{\frac{1}{M}} \text{ with } 0 < p_k \le \sup_k p_k = H < \infty.$$

Proof: For proof (see (Sheikh and Ganie, 2012)).

Lemma 2.2: The Riesz sequence space $r^q(u,p)$ of non-absolute type is linearly isomorphic to the space $\ell(p)$, where $0 < p_k \le H < \infty$.

Proof: For proof (see (Sheikh and Ganie, 2012)).

Lemma 2.3: **(i)** Let$1 < p_k \le H < \infty$ for every $k \in \mathbb{N}$.Then$A \in (r^q(u,p): \ell_\infty)$ if and only if there exists an integer$B > 1$ such that

$$C(B) = \sup_n \sum_k \left|\Delta\left(\frac{a_{nk}}{u_k q_k}\right) Q_k B^{-1}\right|^{p'_k} < \infty, \tag{4}$$

$$\left\{\left(\frac{a_{nk}}{u_k q_k} Q_k B^{-1}\right)^{p_k}\right\} \in \ell_\infty \text{for} n \in \mathbb{N}. \tag{5}$$

(ii) Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$.Then$A \in (r^q(u,p): \ell_\infty)$ if and only if there exists an integer such that

$$\sup_{n,k}\left|\Delta\left(\frac{a_{nk}}{u_k q_k}\right) Q_k\right|^{p_k} < \infty. \tag{6}$$

Proof: For proof (see (Sheikh and Ganie, 2012)).

Lemma 2.4: Let $1 < p_k \le H < \infty$ for every $k \in \mathbb{N}$.Then, $A \in (r^q(u,p): c)$ if and only if (16), (17) and (18) hold and there is a sequence(α_k) of scalars such that

$$\lim_n \Delta\left(\frac{a_{nk} - \alpha_k}{u_k q_k}\right) Q_k = 0 \text{ for every} k \in \mathbb{N}. \tag{7}$$

Proof: For proof (see (Sheikh and Ganie, 2012)).

Theorem 2.5: **(i)** Let$1 < p_k \le H < \infty$ for every $k \in \mathbb{N}$.Then, $A \in (r^q(u,p): bs)$if and only if there exists an integer$B > 1$ such that

$$\sup_n \sum_k \left|\Delta\left(\frac{a_{nk}}{u_k q_k}\right) Q_k B^{-1}\right|^{p'_k} < \infty, \tag{8}$$

$$\sup_k \left|\left(\frac{a_{nk}}{u_k q_k} Q_k B^{-1}\right)\right|^{p'_k} < \infty. \text{ for} n \in \mathbb{N}. \tag{9}$$

(ii) Let$0 < p_k \le 1$ for every $k \in \mathbb{N}$.Then$A \in (r^q(u,p): \text{bs})$if and only if

$$\sup_n \left|\Delta\left(\frac{a_{nk}}{u_k q_k}\right) Q_k\right|^{p_k} < \infty. \tag{10}$$

Proof: Let us define the matrix $E = (e_{nk})$ by $e_{nk} = a(n,k)$ for all $n, k \in \mathbb{N}$. Consider, now the following equality derived from the $n: mth$partial sums of the series$\sum_j \sum_k a_{jk} x_k \, as m \to \infty$,

$\sum_{j=0}^{n}\sum_k a_{jk}x_k = \sum_k e_{nk}x_k$ for all $n, k \in \mathbb{N}$.

Therefore, bearing in mind the fact that the spaces bs and l_∞ are linearly isomorpic, one can easily see that $Ax \in$bs whenever $x \in r^q(u,p)$ if and only if $Ex \in l_\infty$whenever $x \in r^q(u,p)$. Now, the proof directly follows from above Lemma 2.3 with E instead of A.

Theorem 2.6: **(i)** Let $1 < p_k \le H < \infty$ for all$k \in \mathbb{N}$.Then$A \in (r^q(u,p): cs)$ if and only if (7), (8), (9) hold and there is a sequence (α_k) of scalars such that

$$\lim_n \Delta\left(\frac{a_{nk}-\alpha_k}{u_k q_k}\right)Q_k = 0 \text{ for every} k \in \mathbb{N}. \tag{21}$$

(ii) $A \in (r^q(u,p): cs_0)$if and only if(7), (8) , (9) hold $\alpha_k = 0$ for all $k \in \mathbb{N}$.

Proof: This may be obtained by the same kind of argument that has been used in the proof of Lemma 2.4 (above) with Theorem 2.4 instead of Lemma 2.3.

Now, we give our final result which also yields the characterizations of the matrix mappings between the Riesz sequence spaces.

Theorem 2.7: Suppose that the elements of the infinite matrices $C = (c_{nk})$ and $D= (d_{nk})$ which are connected with the relation

$$\boldsymbol{d_{nk} = \frac{1}{Q_n}\sum_{j=0}^{n} u_j q_j c_{jk}} \textbf{for } k, n \in \mathbb{N}.$$

And μ and ρbe any two given sequence spaces. Then, $C \in (\rho: \mu R_u^q)$ if and only if $D \in (\rho: \mu)$.

Proof: Let us take $s = (s_k) \in \rho$ and consider the following equality with (12) that

$$\sum_{k=0}^{m} d_{nk}s_k = \boldsymbol{\frac{1}{Q_n}\sum_{j=0}^{n} u_j q_j \left(\sum_{k=0}^{m} c_{jk}\, s_k\right)} \textbf{ for } m, n \in \mathbb{N}.$$

which yields as $m \to \infty$ that $(Ds) = \{R_u^q(Cs)\}_n$ for $n \in \mathbb{N}$. Now, we immediately deduce from here that $Cs \in \mu R_u^q$ whenever $s \in \rho$ if and only if $Ds \in \mu$ whenever $s \in \rho$ and this completes the proof.

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