International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jpms.htm
2015 Vol. 5 (1) January-March, pp. 78-86/Prasad
Research Article

A MATHEMATICAL STUDY ON SYN-ECO-SYSTEM CONSISTING OF TWO HOSTS AND ONE COMMENSAL WITH MORTALITY RATE FOR THE THIRD SPECIES

***Hari Prasad B.**
Department of Mathematics, Chaitanya Degree College (Autonomous),
Hanamkonda, Telangana State, India-506 001
**Author for Correspondence*

ABSTRACT
The present investigation is a mathematical study on three species syn-eco system with mortality rate for the three species. The system comprises of two hosts S_1, S_2 and one commensal S_3 i.e., S_1 and S_2 both benefit S_3, without getting themselves affected either positively or adversely. Further, S_1 and S_2 are neutral. The model equations constitute a set of three first order non-linear differential equations. Criteria for the asymptotic stability of all the eight equilibrium states are established. Trajectories of the perturbations over the equilibrium states are illustrated. Further, the global stability of the system is established with the aid of suitably constructed Liapunov's function and the numerical solutions for the growth rate equations are computed using Runge-Kutta fourth order scheme.

Keywords: *Commensal, Equilibrium State, Host, Liapunov's Function, Stable, Trajectories, Unstable*

INTRODUCTION
Ecology, a branch evolutionary biology, deals with living species that coexist in a physical environment sustain themselves on common resources. It is a common observation that the species of same nature can not flourish is isolation without any interaction with species of different kinds. Syn-ecology is an ecosystem comprising of two or more distinct species. Species interact with each other in one way or other. The Ecological interactions can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation, Parasitism and so on. Lotka (1925) and Volterra (1931) pioneered theoretical ecology significantly and opened new eras in the field of life and biological sciences. Mathematical Modeling is a vital role in providing insight in to the mutual relationships between the interacting species.

The general concepts of modeling have been discussed by several authors Colinvaux (1986), Kapur (1985), Kushing (1977), Meyer (1985). Srinivas (1991) studied competitive ecosystem of two species and three species with limited and unlimited resources. Later, Narayan and Charyulu (2007) studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Stability analysis of competitive species was carried out by Reddy *et al.,* (2007), Sharma and Charyulu (2008), while Ravindra (2008) investigated mutualism between two species. Acharyulu and Charyulu (2011) derived some productive results on various mathematical models of ecological Ammensalism with multifarious resources in the manifold directions. Further, Kumar (2010) derived some mathematical models of ecological commensalism.

The present author Prasad (2014) studied continuous and discrete models on the three species syn-ecosystems. The present investigation is an analytical study of three species (S_1, S_2, S_3) syn-eco system with mortality rate for the three species. The system comprises of two hosts S_1, S_2 and one commensal S_3 i.e., S_1 and S_2 both benefit S_3, without getting themselves affected either positively or adversely. Further, S_1 and S_2 are neutral. Commensalism is a symbiotic interaction between two populations where one population (S_1) gets benefit from (S_2) while the other (S_2) is neither harmed nor benefited due to the interaction with (S_1). The benefited species (S_1) is called the commensal and the other (S_2) is called the host. Some real-life examples of commensalism are presented below.

(i). A squirrel in an oak tree gets a place to live and food for its survival, while the tree remains neither benefited nor harmed.

(ii). A flatworm attached to the horse crab and eating the crab's food, while the crab is not put to any disadvantage.

Notations and Basic Equations

S_1, S_2: Host of S_3

S_3: Commensal for S_1 and S_2

$N_i(t)$: The population strength of S_i at time t, $i = 1, 2, 3$

t: Time instant

d_3: Natural death rate of S_3

a_i : Natural growth rate of S_i, $i = 1, 2$

a_{ii} : Self inhibition coefficients of S_i, $i = 1, 2, 3$

a_{13}, a_{23} : Interaction coefficients of S_1 due to S_3 and S_2 due to S_3

$e_3 = \dfrac{d_3}{a_{33}}$: Extinction coefficient of S_3

$k_i = \dfrac{a_i}{a_{ii}}$: Carrying capacities of S_i, $i = 1, 2$

Further, the variables N_1, N_2, N_3 are non-negative and the model parameters $a_1, e_3, d_3, a_2, a_{13}, a_{11}, a_{22}$, a_{33}, a_{23}, k_1, k_2 are assumed to be non-negative constants.

The model equations for syn ecosystem are given by the following system of first order non-linear ordinary differential equations.

Equation for the first species (N_1):

$$\frac{dN_1}{dt} = N_1\left(a_1 - a_{11}N_1\right) \tag{1}$$

Equation for the second species (N_2):

$$\frac{dN_2}{dt} = N_2\left(a_2 - a_{22}N_2\right) \tag{2}$$

Equation for the third species (N_3):

$$\frac{dN_3}{dt} = N_3\left(-d_3 - a_{33}N_3 + a_{13}N_1 + a_{23}N_2\right) \tag{3}$$

Equilibrium States

The system under investigation has eight equilibrium states given by

$$\frac{dN_i}{dt} = 0, i = 1, 2, 3 \tag{4}$$

Fully washed out state.

$E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0$

States in which only two of the tree species are washed out while the other one is not.

$E_2 : \bar{N}_1 = k_1, \bar{N}_2 = 0, \bar{N}_3 = 0$

$E_3 : \bar{N}_1 = 0, \bar{N}_2 = k_2, \bar{N}_3 = 0$

$E_4 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = -e_3$

States in which only one of the tree species is washed out while the other two are not.

$E_5 : \bar{N}_1 = k_1, \bar{N}_2 = k_2, \bar{N}_3 = 0$

$E_6 : \bar{N}_1 = k_1 , \bar{N}_2 = 0, \bar{N}_3 = \dfrac{a_{13}k_1}{a_{33}} - e_3$

$E_7 : \bar{N}_1 = 0 , \bar{N}_2 = k_2, \bar{N}_3 = \dfrac{a_{23}k_2}{a_{33}} - e_3$

The co-existent state (or) normal steady state.

$E_8 : \bar{N}_1 = k_1, \bar{N}_2 = k_2, \bar{N}_3 = \dfrac{a_{13}k_1 + a_{23}k_2}{a_{33}} - e_3$

Stability Analysis of Equilibrium States

Let us consider small deviations from the steady state

i.e., $N_i(t) = \bar{N}_i + u_i(t), i = 1,2,3$ (5)

where $u_i(t)$ is a small perturbations in the species S_i.

The basic equations are linearized over the equilibrium state $\bar{N} = \left(\bar{N}_1, \bar{N}_2, \bar{N}_3\right)$ to obtain the equations for the perturbed state as

$$\frac{du_1}{dt} = \left(a_1 - 2a_{11}\bar{N}_1\right)u_1 \quad (6)$$

$$\frac{du_2}{dt} = \left(a_2 - 2a_{22}\bar{N}_2\right)u_2 \quad (7)$$

$$\frac{du_3}{dt} = a_{13}\bar{N}_3 u_1 + a_{23}\bar{N}_3 u_2 + \left(-d_3 - 2a_{33}\bar{N}_3 + a_{13}\bar{N}_1 + a_{23}\bar{N}_2\right)u_3 \quad (8)$$

The characteristic equation for the system is given by

det [A – λI] = 0 (9)

The equilibrium state is stable, if all the roots of the equation (9) are negative in case they are real or have negative real parts, in case they are complex.

Stability of $E_1 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0$

The basic equations are linearized to obtain the equations as

$$\frac{du_1}{dt} = a_1 u_1; \ \frac{du_2}{dt} = a_2 u_2; \ \frac{du_3}{dt} = -d_3 u_3 \quad (10)$$

The characteristic equation is

$$\left(\lambda - a_1\right)\left(\lambda - a_2\right)\left(\lambda + d_3\right) = 0 \quad (11)$$

The characteristic roots of (11) are $a_1, a_2, -d_3$. Since two of these three roots are positive. Hence the state is unstable and the solutions of the equations (10) are

$$u_1 = u_{10}\, e^{a_1 t};\ u_2 = u_{20}\, e^{a_2 t};\ u_3 = u_{30}\, e^{-d_3 t} \quad (12)$$

where u_{10}, u_{20}, u_{30} are the initial values of u_1, u_2, u_3 respectively. The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are

$$\left(\frac{u_1}{u_{10}}\right)^{\frac{1}{a_1}} = \left(\frac{u_2}{u_{20}}\right)^{\frac{1}{a_2}} = \left(\frac{u_3}{u_{30}}\right)^{-\frac{1}{d_3}}$$

Stability of $E_2 : \bar{N}_1 = k_1, \bar{N}_2 = 0, \bar{N}_3 = 0$

In this state, the basic equations can be linearized, we get

$$\frac{du_1}{dt} = -a_1 u_1;\ \frac{du_2}{dt} = a_2 u_2;\ \frac{du_3}{dt} = \left(a_{13}k_1 - d_3\right)u_3 \tag{13}$$

The characteristic roots are $-a_1,\ a_2$ and $a_{13}k_1 - d_3$. Since one of these three roots is positive, hence the state is unstable and the solutions are

$$u_1 = u_{10}e^{-a_1 t};\ u_2 = u_{20}e^{a_2 t};\ u_3 = u_{30}e^{(a_{13}k_1 - d_3)t} \tag{14}$$

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$\left(\frac{u_1}{u_{10}}\right)^{-\frac{1}{a_1}} = \left(\frac{u_2}{u_{20}}\right)^{\frac{1}{a_2}} = \left(\frac{u_3}{u_{30}}\right)^{\frac{1}{a_{13}k_1 - d_3}}$$

Stability of $E_3 : \bar{N}_1 = 0,\ \bar{N}_2 = k_2,\ \bar{N}_3 = 0$

The basic equations can be linearized, we get

$$\frac{du_1}{dt} = a_1 u_1;\ \frac{du_2}{dt} = -a_2 u_2;\ \frac{du_3}{dt} = \left(a_{23}k_2 - d_3\right)u_3 \tag{15}$$

The characteristic roots are $a_1,\ -a_2$ and $a_{23}k_2 - d_3$. Since one of these three roots is positive, hence the state is unstable. The equations (15) yield the solutions,

$$u_1 = u_{10}e^{a_1 t};\ u_2 = u_{20}e^{-a_2 t};\ u_3 = u_{30}e^{(a_{23}k_2 - d_3)t} \tag{16}$$

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are

$$\left(\frac{u_1}{u_{10}}\right)^{\frac{1}{a_1}} = \left(\frac{u_2}{u_{20}}\right)^{-\frac{1}{a_2}} = \left(\frac{u_3}{u_{30}}\right)^{\frac{1}{a_{23}k_2 - d_3}}$$

Stability of $E_4 : \bar{N}_1 = 0,\ \bar{N}_2 = 0,\ \bar{N}_3 = -e_3$

In this state, the basic equations can be linearized, we have

$$\frac{du_1}{dt} = a_1 u_1;\ \frac{du_2}{dt} = a_2 u_2;\ \frac{du_3}{dt} = -a_{13}e_3 u_1 - a_{23}e_3 u_2 + d_3 u_3 \tag{17}$$

The characteristic roots are $a_1,\ a_2,\ d_3$. Since all the three roots are positive, hence the state is unstable. The equations (17) yield the solutions,

$$u_1 = u_{10}e^{a_1 t}; u_2 = u_{20}e^{a_2 t}; u_3 = A_1 u_{10}e^{a_1 t} + A_2 u_{20}e^{a_2 t} + \left(u_{30} - A_1 u_{10} - A_2 u_{20}\right)e^{d_3 t} \tag{18}$$

Where $A_1 = \dfrac{a_{13}e_3}{d_3 - a_1};\ A_2 = \dfrac{a_{23}e_3}{d_3 - a_2}$; with $d_3 \neq a_1, a_2$ (19)

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$\left(\frac{u_1}{u_{10}}\right)^{a_2} = \left(\frac{u_2}{u_{20}}\right)^{a_1};\ u_3 = A_1 u_{10}\left(\frac{u_2}{u_{20}}\right)^{\frac{a_1}{a_2}} + A_2 u_2 + \left(u_{30} - A_1 u_{10} - A_2 u_{20}\right)\left(\frac{u_2}{u_{20}}\right)^{\frac{d_3}{a_2}}$$

Stability of $E_5 : \bar{N}_1 = k_1,\ \bar{N}_2 = k_2,\ \bar{N}_3 = 0$

In this state, the basic equations can be linearized, we have

$$\frac{du_1}{dt} = -a_1 u_1;\ \frac{du_2}{dt} = -a_2 u_2;\ \frac{du_3}{dt} = \left(a_{13}k_1 + a_{23}k_2 - d_3\right)u_3 \tag{20}$$

The characteristic roots are $-a_1,\ -a_2$ and $a_{13}k_1 + a_{23}k_2 - d_3$.

Case I : When $a_{13}k_1 + a_{23}k_2 > d_3$

In this case, one of the three roots is positive, hence the state is unstable. The equations (20) yield the solutions,

$$u_1 = u_{10} e^{-a_1 t}; u_2 = u_{20} e^{-a_2 t}; u_3 = u_{30} e^{(a_{13}k_1 + a_{23}k_2 + d_3)t} \quad (21)$$

Case II : When $a_{13}k_1 + a_{23}k_2 < d_3$

In this case, all the three roots are negative, hence the state is stable and the equations (20) yield the solutions.

$$u_1 = u_{10} e^{-a_1 t}; u_2 = u_{20} e^{-a_2 t}; u_3 = u_{30} e^{-(a_{13}k_1 + a_{23}k_2 + d_3)t} \quad (22)$$

It can be noticed that $u_1 \to 0, u_2 \to 0$ and $u_3 \to 0$ as $t \to \infty$

Case III : When $a_{13}k_1 + a_{23}k_2 = d_3$

In this case the state is neutrally stable and the solution curves of (20) are given by

$$u_1 = u_{10} e^{-a_1 t}; u_2 = u_{20} e^{-a_2 t}; u_3 = u_{30} \quad (23)$$

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$\left(\frac{u_1}{u_{10}}\right)^{-\frac{1}{a_1}} = \left(\frac{u_2}{u_{20}}\right)^{-\frac{1}{a_2}} = \left(\frac{u_3}{u_{30}}\right)^{\frac{1}{a_{13}k_1 + a_{23}k_2 - d_3}}$$

Stability of $E_6 : \bar{N}_1 = k_1 , \bar{N}_2 = 0, \bar{N}_3 = \frac{a_{13}k_1}{a_{33}} - e_3$

In this state, the basic equations can be quasi-linearized, we have

$$\frac{du_1}{dt} = -a_1 u_1; \frac{du_2}{dt} = a_2 u_2; \frac{du_3}{dt} = \frac{a_{13}}{a_{33}}(a_{13}k_1 - d_3)u_1 + \frac{a_{23}}{a_{33}}(a_{13}k_1 - d_3)u_2 + (d_3 - a_{13}k_1)u_3 \quad (24)$$

The characteristic roots are $-a_1$, a_2 and $d_3 - a_{13}k_1$. Since one of these three roots is positive, hence the state is unstable. The equations (24) yield the solutions,

$$u_1 = u_{10} e^{-a_1 t}; u_2 = u_{20} e^{a_2 t}; u_3 = B_1 u_{10} e^{-a_1 t} + B_2 u_{20} e^{a_2 t} + (u_{30} - B_1 u_{10} - B_2 u_{20}) e^{(d_3 - a_{13}k_1)t} \quad (25)$$

Where $B_1 = \frac{a_{13}(d_3 - a_{13}k_1)}{a_{33}(a_1 + d_3 - a_{13}k_1)}$; $B_2 = \frac{a_{23}(a_{13}k_1 - d_3)}{a_{33}(a_{13}k_1 + a_2 - d_3)}$ (26)

With $a_{13}k_1 \neq a_1 + d_3$; $a_2 + a_{13}k_1 \neq d_3$ (27)

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$\left(\frac{u_1}{u_{10}}\right)^{-a_2} = \left(\frac{u_2}{u_{20}}\right)^{a_1}; u_3 = B_1 u_{10}\left(\frac{u_2}{u_{20}}\right)^{-\frac{a_1}{a_2}} + B_2 u_2 + (u_{30} - B_1 u_{10} - B_2 u_{20})\left(\frac{u_2}{u_{20}}\right)^{\frac{d_3 - a_{13}k_1}{a_2}}$$

Stability of $E_7 : \bar{N}_1 = 0 , \bar{N}_2 = k_2, \bar{N}_3 = \frac{a_{23}k_2}{a_{33}} - e_3$

In this state, the basic equations can be quasi-linearized, we get

$$\frac{du_1}{dt} = a_1 u_1; \frac{du_2}{dt} = -a_2 u_2; \frac{du_3}{dt} = \frac{a_{13}}{a_{33}}(a_{23}k_2 - d_3)u_1 + \frac{a_{23}}{a_{33}}(a_{23}k_2 - d_3)u_2 + (d_3 - a_{23}k_2)u_3 \quad (28)$$

The characteristic roots are a_1, $-a_2$ and $d_3 - a_{23}k_2$. Since one of these three roots is positive, hence the state is unstable. The equations (28) yield the solutions,

$$u_1 = u_{10} e^{a_1 t}; u_2 = u_{20} e^{-a_2 t}; u_3 = C_1 u_{10} e^{a_1 t} + C_2 u_{20} e^{-a_2 t} + (u_{30} - C_1 u_{10} - C_2 u_{20}) e^{(d_3 - a_{23}k_2)t} \quad (29)$$

Where $C_1 = \frac{a_{13}(a_{23}k_2 - d_3)}{a_{33}(a_1 + a_{23}k_2 - d_3)}$; $C_2 = \frac{a_{23}(d_3 - a_{23}k_2)}{a_{33}(a_2 + d_3 - a_{23}k_2)}$ (30)

With $a_1 + a_{23}k_2 \neq d_3;\ a_2 + d_3 \neq a_{23}k_2$ (31)

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$\left(\frac{u_1}{u_{10}}\right)^{-a_2} = \left(\frac{u_2}{u_{20}}\right)^{a_1}; \qquad u_3 = C_1 u_{10}\left(\frac{u_2}{u_{20}}\right)^{-\frac{a_1}{a_2}} + C_2 u_2 + \left(u_{30} - C_1 u_{10} - C_2 u_{20}\right)\left(\frac{u_2}{u_{20}}\right)^{\frac{a_{23}k_2 - d_3}{a_2}}$$

Stability of $E_8 : \bar{N}_1 = k_1, \bar{N}_2 = k_2, \bar{N}_3 = \dfrac{a_{13}k_1 + a_{23}k_2}{a_{33}} - e_3$

In this state, the basic equations can be quasi-linearized, we have

$$\frac{du_1}{dt} = -a_1 u_1; \frac{du_2}{dt} = -a_2 u_2;$$

$$\frac{du_3}{dt} = \frac{a_{13}}{a_{33}}\left(a_{13}k_1 + a_{23}k_2 - d_3\right)u_1 + \frac{a_{23}}{a_{33}}\left(a_{13}k_1 + a_{23}k_2 - d_3\right)u_2 + \left(d_3 - a_{13}k_1 - a_{23}k_2\right)u_3 \qquad (32)$$

The characteristic roots are $-a_1,\ -a_2$ and $d_3 - a_{13}k_1 - a_{23}k_2$.

Case I : When $d_3 > a_{13}k_1 + a_{23}k_2$

In this case, one of the three roots is positive, hence the state is unstable. The equations (32) yield the solutions,

$$u_1 = u_{10}e^{-a_1 t}; u_2 = u_{20}e^{-a_2 t}; u_3 = D_1 u_{10}e^{-a_1 t} + D_2 u_{20}e^{-a_2 t} + \left(u_{30} - D_1 u_{10} - D_2 u_{20}\right)e^{(d_3 - a_{13}k_1 - a_{23}k_2)t} \qquad (33)$$

Where $D_1 = \dfrac{a_{13}(a_{13}k_1 - a_{23}k_2 - d_3)}{a_{33}(a_{13}k_1 + a_{23}k_2 - a_1 - d_3)};\ D_2 = \dfrac{a_{23}(a_{13}k_1 - a_{23}k_2 - d_3)}{a_{33}(a_{13}k_1 + a_{23}k_2 - a_2 - d_3)}$ (34)

With $a_{13}k_1 + a_{23}k_2 \neq a_1 + d_3;\ a_{13}k_1 + a_{23}k_2 \neq a_2 + d_3$ (35)

Case II : When $d_3 < a_{13}k_1 + a_{23}k_2$

In this case, all the three roots are negative, hence the state is stable and the equations (32) yield the solutions,

$$u_1 = u_{10}e^{-a_1 t}; u_2 = u_{20}e^{-a_2 t}; u_3 = D_1 u_{10}e^{-a_1 t} + D_2 u_{20}e^{-a_2 t} + \left(u_{30} - D_1 u_{10} - D_2 u_{20}\right)e^{(d_3 - a_{13}k_1 - a_{23}k_2)t} \qquad (36)$$

It can be noticed that $u_1 \to 0, u_2 \to 0$ and $u_3 \to 0$ as $t \to \infty$

Case III : When $d_3 = a_{13}k_1 + a_{23}k_2$

In this case, the state is neutrally stable and the solution curves of (32) are given by

$$u_1 = u_{10}e^{-a_1 t}; u_2 = u_{20}e^{-a_2 t}; u_3 = u_{30} \qquad (37)$$

The trajectories in the $u_1 - u_2$ and $u_2 - u_3$ planes are given by

$$\left(\frac{u_1}{u_{10}}\right)^{a_2} = \left(\frac{u_2}{u_{20}}\right)^{a_1};\ u_3 = D_1 u_{10}\left(\frac{u_2}{u_{20}}\right)^{\frac{a_1}{a_2}} + D_2 u_2 + \left(u_{30} - D_1 u_{10} - D_2 u_{20}\right)\left(\frac{u_2}{u_{20}}\right)^{\frac{a_{13}k_1 + a_{23}k_2 - d_3}{a_2}}$$

Liapunov's Function for Global Stability

We discussed the local stability of all eight equilibrium states. From which only two states E_5 and E_8 are stable and rest of them are unstable. We now examine the global stability of dynamical system (1), (2) and (3) at these states by suitable Liapunov's functions.

Theorem 1: The equilibrium state $E_5\left(k_1, k_2, 0\right)$ is globally asymptotically stable.

Proof: Let us consider the following Liapunov's function

$$L(N_1,N_2)=N_1-\bar{N}_1-\bar{N}_1\ln\left(\frac{N_1}{\bar{N}_1}\right)+\left[N_2-\bar{N}_2-\bar{N}_2\ln\left(\frac{N_2}{\bar{N}_2}\right)\right]$$

Now, the time derivative of L, along with solutions of (1) and (2) can be written as,

$$\frac{dL}{dt}=\left(\frac{N_1-\bar{N}_1}{N_1}\right)\frac{dN_1}{dt}+\left(\frac{N_2-\bar{N}_2}{N_2}\right)\frac{dN_2}{dt}$$

$$\frac{dL}{dt}=(N_1-\bar{N}_1)(a_{11}\bar{N}_1-a_{11}N_1)+(N_2-\bar{N}_2)(a_{22}\bar{N}_2-a_{22}N_2)$$

$$\frac{dL}{dt}=-\left[a_{11}(N_1-\bar{N}_1)^2+a_{22}(N_2-\bar{N}_2)^2\right]<0$$

Hence, the steady state is globally asymptotically stable.

Theorem 2: The equilibrium state $E_8(\bar{N}_1,\bar{N}_2,\bar{N}_3)$ is globally asymptotically stable.

Proof: Let us consider the following Liapunov's function

$$L(N_1,N_2,N_3)=N_1-\bar{N}_1-\bar{N}_1\ln\left(\frac{N_1}{\bar{N}_1}\right)+l_1\left[N_2-\bar{N}_2-\bar{N}_2\ln\left(\frac{N_2}{\bar{N}_2}\right)\right]+l_2\left[N_3-\bar{N}_3-\bar{N}_3\ln\left(\frac{N_3}{\bar{N}_3}\right)\right]$$

where l_1 and l_2 are suitable constants to be determined in the subsequent steps. Now, the time derivative of L, along with solutions of the equations (1) - (3) can be written as,

$$\frac{dL}{dt}=\left(\frac{N_1-\bar{N}_1}{N_1}\right)\frac{dN_1}{dt}+l_1\left(\frac{N_2-\bar{N}_2}{N_2}\right)\frac{dN_2}{dt}+l_2\left(\frac{N_3-\bar{N}_3}{N_3}\right)\frac{dN_3}{dt}$$

$$=(N_1-\bar{N}_1)(a_1-a_{11}N_1)+l_1(N_2-\bar{N}_2)(a_2-a_{22}N_2)+l_2(N_3-\bar{N}_3)(-d_3-a_{33}N_3+a_{13}N_1+a_{23}N_2)$$

$$=-a_{11}(N_1-\bar{N}_1)^2-a_{22}l_1(N_2-\bar{N}_2)^2+l_2(N_3-\bar{N}_3)\left[a_{13}(N_1-\bar{N}_1)+a_{23}(N_2-\bar{N}_2)-a_{33}(N_3-\bar{N}_3)\right]$$

Choosing $l_1=\frac{a_{11}a_{23}^2}{a_{22}a_{13}^2}>0,\ l_2=\frac{4a_{11}a_{33}}{a_{13}^2}>0$ and with some algebraic manipulation, we get

$$\frac{dL}{dt}=-\sqrt{a_{11}}\left[(N_1-\bar{N}_1)+\frac{a_{23}}{a_{13}}(N_2-\bar{N}_2)+\frac{2}{a_{13}}(N_3-\bar{N}_3)\right]^2<0,\ \text{when } N_1+\bar{N}_2<N_2+\bar{N}_1$$

Hence, the steady state is globally asymptotically stable.

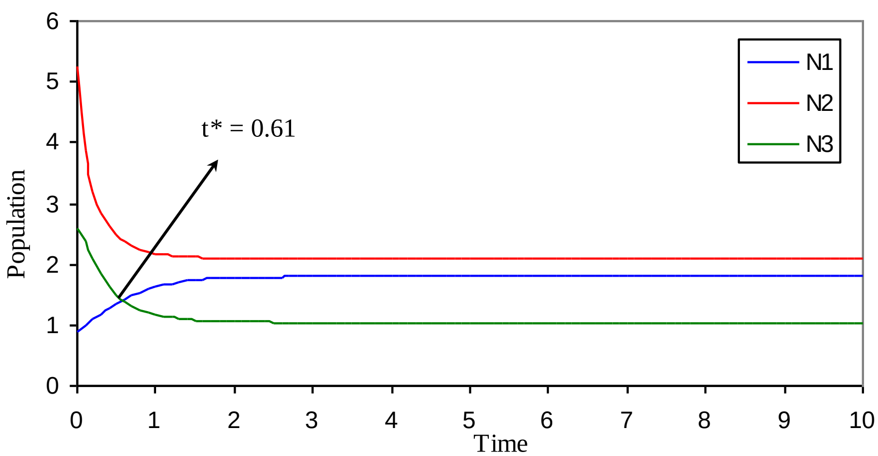

Figure 1: Variation of population against time for N_{10}=0.88, N_{20}=5.24, N_{30}=2.6

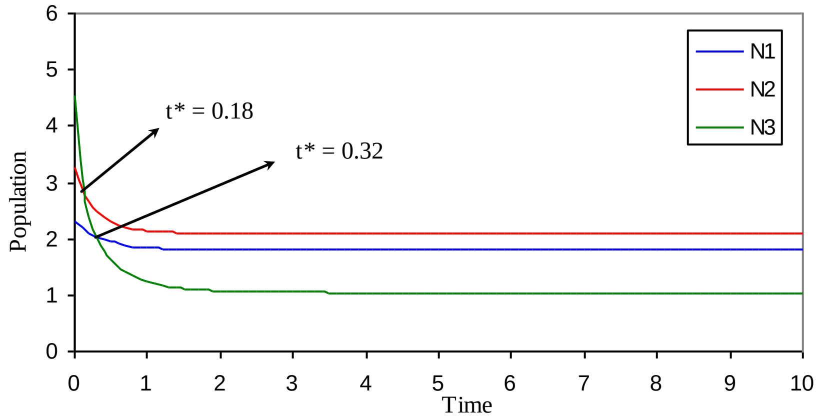

Figure 2: Variation of population against time for N_{10}=2.32, N_{20}=3.28, N_{30}=4.56

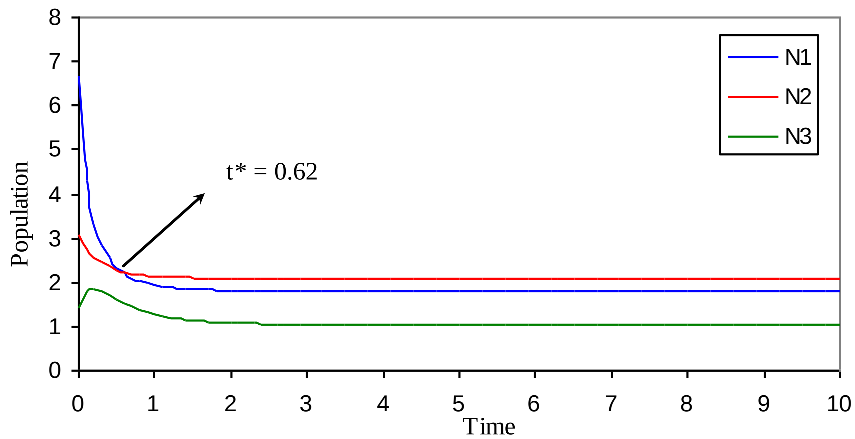

Figure 3: Variation of population against time for N_{10}=6.68, N_{20}=3.08, N_{30}=1.44

A Numerical Approach of the Growth Rate Equations

The numerical solutions of the growth rate equations (1)-(3) computed employing the fourth order Runge-Kutta method. The results are illustrated in Figures 1 to 6. Let us consider the fixed parameters as, a_1=2.3, a_2=2.72, d_3=1.572, a_{11}=1.28, a_{22}=1.3, a_{33} =2.1, a_{12}=6.16, a_{13}=0.7, a_{23}=1.2.

Observations of the above Graphs

Case 1: In this case the first species has the least initial value. Initially the third species dominates over the first till the time instant $t^* = 0.61$ and thereafter the dominance is reversed. This is illustrated in Figure 1.

Case 2: In this case the initial values of S_1, S_2, S_3 are in increasing order. Initially the third species dominates over the second and first till the time instant $t^* = 0.18$ and $t^* = 0.32$ and thereafter the dominance is reversed. Further it is evident that all the three species asymptotically converge to the equilibrium point as shown in Figure 2.

Case 3: In this case the initial values of S_1, S_2, S_3 are in decreasing order. The first species dominates over the second initially up to the time $t^* = 0.62$ after which the dominance is reversed. In course of time we notice a steady variation with no appreciable growth rate in all the three species (Figure 3).

CONCLUSION

The present paper deals with an investigation on the stability of a syn eco-system consisting of two hosts and one commensal with mortality rate for the third species. In this paper we established all possible equilibrium states. It is conclude that, in all eight equilibrium states, only the two states E_5 and E_8 are stable. Further the global stability is established with the help of suitable Liapunov's function and the numerical solutions are computed using Runge-Kutta fourth order method.

ACKNOWLEDGMENT
I thank to Professor (Retd), N.Ch. Pattabhi Ramacharyulu, Department of Mathematics, NIT, Warangal (T.S.), India for his valuable suggestions and encouragement.

REFERENCES

Acharyulu KVLN and Charyulu NChPR (2011). An Ammensal-Prey with three species Ecosystem. *International Journal of Computational Cognition* **9** 30-39.

Colinvaux AP (1986). *Ecology* (John Wiley) New York.

Kapur JN (1985). *Mathematical Modeling in Biology and Medicine*. Affiliated East West.

Kapur JN (1985). *Mathematical Modeling* (Wiley Easter).

Kumar NP (2010). *Some Mathematical Models of Ecological Commensalism*. Acharya Nagarjuna University Ph.D Thesis.

Kushing JM (1977). *Integro-Differential Equations and Delay Models in Population Dynamics*. Lecture Notes in Bio-Mathematics Springer Verlag 20.

Lotka AJ (1925). *Elements of Physical Biology* (Williams and Wilking) Baltimore.

Meyer WJ (1985). *Concepts of Mathematical Modeling* (Mc.Graw hill).

Narayan KL and Charyulu NChPR (2007). A Prey-Predator Model with Cover for Prey and Alternate Food for the Predator and Time Delay. *International Journal of Scientific Computing* **1** 7-14.

Prasad BH (2014). On the Stability of a Three Species Syn-Eco-System with Mortality Rate for the Second Species. *International Journal of Social Science & Interdisciplinary Research* **3** 35-45.

Prasad BH (2014). The Stability Analysis of a Three Species Syn-Eco-System with Mortality Rates. *Contemporary Mathematics and Statistics* **2** 76-89.

Prasad BH (2014). A Study on Discrete Model of Three Species Syn-Eco-System with Limited Resources. *Int. Journal Modern Education and Computer Science* **11** 38-44.

Prasad BH (2014). A Discrete Model of a Typical Three Species Syn -Eco-System with Unlimited Resources for the First and Third Species. *Asian Academic Research Journal of Multidisciplinary* **1** 36-46.

Reddy RA, Charyulu NChPR and Gandhi BK (2007). A Stability Analysis of Two Competitive Interacting Species with Harvesting of Both the Species at a Constant Rate. *International Journal of Scientific Computing* **1** 57-68.

Ravindra Reddy R (2008). *A Study on Mathematical Models of Ecological Mutualism between Two Interacting Species*. Osmania University Ph.D Thesis.

Sharma BBR and Charyulu NChPR (2008). Stability Analysis of Two Species Competitive Eco-system. *International Journal of Logic Based Intelligent Systems* **2** 79-86.

Srinivas NC (1991). *Some Mathematical Aspects of Modeling in Bio-medical Sciences*. Kakatiya University Ph.D Thesis.

Volterra V (1931). *Leconssen La Theorie Mathematique De La Leitte Pou Lavie* (Gauthier-Villars) Paris.