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Review Article

THE NON-HOMOGENEOUS QUINTIC EQUATION WITH FIVE UNKNOWNS $x^4 - y^4 = 65(z^2 - w^2)p^3$

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ABSTRACT

The non-homogeneous quintic equation with five unknowns given by $x^4 - y^4 = 65(z^2 - w^2)p^3$ is considered and analyzed for its non– zero distinct integer solutions. A few interesting relations between the solutions and special numbers, namely, polygonal numbers, pyramidal numbers, centered pyramidal numbers, pronic numbers are presented.

Keywords: *Non - homogeneous Quintic, Quintic with Five Unknowns, Integral Solutions, Special Numbers*

2010 Mathematics subject classification: 11D41

Notations

Special numbers	**Notations**	**Definitions**
Regular Polygonal Number	$t_{m,n}$	$n\left(1+\frac{(n-1)(m-2)}{2}\right)$
Pronic Number	Pr_n	$n(n+1)$
Pyramidal number	P_n^m	$\frac{n(n+1)}{6}[(m-2)n+(5-m)]$
Centered Pyramindal number	$CP_{m,n}$	$\frac{m(n-1)n(n+1)}{6}+n$
Stella octangular Number	SO_n	$n(2n^2-1)$

INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, Quintic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity (Dickson, 1952; Mordell, 1969).

For illustration, one may refer (Gopalan and Vijayashankar, 2010; Gopalan and Vijayashankar, 2010; Gopalan *et al.*, 2013) for Quintic equation with three unknowns and (Gopalan and Vijayashankar, 2011; Gopalan and Vijayashankar; Gopalan *et al.*, 2013; Gopalan *et al.*, 2013; Gopalan *et al.*, 2013; Vidhyalakshmi *et al.*, 2013; Vidhyalakshmi *et al.*, 2013) for Quintic equation with five unknowns.

This paper concerns with the problem of the non-homogeneous Quintic equation with five unknowns given by $x^4 - y^4 = 65(z^2 - w^2)p^3$. A few relations among the solutions are presented.

Method of Analysis

The non-homogeneous quintic equation with 5 unknowns to be solved is given by

$$x^4 - y^4 = 65(z^2 - w^2)p^3 \tag{1}$$

Assume $x = u+v, y = u-v, z = 2u+v$ and $w = 2u-v$ (2)

Substituting (2) in (1), it leads to

$$u^2+v^2=65p^3 \quad (3)$$

Pattern-1

Assume $p=a^2+b^2$ (4)

where a and b are non-zero distinct integers.

Write 65 as $65=(8+i)(8-i)$ (5)

Using (4) & (5) in (3) and employing the method of factorization, define

$$u+iv=(8+i)(a+ib)^3 \quad (6)$$

Equating the real and imaginary parts of (6), we get

$$u=8a^3-24ab^2-3a^2b+b^3$$
$$v=a^3-3ab^2+24a^2b-8b^3 \quad (7)$$

Substituting (7) in (2), the integral solutions of (1) are given by

$$x(a,b)=x=9a^3-27ab^2+21a^2b-7b^3$$
$$y(a,b)=y=7a^3-21ab^2-27a^2b+9b^3 \quad (8)$$

$$z(a,b)=z=17a^3-51ab^2+18a^2b-6b^3$$
$$w(a,b)=w=15a^3-45ab^2-30a^2b+10b^3 \quad (9)$$

along with (4)

Properties

- $x(a,a)+y(a,a)-w(a,a)-14CP_{6,a}=0$
- $x(a,1)+y(a,1,)-w(a,1)-CP_{6,a}-24t_{4,a}+8\equiv 0 \pmod 3$
- $4\{w(a,2a)-z(a,2a)\}$ is a cubical integer
- $z(a+1,a)-w(a+1,a)-2CP_{6,a+1}+16CP_{6,a}+12P_a^5-48\,\Pr_a(a+1)=0$

Note 1

In (2), the representations of z and w may be taken as

$$z=2uv+1,\ w=2uv-1 \quad (*)$$

In this case, the values of z and w are given by

$$z(a,b)=z=16a^6-16b^6+378a^5b+378ab^5-240a^4b^2+240a^2b^4-1260a^3b^3+1$$
$$w(a,b)=w=16a^6-16b^6+378a^5b+378ab^5-240a^4b^2+240a^2b^4-1260a^3b^3-1 \quad (10)$$

Thus (4),(8)and (10) represent a different set of solutions to (1)

Note 2

Observe that z and w in (2) may also be considered as

$$z=uv+2,\ w=uv-2 \quad (**)$$

For this choice, the corresponding values of z and w are obtained as

$$z(a,b)=z=8a^6-8b^6+189a^5b+189ab^5-120a^4b^2+120a^2b^4-630a^3b^3+2$$
$$w(a,b)=w=8a^6-8b^6+189a^5b+189ab^5-120a^4b^2+120a^2b^4-630a^3b^3-2 \quad (11)$$

Thus (4),(8) and (11) represent an another set of integer solutions to (1)

Pattern-2

In (3), write 65 as $65=(-8+i)(-8-i)$

Following the procedure in pattern1, the corresponding integer solutions are

$$\begin{aligned} x(a,b) &= x = -7a^3 + 21ab^2 - 27a^2b + 9b^3 \\ y(a,b) &= y = -9a^3 + 27ab^2 + 21a^2b - 7b^3 \\ z(a,b) &= z = -15a^3 + 45ab^2 - 30a^2b + 10b^3 \\ w(a,b) &= w = -17a^3 + 51ab^2 + 18a^2b - 6b^3 \end{aligned} \qquad (12)$$

along with (4)

Note 3

For this choice of z and w given by (*) and (**), corresponding two sets (I and II) of values of z and w are as follows:

SET I:

$$z(a,b) = z = -16a^6 + 16b^6 + 378a^5b + 378ab^5 + 240a^4b^2 - 240a^2b^4 - 1260a^3b^3 + 1$$

$$w(a,b) = w = -16a^6 + 16b^6 + 378a^5b + 378ab^5 + 240a^4b^2 - 240a^2b^4 - 1260a^3b^3 - 1$$

SET II:

$$z(a,b) = z = -8a^6 + 8b^6 + 189a^5b + 189ab^5 + 120a^4b^2 - 120a^2b^4 - 630a^3b^3 + 2$$

$$w(a,b) = w = -8a^6 + 8b^6 + 189a^5b + 189ab^5 + 120a^4b^2 - 120a^2b^4 - 630a^3b^3 - 2$$

Considering (4),(12) along with above sets I and II inturn, we have two more choices of solutions to (1).

Pattern-3

Instead of (5), write 65 as $65 = (1 + i8)(1 - i8)$

Following the procedure presented in pattern-1, the corresponding integer solutions of (1) are

$$\begin{aligned} x(a,b) &= x = 9a^3 - 27ab^2 - 21a^2b + 7b^3 \\ y(a,b) &= y = -7a^3 + 21ab^2 - 27a^2b + 9b^3 \\ z(a,b) &= z = 10a^3 - 30ab^2 - 45a^2b + 15b^3 \\ w(a,b) &= w = -6a^3 + 18ab^2 - 51a^2b + 17b^3 \end{aligned} \qquad (13)$$

along with (4).

Properties

- $12\{x(a,a) - z(a,a)\}$ is a cubical integer
- $7\{x(a,1) - z(a,1)) + CP_{6,a} + t_{10,a} + 8\}$ is a perfect square
- $w(1,a) - y(1,a) - 8CP_{6,a} + t_{8,a} - 1 \equiv 0 \pmod{26}$
- $6\{y(2a,2) - w(2a,2) + 8CP_{6,a} + t_{52,a} - 21t_{4,a} + 64\}$ is a nasty number

Note 4

For this choice of z and w given by (*) and (**), corresponding two sets (I and II) of values of z and w are as follows:

SET I:

$$z(a,b) = z = 16a^6 - 16b^6 - 378a^5b - 378ab^5 - 240a^4b^2 + 240a^2b^4 + 1260a^3b^3 + 1$$

$$w(a,b) = w = 16a^6 - 16b^6 - 378a^5b - 378ab^5 - 240a^4b^2 + 240a^2b^4 + 1260a^3b^3 - 1$$

SET II:

$$z(a,b) = z = 8a^6 - 8b^6 - 189a^5b - 189ab^5 - 120a^4b^2 + 120a^2b^4 + 630a^3b^3 + 2$$

$$w(a,b) = w = 8a^6 - 8b^6 - 189a^5b - 189ab^5 - 120a^4b^2 + 120a^2b^4 + 630a^3b^3 - 2$$

Considering (4),(13) along with above sets I and II in turn, we have two more choices of solutions to (1).

Pattern-4

In (3), write 65 as $65 = (-1 + i8)(-1 - i8)$

Following the procedure presented in pattern-1, the corresponding integer solutions of (1) are

$$x(a,b) = x = 7a^3 - 21ab^2 - 27a^2b + 9b^3$$
$$y(a,b) = y = -9a^3 + 27ab^2 - 21a^2b + 7b^3 \qquad (14)$$
$$z(a,b) = z = 6a^3 - 18ab^2 - 51a^2b + 17b^3$$
$$w(a,b) = w = -10a^3 + 30ab^2 - 45a^2b + 15b^3$$

along with (4).

Note 5

For this choice of z and w given by (*) and (**), corresponding two sets (I and II) of values of z and w are as follows:

SET I:

$$z(a,b) = z = -16a^6 + 16b^6 - 378a^5b - 378ab^5 + 240a^4b^2 - 240a^2b^4 + 1260a^3b^3 + 1$$
$$w(a,b) = w = -16a^6 + 16b^6 - 378a^5b - 378ab^5 + 240a^4b^2 - 240a^2b^4 + 1260a^3b^3 - 1$$

SET II:

$$z(a,b) = z = -8a^6 + 8b^6 - 189a^5b - 189ab^5 + 120a^4b^2 - 120a^2b^4 + 630a^3b^3 + 2$$
$$w(a,b) = w = -8a^6 + 8b^6 - 189a^5b - 189ab^5 + 120a^4b^2 - 120a^2b^4 + 630a^3b^3 - 2$$

Considering (4),(14) along with above sets I and II inturn, we have two more choices of solutions to (1).

Pattern-5

In (3), write 65 as $65 = (7 + i4)(7 - i4)$

Following the procedure presented in pattern-1, the corresponding integer solutions of (1) are

$$x(a,b) = x = 11a^3 - 33ab^2 + 9a^2b - 3b^3$$
$$y(a,b) = y = 3a^3 - 9ab^2 - 33a^2b + 11b^3 \qquad (15)$$
$$z(a,b) = z = 18a^3 - 54ab^2 - 3a^2b + b^3$$
$$w(a,b) = w = 10a^3 - 30ab^2 - 45a^2b + 15b^3$$

along with (4).

Properties

- $z(a,1) - 6y(a,1) - 195t_{4,a} + 65 = 0$
- $(y+z)^3 - x^3 - w^3 = 3xw(y+z)$
- $(z+w) = 2(x+y)$
- $(y+z)^3 - x^3 - w^3 = 3(2x^2 + 2xy - xz)(y+z)$

Note 6

For this choice of z and w given by (*) and (**), corresponding two sets(I and II) of values of z and w are as follows:

SET I:

$$z(a,b) = z = 56a^6 - 56b^6 + 198a^5b + 198ab^5 - 840a^4b^2 + 840a^2b^4 - 660a^3b^3 + 1$$
$$w(a,b) = w = 56a^6 - 56b^6 + 198a^5b + 198ab^5 - 840a^4b^2 + 840a^2b^4 - 660a^3b^3 - 1$$

SET II:

$$z(a,b) = z = 28a^6 - 28b^6 + 99a^5 b + 99ab^5 - 420a^4 b^2 + 420a^2 b^4 - 330a^3 b^3 + 2$$

$$w(a,b) = w = 28a^6 - 28b^6 + 99a^5 b + 99ab^5 - 420a^4 b^2 + 420a^2 b^4 - 330a^3 b^3 - 2$$

Considering (4),(15) along with above sets I and II in turn, we have two more choices of solutions to (1).

Pattern-6

In (3), write 65 as $65 = (-7 + i4)(-7 - i4)$

Following the procedure presented in pattern-1, the corresponding integer solutions of (1) are

$$\begin{aligned} x(a,b) &= x = -3a^3 + 9ab^2 - 33a^2 b + 11b^3 \\ y(a,b) &= y = -11a^3 + 33ab^2 + 9a^2 b - 3b^3 \\ z(a,b) &= z = -10a^3 + 30ab^2 - 45a^2 b + 15b^3 \\ w(a,b) &= w = -18a^3 + 54ab^2 - 3a^2 b + b^3 \end{aligned} \qquad (16)$$

Note 7

For this choice of z and w given by (*) and (**), corresponding two sets (I and II) of values of z and w are as follows:

SET I:

$$z(a,b) = z = -56a^6 + 56b^6 + 198a^5 b + 198ab^5 + 840a^4 b^2 - 840a^2 b^4 - 660a^3 b^3 + 1$$

$$w(a,b) = w = -56a^6 + 56b^6 + 198a^5 b + 198ab^5 + 840a^4 b^2 - 840a^2 b^4 - 660a^3 b^3 - 1$$

SET II:

$$z(a,b) = z = -28a^6 + 28b^6 + 99a^5 b + 99ab^5 + 420a^4 b^2 - 420a^2 b^4 - 330a^3 b^3 + 2$$

$$w(a,b) = w = -28a^6 + 28b^6 + 99a^5 b + 99ab^5 + 420a^4 b^2 - 420a^2 b^4 - 330a^3 b^3 - 2$$

Considering (4), (16) along with above sets I and II in turn, we have two more choices of solutions to (1).

Pattern-7

In (3), write 65 as $65 = (4 + i7)(4 - i7)$

Following the procedure presented in pattern-1, the corresponding integer solutions of (1) are

$$\begin{aligned} x(a,b) &= x = 11a^3 - 33ab^2 - 9a^2 b + 3b^3 \\ y(a,b) &= y = -3a^3 + 9ab^2 - 33a^2 b + 11b^3 \\ z(a,b) &= z = 15a^3 - 45ab^2 - 30a^2 b + 10b^3 \\ w(a,b) &= w = a^3 - 3ab^2 - 54a^2 b + 18b^3 \end{aligned} \qquad (17)$$

Properties

- $y(2,a) + 3w(2,a) - 65CP_{6,a} \equiv 0 \pmod{780}$
- $y(a,1) + 3w(a,1) + 195t_{4,a} - 65 = 0$
- $z(a,a) + y(a,a) - w(a,a) - x(a,a) = 0$
- $z(2,a) + y(2,a) - x(2,a) - 18CP_{6,a} + 6t_{4,a} - 8 \equiv 0 \pmod{216}$

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Note 8

For this choice of z and w given by (*) and (**), corresponding two sets(I and II) of values of z and w are as follows:

SET I:

$$z(a,b)=z=56a^6-56b^6-198a^5b-198ab^5-840a^4b^2+840a^2b^4+660a^3b^3+1$$

$$w(a,b)=w=56a^6-56b^6-198a^5b-198ab^5-840a^4b^2+840a^2b^4+660a^3b^3-1$$

SET II:

$$z(a,b)=z=28a^6-28b^6-99a^5b-99ab^5-420a^4b^2+420a^2b^4+330a^3b^3+2$$

$$w(a,b)=w=28a^6-28b^6-99a^5b-99ab^5-420a^4b^2+420a^2b^4+330a^3b^3-2$$

Considering (4), (17) along with above sets I and II in turn, we have two more choices of solutions to (1).

Pattern-8

In (3), write 65 as $\qquad 65=(-4+i7)(-4-i7)$

Following the procedure presented in pattern-1, the corresponding integer solutions of (1) are

$$\begin{aligned} x(a,b)&=x=3a^3-9ab^2-33a^2b+11b^3 \\ y(a,b)&=y=-11a^3+33ab^2-9a^2b+3b^3 \end{aligned} \qquad (18)$$

$$z(a,b)=z=-a^3+3ab^2-54a^2b+18b^3$$

$$w(a,b)=w=-15a^3+45ab^2-30a^2b+10b^3$$

Note 9

For this choice of z and w given by (*) and (**), corresponding two sets (I and II) of values of z and w are as follows:

SET I:

$$z(a,b)=z=-56a^6+56b^6-198a^5b-198ab^5+840a^4b^2-840a^2b^4+660a^3b^3+1$$

$$w(a,b)=w=-56a^6+56b^6-198a^5b-198ab^5+840a^4b^2-840a^2b^4+660a^3b^3-1$$

SET II:

$$z(a,b)=z=-28a^6+28b^6-99a^5b-99ab^5+420a^4b^2-420a^2b^4+330a^3b^3+2$$

$$w(a,b)=w=-28a^6+28b^6-99a^5b-99ab^5+420a^4b^2-420a^2b^4+330a^3b^3-2$$

Considering (4), (18) along with above sets I and II in turn, we have two more choices of solutions to (1).

Pattern-9

(3) can be written as $u^2+v^2=65p^3*1$ (19)

write 65 as $65=(8+i)(8-i)$ (20)

and $1=\dfrac{(5+i12)(5-i12)}{169}$ (21)

Using (4),(20) & (21) in (19) and employing the method of factorization, define

$$u+iv=(8+i)(a+ib)^3\,\frac{(5+i12)}{13}$$

Equating the real and imaginary parts, we get

$$u=\frac{1}{13}(28a^3-84ab^2-303a^2b+101b^3)$$

$$v=\frac{1}{13}(101a^3-303ab^2+84a^2b-28b^3)$$

Since our interest is to find integer solutions, replace a by 13a and b by 13b in the values of u and v , we get

$$u=4732a^3-14196ab^2-51207a^2b+17069b^3$$

$$v=17069a^3-51207ab^2+14196a^2b-4732b^3$$

Hence in view of (2), the integral solutions of (1) are given by

$$\begin{aligned} x(a,b)=x&=21801a^3-65403ab^2-37011a^2b+12337b^3\\ y(a,b)=y&=-12337a^3+37011ab^2-65403a^2b+21801b^3\\ z(a,b)=z&=26533a^3-79599ab^2-88218a^2b+29406b^3\\ w(a,b)=w&=-7605a^3+22815ab^2-116610a^2b+38870b^3 \end{aligned} \quad (22)$$

$$p(a,b)=p=169a^2+169b^2 \quad (23)$$

Note 10

For this choice of z and w given by (*) and (**), corresponding two sets(I and II) of values of z and w are as follows:

SET I:

$$z(a,b)=z=161541016a^6-161541016b^6-1613753622a^5b-1613753622ab^5-2423115240a^4b^2$$
$$-2423115240a^2b^4+5379178740a^3b^3+1$$

$$w(a,b)=w=161541016a^6-161541016b^6-1613753622a^5b-1613753622ab^5-2423115240a^4b^2$$
$$-2423115240a^2b^4+5379178740a^3b^3-1$$

SET II:

$$z(a,b)=z=80770508a^6-80770508b^6-806876811a^5b-806876811ab^5-1211557620a^4b^2$$
$$-1211557620a^2b^4+2689589370a^3b^3+2$$

$$w(a,b)=w=80770508a^6-80770508b^6-806876811a^5b-806876811ab^5-1211557620a^4b^2$$
$$-1211557620a^2b^4+2689589370a^3b^3-2$$

Considering (22), (23) along with above sets I and II in turn, we have two more choices of solutions to (1).

Properties

- $8[(y+z)^3-w^3-3xw(y+z)]=(z+w-2y)^3$
- $(y+z)^3-x^3-w^3=3xw(2x+3y-w)$

- $xw(2x+3y-w)=(2x^2+2xy-xz)(y+z)$
- $(2(z-x)+(w-y))^3=x^3+w^3+3xw(y+z)$

Note that 1 may also have the following representations:

1) $1=\frac{(12+i5)(12-i5)}{169}$

2) $1=\frac{(7+i24)(7-i24)}{625}$

3) $1=\frac{(24+i7)(24-i7)}{625}$

Applying the procedure presented above, we get different choices of solutions to (1).

CONCLUSION

In this paper, we have illustrated different methods of obtaining non-zero integer solutions to the quintic equation with 5 unknowns given by $x^4-y^4=65(z^2-w^2)p^3$. As the quintic Diophantine equation are rich in variety one may consider other forms of quintic equation with variable ≥ 5 and search for their corresponding integer solutions along with the corresponding properties.

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