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Review Article

THE NON-HOMOGENEOUS QUINTIC EQUATION WITH FIVE UNKNOWNS $x^4-y^4+2(x^2-y^2)(x-y)^2=14(z^2-w^2)p^3$

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ABSTRACT
The non-homogeneous Quintic equation with five unknowns given by $x^4-y^4+2(x^2-y^2)(x-y)^2=14(z^2-w^2)p^3$ is considered and analyzed for its non– zero distinct integer solutions. A few interesting relations between the solutions and special numbers namely, Polygonal numbers, Pyramidal numbers, Stella octangular numbers, octahedral numbers, rhombic dodecagonal numbers are presented.

Keywords: *Non - homogeneous Quintic, Quintic with Five Unknowns, Integral Solutions, Special Numbers*

2010 Mathematics subject classification: 11D41

Notations

Special numbers	**Notations**	**Definitions**
Regular Polygonal Number	$t_{m,n}$	$n\left(1+\frac{(n-1)(m-2)}{2}\right)$
Octahedral Number	OH_n	$\frac{1}{3}n(2n^2+1)$
Stella Octangular Number	SO_n	$n(2n^2-1)$
Rhombic Dodecagonal Number	RD_n	$(2n-1)(2n^2-2n+1)$
Pyramidal Number	P_n^m	$\frac{n(n+1)}{6}((m-2)n+(5-n))$

INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, Quintic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity (Dickson, 1952; Mordell, 1969). For illustration, one may refer (Gopalan and Vijayashankar, 2010; Gopalan and Vijayashankar, 2010; Gopalan *et al.*, 2013) for Quintic equation with three unknowns and (Gopalan and Vijayashankar, 2011; Gopalan and Vijayashankar, Gopalan *et al.*, 2013; Gopalan *et al.*, 2013; Gopalan *et al.*, 2013; Vidhyalakshmi *et al.*, 2013; Vidhyalakshmi *et al.*, 2013) for Quintic equation with five unknowns. This paper concerns with the problem of the non-homogeneous Quintic equation with five unknowns given by $x^4-y^4+2(x^2-y^2)(x-y)^2=14(z^2-w^2)p^3$. A few relations among the solutions are presented.

Method of Analysis

The non-homogeneous quintic equation with 5 unknowns to be solved is given by

$$x^4-y^4+2(x^2-y^2)(x-y)^2=14(z^2-w^2)p^3 \tag{1}$$

Assume $x=u+v, y=u-v, z=2u+v$ and $w=2u-v$ (2)

Substituting (2) in (1), it leads to

$$u^2 + 5v^2 = 14p^3 \tag{3}$$

(3) is solved through different approaches and different patterns of solutions of (1) obtained are presented below

Pattern-1

Assume $p = a^2 + 5b^2$ (4)

where a and b are non-zero distinct integers.

Write 14 as $14 = (3 + i\sqrt{5})(3 - i\sqrt{5})$ (5)

Using (4) & (5) in (3) and employing the method of factorization, define

$$u + i\sqrt{5}v = (3 + i\sqrt{5})(a + i\sqrt{5}b)^3 \tag{6}$$

Equating the real and imaginary parts of (6), we get

$$u = 3a^3 - 15a^2b - 45ab^2 + 25b^3 \tag{7}$$

$$v = a^3 + 9a^2b - 15ab^2 - 15b^3 \tag{8}$$

Substituting (7) and (8) in (2), the integral solutions of (1) are given by

$$x(a,b) = x = 4a^3 - 6a^2b - 60ab^2 + 10b^3 \tag{9}$$

$$y(a,b) = y = 2a^3 - 24a^2b - 30ab^2 + 40b^3 \tag{10}$$

$$z(a,b) = z = 7a^3 - 21a^2b - 105ab^2 + 35b^3 \tag{11}$$

$$w(a,b) = w = 5a^3 - 39a^2b - 75ab^2 + 65b^3 \tag{12}$$

along with (4)

Properties

- $y(1,n) + w(1,n) - z(1,n) - 42OH_n + 14SO_n - 28P_n^5 + 28t_{3,n} \equiv 0 \pmod 2$
- $x(1,n) + y(1,n) + z(1,n) + w(1,n) - 37RD_n - 4P_n^5 + 100t_{3,n} \equiv -1 \pmod 8$
- $x(n,1) - RD_n \equiv 11 \pmod{64}$
- $4y(a,b) + z(a,b) - 3w(a,b) = 0$
- $64y^3(a,b) + z^3(a,b) - 27w^3(a,b) + 36y(a,b)z(a,b)w(a,b) = 0$

Note 1

In (2), the representations of z and w may be taken as

$$z = 2uv + 1, \qquad w = 2uv - 1 \tag{13}$$

In this case, the values of z and w are given by

$$z(a,b) = z = 6a^6 + 24a^5b - 450a^4b^2 - 400a^3b^3 + 2250a^2b^4 + 600ab^5 - 750b^6 + 1 \tag{14}$$

$$w(a,b) = w = 6a^6 + 24a^5b - 450a^4b^2 - 400a^3b^3 + 2250a^2b^4 + 600ab^5 - 750b^6 - 1 \tag{15}$$

Thus (9), (10), (14), (15) and (4) represent a different set of solutions to (1)

Note 2

Observe that z and w in (2) may also be taken as

$$z = uv + 2, \qquad w = uv - 2 \tag{16}$$

For this choice, the corresponding values of z and w are obtained as

$$z(a,b) = z = 3a^6 + 12a^5b - 225a^4b^2 - 200a^3b^3 + 1125a^2b^4 + 300ab^5 - 375b^6 + 2 \tag{17}$$

$$w(a,b) = w = 3a^6 + 12a^5b - 225a^4b^2 - 200a^3b^3 + 1125a^2b^4 + 300ab^5 - 375b^6 - 2 \tag{18}$$

Thus (9), (10), (17), (18) and (4) represent an another set of integer solutions to (1)

Pattern-2

Instead of (5), write 14 as

$$14=(-3+i\sqrt{5})(-3-i\sqrt{5})$$

Following the procedure presented in pattern-1, the corresponding integer solutions of (1) are

$$x(a,b)=x=-2a^3-24a^2b+30ab^2+40b^3 \tag{19}$$

$$y(a,b)=y=-4a^3-6a^2b+60ab^2+10b^3 \tag{20}$$

$$z(a,b)=z=-5a^3-39a^2b+75ab^2+65b^3 \tag{21}$$

$$w(a,b)=w=-7a^3-21a^2b+105ab^2+35b^3 \tag{22}$$

along with (4)

Note 3

For the choices of z and w given by (13) and (16), the corresponding two sets (I and II) of values of z and w are as follows:

Set I:

$$z(a,b)=z=-6a^6+24a^5b+450a^4b^2-400a^3b^3-2250a^2b^4+600ab^5+750b^6+1$$

$$w(a,b)=w=-6a^6+24a^5b+450a^4b^2-400a^3b^3-2250a^2b^4+600ab^5+750b^6-1$$

Set II:

$$z(a,b)=z=-3a^6+12a^5b+225a^4b^2-200a^3b^3-1125a^2b^4+300ab^5+375b^6+2$$

$$w(a,b)=w=-3a^6+12a^5b+225a^4b^2-200a^3b^3-1125a^2b^4+300ab^5+375b^6-2$$

Considering (19), (20), (4) with the above sets, we have two more choices of integer solutions to (1)

Pattern-3

(3) can be written as $u^2+4v^2=14p^3*1$ (23)

Write 14 as $14=(3+i\sqrt{5})(3-i\sqrt{5})$ (24)

and 1 as $1=\dfrac{(1+i4\sqrt{5})(1-i4\sqrt{5})}{81}$ (25)

Using (4),(24) & (25) in (23) and employing the method of factorization, define

$$u+i\sqrt{5}v=(3+i\sqrt{5})(a+i\sqrt{5}b)^3\frac{(1+i4\sqrt{5})}{9}$$

Equating the real and imaginary parts, we get

$$u=\frac{1}{9}(-17a^3-195a^2b+255ab^2+325b^3)$$

$$v=\frac{1}{9}(13a^3-51a^2b-195ab^2+85b^3)$$

Replacing a by 3a and b by 3b in the above set of equations and in (4) the values of u, v and p becomes,

$$u=-51a^3-585a^2b+765ab^2+975b^3$$

$$v=39a^3-153a^2b-585ab^2+255b^3$$

$$p=9a^2+45b^2 \tag{26}$$

Hence in view of (2), the integral solutions of (1) are given by

$$x(a,b) = x = -12a^3 - 738a^2b + 180ab^2 + 1230b^3 \quad (27)$$

$$y(a,b) = y = -90a^3 - 432a^2b + 1350ab^2 + 720b^3 \quad (28)$$

$$z(a,b) = z = -63a^3 - 1323a^2b + 945ab^2 + 2205b^3 \quad (29)$$

$$w(a,b) = w = -141a^3 - 1017a^2b + 2115ab^2 + 1695b^3 \quad (30)$$

along with (26)

Properties

- $x(a,b) + w(a,b) - y(a,b) - z(a,b) = 0$
- $2x(a,b) + 2y(a,b) - z(a,b) - w(a,b) = 0$
- $x(a,b) + 3y(a,b) - 2w(a,b) = 0$
- $x^3(a,b) + 27y^3(a,b) - 8w^3(a,b) + 18x(a,b)y(a,b)w(a,b) = 0$
- $3x(a,b) + y(a,b) - 2z(a,b) = 0$
- $27x^3(a,b) + 3y^3(a,b) - 8z^3(a,b) + 18x(a,b)y(a,b)z(a,b) = 0$

Note 4

For the choices of z and w given by (13) and (16), the corresponding two sets (I and II) of values of z and w are as follows:

Set I:

$$z(a,b) = z = -3978a^6 - 30024a^5b + 298350a^4b^2 + 500400a^3b^3 - 14917500a^2b^4 - 750600ab^5 + 497250b^6 + 1$$

$$w(a,b) = w = -3978a^6 - 30024a^5b + 298350a^4b^2 + 500400a^3b^3 - 14917500a^2b^4 - 750600ab^5 + 497250b^6 - 1$$

Set II:

$$z(a,b) = z = -1989a^6 - 15012a^5b + 149175a^4b^2 + 250200a^3b^3 - 745875a^2b^4 - 375300ab^5 + 248625b^6 + 2$$

$$w(a,b) = w = -1989a^6 - 15012a^5b + 149175a^4b^2 + 250200a^3b^3 - 745875a^2b^4 - 375300ab^5 + 248625b^6 - 2$$

Considering (26), (27), (28) with the above sets, we have two more choices of integer solutions to (1)

Pattern-4

Instead of (17), write 1 as

$$1 = \frac{(2 + i\sqrt{5})(2 - i\sqrt{5})}{9}$$

Following the procedure presented in pattern-3, the corresponding integer solutions of (1) are

$$x(a,b) = x = 54a^3 - 648a^2b - 810ab^2 + 1080b^3 \quad (31)$$

$$y(a,b) = y = -36a^3 - 702a^2b + 540ab^2 + 1170b^3 \quad (32)$$

$$z(a,b) = z = 63a^3 - 1323a^2b - 945ab^2 + 2205b^3 \quad (33)$$

$$w(a,b) = w = -27a^3 - 1377a^2b + 405ab^2 + 2295b^3 \quad (34)$$

along with (26)

Note 5

For the choices of z and w given by (13) and (16), the corresponding two sets (I and II) of values of z and w are as follows:

Set I:

$$z(a,b) = z = 810a^6 - 60264a^5b - 60750a^4b^2 + 1004400a^3b^3 + 303750a^2b^4 - 1506600ab^5 - 101250b^6 + 1$$

$$w(a,b) = w = 810a^6 - 60264a^5b - 60750a^4b^2 + 1004400a^3b^3 + 303750a^2b^4 - 1506600ab^5 - 101250b^6 - 1$$

Set II:

$$z(a,b)=z=405a^6-30132a^5b-30375a^4b^2+502200a^3b^3+151875a^2b^4-753300ab^5-50625b^6+2$$

$$w(a,b)=w=405a^6-30132a^5b-30375a^4b^2+502200a^3b^3+151875a^2b^4-753300ab^5-50625b^6-2$$

Considering (26), (31), (32) with the above sets, we have two more choices of integer solutions to (1)

Pattern-5

Also (17) can be written as

$$1=\frac{(2+i3\sqrt{5})(2-i3\sqrt{5})}{49}$$

Following the procedure as presented above, the values of x,y,z,w and p are given by

$$x(a,b)=x=98a^3-9408a^2b-1470ab^2+15680b^3 \quad (35)$$

$$y(a,b)=y=-980a^3-6762a^2b+14700ab^2+11270b^3 \quad (36)$$

$$z(a,b)=z=-343a^3-17493a^2b+5145ab^2+29155b^3 \quad (37)$$

$$w(a,b)=w=-1421a^3-14847a^2b+21315ab^2+24745b^3 \quad (38)$$

$$p(a,b)=p=49a^2+245b^2 \quad (39)$$

Note 6

For the choices of z and w given by (13) and (16), the corresponding two sets (I and II) of values of z and w are as follows:

Set I:

$$z(a,b)=z=-475398a^6-7548744a^5b+35654850a^4b^2+125812400a^3b^3$$
$$-178274250a^2b^4-188718600ab^5+59424750b^6+1$$

$$w(a,b)=w=-475398a^6-7548744a^5b+35654850a^4b^2+125812400a^3b^3$$
$$-178274250a^2b^4-188718600ab^5+59424750b^6-1$$

Set II:

$$z(a,b)=z=-237699a^6-3774372a^5b+17827425a^4b^2+62906200a^3b^3$$
$$-89137125a^2b^4-94359300ab^5+29712375b^6+2$$

$$w(a,b)=w=-237699a^6-3774372a^5b+17827425a^4b^2+62906200a^3b^3$$
$$-89137125a^2b^4-94359300ab^5+29712375b^6-2$$

Considering (35), (36), (39) with the above sets, we have two more choices of integer solutions to (1).

CONCLUSION

In this paper, we have illustrated different methods of obtaining non-zero integer solutions to the Quintic equation with 5 unknowns given by $x^4-y^4+2(x^2-y^2)(x-y)^2=14(z^2-w^2)p^3$. As the Quintic Diophantine equation are rich in variety one may consider other forms of Quintic equation with variable ≥ 5 and search for their corresponding integer solutions along with the corresponding properties.

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