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Review Article

ON THE CUBIC DIOPHANTINE EQUATION WITH FIVE UNKNOWNS $x^3+y^3+(x+y)(x-y)^2=16(z+w)p^2$

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ABSTRACT
The cubic Diophantine equation with five unknowns represented by $x^3+y^3+(x+y)(x-y)^2=16p^2(z+w)$ is analyzed for its patterns of non – zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

Keywords: *Cubic Equation, Integral Solutions, Special Polygonal Numbers, Pyramidal Numbers*

INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, cubic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity (Dickson, 1952; Mordell, 1969; Carmichael, 1959). For illustration, one may refer (Gopalan and Premalatha, 2009; Gopalan and Pandichelvi, 2010; Gopalan and Sivagami, 2010; Gopalan and Premalatha, 2010; Gopalan and KaligaRani, 2010; Gopalan and Premalatha, 2010; Gopalan *et al.*, 2012; Gopalan *et al.*, 2012; Gopalan *et al.*, 2012) for homogeneous and non-homogeneous cubic equations with three, four and five unknowns. This paper concerns with the problem of determining non-trivial integral solution of the non- homogeneous cubic equation with five unknowns given by $x^3+y^3+(x+y)(x-y)^2=16p^2(z+w)$ .A few relations between the solutions and the special numbers are presented.

Notations Used

- $t_{m,n}$ - Polygonal number of rank n with size m.
- P_n^m - Pyramidal number of rank n with size m.
- gn_a - Gnomonic number of rank a
- J_n - Jacobsthal number of rank n
- j_n - Jacobsthal-Lucas number of rank n

Method of Analysis

The cubic Diophantine equation with five unknowns to be solved for its non-zero distinct integral solutions is given by

$$x^3+y^3+(x+y)(x-y)^2=16p^2(z+w) \tag{1}$$

Introducing the linear transformations

$$x=u+v, y=u-v, z=u+S, w=u-S \tag{2}$$

in (1), leads to

$$u^2+7v^2=16p^2 \tag{3}$$

We present below different methods of solving (3) and thus obtain different patterns of integral solutions to (1).

Pattern-I

Assume $p=p(a,b)=a^2+7b^2$ (4)

where a and b are non zero distinct integers.

Write 16 as $16 = (3+i\sqrt{7})(3-i\sqrt{7})$ (5)

Substituting (4) & (5) in (3) and employing factorization, define

$(u+i\sqrt{7}v) = (3+i\sqrt{7})(a+i\sqrt{7}b)^2$

Equating the real and imaginary parts, we have

$u = u(a,b) = 3a^2 - 21b^2 - 14ab$

$v = v(a,b) = a^2 - 7b^2 + 6ab$

Hence in view of (2) and (4), the non-zero distinct integral solutions of (1) are

$x = x(a,b) = 4a^2 - 28b^2 - 8ab$

$y = y(a,b) = 2a^2 - 14b^2 - 20ab$

$z = z(a,b,S) = 3a^2 - 21b^2 - 14ab + S$

$w = w(a,b,S) = 3a^2 - 21b^2 - 14ab - S$

$p = p(a,b) = a^2 + 7b^2$

Properties

1. $x(a,1) - t_{10,a} \equiv -3 (\bmod 5)$

2. $x(a,1) + z(a,1,52) \equiv 3 (\bmod 16)$

3. $y(-a,1) - 4t_{3,a} \equiv 4 (\bmod 18)$

4. $y(a,1) - t_{6,a} \equiv 5 (\bmod 19)$

5. $y(a,1) + w(a,1,-35) - t_{12,a} + 30a = 0$

6. $x(-a,1) - 8t_{3,a} - 4a + j_5 - J_3 = 0$

7. $-[x(a,a)p(a,a)]$ is a bi quadratic integer.

Pattern-II

Instead of (5), write 16 as $16 = (-3+i\sqrt{7})(-3-i\sqrt{7})$

Following the procedure similar to pattern-I, the corresponding non-zero distinct integer solutions of (1) are found to be

$x = x(a,b) = -2a^2 + 14b^2 - 20ab$

$y = y(a,b) = -4a^2 + 28b^2 + 8ab$

$z = z(a,b,S) = -3a^2 + 21b^2 - 14ab + S$

$w = w(a,b,S) = -3a^2 + 21b^2 - 14ab - S$

$p = p(a,b) = a^2 + 7b^2$

Properties

1. $y(1,2a-1) - 2x(1,2a-1) - 32gn_a = 0$

2. $3p(a,a+1) - z(a,a+1,S) - 6t_{4,a} - 28t_{3,a} + S = 0$

3. $4[z(a,b,S^3) - w(a,b,S^3)]$ is a cubical integer.

4. $y(6a,a-1) - 2x(6a,a-1) - 32S_a + j_5 + J_2 = 0$

Pattern-III

Rewrite (3) as $u^2 = 16p^2 - 7v^2$ (6)

Introducing the linear transformations

$$p = X + 7T, v = X + 16T \quad (7)$$

in (6) , it leads to

$$u^2 = 9X^2 + (-9)112T^2$$

Replacing u by 3U, we get (8)

$$X^2 = 112T^2 + U^2 \quad (9)$$

which is satisfied by

$$T = 2rs$$

$$U = 112r^2 - s^2$$

$$X = 112r^2 + s^2$$

In view of (7) & (8) , we have

$$u = 336r^2 - 3s^2$$

$$p = 112r^2 + s^2 + 14rs$$

$$v = 112r^2 + s^2 + 32rs$$

Substituting the values of u, v and p in (2), the corresponding non-zero integer solutions are given by

$$x = x(r,s) = 448r^2 - 2s^2 + 32rs$$

$$y = y(r,s) = 224r^2 - 4s^2 - 32rs$$

$$z = z(r,s,S) = 336r^2 - 3s^2 + S$$

$$w = w(r,s,S) = 336r^2 - 3s^2 - S$$

$$p = p(r,s) = 112r^2 + s^2 + 14rs$$

Properties

1. $x(r,1) - 448t_{4,r} \equiv -2 \pmod{32}$

2. $y(r,1) - t_{450,r} \equiv -4 \pmod{191}$

3. $x(r,r) - 2y(r,r) - 96t_{4,r}$ is a nasty number.

Each of the following expressions represents the perfect square:

1. $3x(r,1) - 2z(r,1,S) - 192t_{3,r} + 2S$

2. $x(s,s) - y(s,s) - J_2$

Multiplying each of the above by 6, we obtain Nasty number.

Note 1

The linear transformation (7) can also be taken as

$$p = X - 7T, v = X - 16T$$

By following the procedure as in the above pattern, we get the non-zero distinct integer solutions are given by

$$x = x(r,s) = 448r^2 - 2s^2 - 32rs$$

$$y = y(r,s) = 224r^2 - 4s^2 + 32rs$$

$$z = z(r,s,S) = 336r^2 - 3s^2 + S$$

$$w = w(r,s,S) = 336r^2 - 3s^2 - S$$

$$p = p(r,s) = 112r^2 + s^2 - 14rs$$

Pattern-IV

Equation (9) can be written in the form $X^2 - U^2 = 112T^2$

Define $(X+U)(X-U)=112T^2$ (10)

Now consider
$$X+U=T^2$$
$$X-U=112$$

Solving the above two equations, we obtain

$$X=\frac{T^2+112}{2}$$

$$U=\frac{T^2-112}{2}$$

Since our interest is on finding integer solutions, it is noted that the values of X & U are integers when T is even.

In other words, choosing T=2k and proceeding as in pattern-III the corresponding non-zero integer solutions are

$$x=x(k)=8k^2+32k-112$$
$$y=y(k)=4k^2-32k-224$$
$$z=z(k,S)=6k^2-168+S$$
$$w=w(k,S)=6k^2-168-S$$
$$p=p(k)=2k^2+14k+56$$

Properties

The following expressions represent the nasty number:

1. $2x(1)+2y(1)+672$
2. $x(1)-2y(1)-336$
3. $4p(1)-x(1)-336$

Each of the following expressions represents a perfect square:

1. $p(1)-x(1)$
2. $-[z(k,S)+w(k,S)]+12t_{4,k}-12$

Note 2

The system (10) can also be written as

$$X+U=2T^2$$
$$X-U=56$$

Following the procedure similar to pattern-IV, the corresponding integral solutions are obtained to be

$$x=x(T)=4T^2+16T-56$$
$$y=y(T)=2T^2-16T-112$$
$$z=z(T,S)=3T^2-84+S$$
$$w=w(T,S)=3T^2-84-S$$
$$p=p(T)=T^2+7T+28$$

Note 3

Rewrite (10),

$$X+U=8T^2$$
$$X-U=14$$

By repeating the process as in pattern-IV, the non-zero distinct integer solutions are found to be

$$x = x(T) = 16T^2 + 16T - 14$$

$$y = y(T) = 8T^2 - 16T - 28$$

$$z = z(T,S) = 12T^2 - 21 + S$$

$$w = w(T,S) = 12T^2 - 21 - S$$

$$p = p(T) = 4T^2 + 7T + 7$$

CONCLUSION

To conclude, one may search for other patterns of integral solutions of (1).

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