International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jpms.htm
2015 Vol. 5 (1) January-March, pp. 32-34/Roy
Research Article

ON THE FORMULA FOR PERIHELION PRECESSION IN THE TWO BODY PROBLEM AT THE POST NEWTONIAN APPROXIMATION OF GENERAL RELATIVITY

*Rajat Roy
Department of Electronics and Electrical Communication Engineering, Indian Institute of Technology, Kharagpur, India-721302
**Author for Correspondence*

ABSTRACT
It is demonstrated that the formula for perihelion precession in the case of the two body problem of comparable masses in general relativity (or rather the post Newtonian approximation of it) which is supposedly identical to that of a single massive gravitating body problem is incorrect. A term by term analysis of the derivation given in existing literature reveals the error and this can be as high as $1/24$ th of the value that the formula predicts. A critical discussion on the definition of the centre of inertia is also presented.

Keywords*: *Perihelion Precession, Post Newtonian Approximation, Two Body Problem

INTRODUCTION

In a recent book by Padmanabhan (2010) the author states that the two body problem in the post Newtonian approximation of general relativity which yields a formula for perihelion precession rate needs to be reanalyzed. The formula is identical to that of a test body in a Schwarzschild metric with the mass M in the metric being replaced by the sum of masses $m_1 + m_2$ of the two body problem. In the words of the above author we find this: "No simple reason for this coincidence is known and it is an issue worth thinking about". The method of derivation is given following the classic book by Landau and Lifshitz (1975) and we will reproduce this in the first subsection below pointing out exactly where the error occurs and hence get an idea of the magnitude of this error. One can also find that at some point in the derivation the concept of centre of inertia (mass) along with a reference system whose origin is fixed to this centre of inertia is being invoked. The justification of using this concept with respect to conservation laws available in literature in the post Newtonian approximation of general relativity is critically studied in the second subsection.

The Formula and the Method of its Derivation by Landau and Lifshitz (1975)

On page 366 of this above quoted reference the Lagrangian of the two body system is given as

$$L = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} + \frac{k m_1 m_2}{r} + \frac{1}{8c^2}\left(m_1 v_1^4 + m_2 v_2^4\right) - \frac{k^2 m_1 m_2 (m_1 + m_2)}{2c^2 r^2} + \frac{k m_1 m_2}{2c^2 r}\left[3\left(v_1^2 + v_2^2\right) - 7\vec{v}_1 . \vec{v}_2 - \left(\vec{v}_1 . \vec{n}\right)\left(\vec{v}_2 . \vec{n}\right)\right], \quad (1)$$

where all the notations have the same meaning as explained in the reference. From this the Hamiltonian is obtained by transforming to the centre of inertia frame which is defined on page 183 of the same book as

$$\vec{p}_2 = m_2 \vec{v}_2 = -\vec{p}_1 = -m_1 \vec{v}_1 = \vec{p} \quad (2)$$

where $\vec{p}_1$ and $\vec{p}_2$ are the momentum of the individual particles. The Hamiltonian function is thus obtained as

$$H = \frac{p^2}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right) - \frac{km_1m_2}{r} - \frac{p^4}{8c^2}\left(\frac{1}{m_1{}^3}+\frac{1}{m_2{}^3}\right) + \frac{k^2m_1m_2(m_1+m_2)}{2c^2r^2} - \frac{k}{2c^2r}\left[3p^2\left(\frac{m_2}{m_1}+\frac{m_1}{m_2}\right)+7p^2+\left(\vec{p}.\vec{n}\right)^2\right] \quad (3)$$

Since the Lagrangian has no explicit time dependence the Hamiltonian is equal to a constant E that is an integral of motion called the energy integral. With the radial component of momentum denoted by p_r and the angular momentum by M we must replace p^2 by $p_r{}^2 + M^2/r^2$ to obtain up to the post Newtonian order (or second order)

$$E = \frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\left(p_r{}^2+\frac{M^2}{r^2}\right) - \frac{km_1m_2}{r} - \frac{1}{8c^2}\left(\frac{1}{m_1{}^3}+\frac{1}{m_2{}^3}\right)\left(\frac{2m_1m_2}{m_1+m_2}\right)^2\left(E+\frac{km_1m_2}{r}\right)^2 + \frac{k^2m_1m_2(m_1+m_2)}{2c^2r^2} - \frac{k}{2c^2r}\left[3\left(\frac{m_2}{m_1}+\frac{m_1}{m_2}\right)+7\right]\frac{2m_1m_2}{m_1+m_2}\left(E+\frac{km_1m_2}{r}\right) - \frac{k}{2c^2r}p_r{}^2 \quad (4)$$

All these equations have been derived on page 366 of Landau and Lifshitz (1975). After this p_r is determined from Eq. (4) and with the definition $S_r = \int p_r dr$ we obtain for S_r the radial part of action (see page 450 of Padmanabhan (2010)) an expression given in both the references consulted so far as

$$S_r = \int\sqrt{A + \frac{B}{r} - \left(M^2 - \frac{6k^2m_1{}^2m_2{}^2}{c^2}\right)\frac{1}{r^2}}\,dr \quad (5)$$

where A and B are some constant coefficients whose explicit determination is unnecessary for determining the perihelion advance. We feel that the error occurs precisely at this point. The terms which have $1/r^2$ dependence in Eq. (4) above appears within the parenthesis of Eq. (5) after division by a common factor $\frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)-\frac{k}{2c^2r}$. Let us check each of the terms one by one.

The first term is: $$+\frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\frac{M}{r^2}$$

The second term is: $$-\frac{1}{8c^2}\left(\frac{1}{m_1{}^3}+\frac{1}{m_2{}^3}\right)\frac{4k^2m_1{}^4m_2{}^4}{(m_1+m_2)^2r^2}$$

The third term is: $$+\frac{k^2m_1m_2(m_1+m_2)}{2c^2r^2}$$

The fourth term is: $$-\frac{3k^2}{c^2r^2}\left(\frac{m_2}{m_1}+\frac{m_1}{m_2}\right)\frac{m_1{}^2m_2{}^2}{m_1+m_2}$$

The fifth term is: $$-\frac{7k^2m_1{}^2m_2{}^2}{c^2r^2(m_1+m_2)}$$

All these terms add up to give instead of Eq. (5) the following expression for S_r

© Copyright 2014 | Centre for Info Bio Technology (CIBTech)

$$S_r = \int \sqrt{A + \frac{B}{r} - \left(M^2 - \frac{6k^2 m_1^2 m_2^2}{c^2} + \frac{k^2 m_1^3 m_2^3}{c^2 (m_1 + m_2)^2} \right) \frac{1}{r^2}} dr \qquad (6)$$

Thus instead of the perihelion advance formula as given by Landau and Lifshitz (1975) which is $\delta\phi = \frac{6\pi k(m_1 + m_2)}{c^2 a(1-e^2)}$ the actual advance should read as $\delta\phi = \frac{6\pi k(m_1 + m_2)}{c^2 a(1-e^2)} - \frac{\pi k m_1 m_2}{c^2 (m_1 + m_2) a(1-e^2)}$. This extra term vanishes in the limit of the test particle $m_2 = 0$ and becomes a maximum when $m_2 = m_1$ taking a value $1/24$ th of the term $\frac{6\pi k(m_1 + m_2)}{c^2 a(1-e^2)}$.

The Definition of the Centre of Inertia used above and its Inadequacy in the Post Newtonian Approximation (PNA)

The centre of inertia frame defined by Eq. (2) and used above to obtain the Hamiltonian function may not be suitable at the post Newtonian approximation. There is no harm in writing $\vec{p}_2 = -\vec{p}_1 = \vec{p}$ but to say $\vec{p}_2 = m_2 \vec{v}_2$ and $\vec{p}_1 = m_1 \vec{v}_1$ is incorrect although without this Eq. (3) above cannot be obtained from the Lagrangian. The correct expressions for $\vec{p}_2$ and $\vec{p}_1$ are given for example by Chandrasekhar and Contopoulos (1967)

$$\vec{p}_1 = m_1 \vec{v}_1 \left(1 + \frac{v_1^2}{2c^2} + \frac{3k}{c^2} \frac{m_2}{r} \right) - \frac{k m_1 m_2}{2c^2 r} \left[7\vec{v}_2 + \left(\vec{n} . \vec{v}_2 \right) \vec{n} \right] \qquad (7)$$

$$\vec{p}_2 = m_2 \vec{v}_2 \left(1 + \frac{v_2^2}{2c^2} + \frac{3k}{c^2} \frac{m_1}{r} \right) - \frac{k m_1 m_2}{2c^2 r} \left[7\vec{v}_1 + \left(\vec{n} . \vec{v}_1 \right) \vec{n} \right] \qquad (8)$$

This might in our opinion add more correction terms to the computed value of perihelion precession.

CONCLUSION

One must remember that the precession formula has immense practical application in Binary systems consisting of two compact bodies. For example it has been used by Taylor and Weisberg (1982) to fit the data of electromagnetic pulse arrival times of PSR 1913+16. Unless we use correct formulas the general relativistic corrections to the elliptic orbit will never be correctly evaluated.

REFERENCES

Chandrasehkar S and Contopoulos G (1967). On a Post-Galilean Transformation Appropriate to the Post-Newtonian Theory of Einstein, Infeld and Hoffmann. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* **298** 123-141.

Landau LD and Lifshitz EM (1975). *The Classical Theory of Fields: Course of Theoretical Physics* (Reed Educational and Professional Publishing Ltd., Indian Reprint, Reed Elsevier India 2010) **2**.

Padmanabhan T (2010). *Gravitation Foundations and Frontiers* (Cambridge University Press) New Delhi.

Taylor JH and Weisberg JM (1982). A new test of general relativity: Gravitational radiation and the binary pulsar PSR 1913+16. *The Astrophysical Journal* **253** 908-920.

© Copyright 2014 | Centre for Info Bio Technology (CIBTech)