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Research Article

REGULAR SOLUTIONS TO THE BOLTZMANN EQUATION ON A ROBERTSON-WALKER SPACETIME

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ABSTRACT
A regular solution for the Boltzmann equation is proved using the Sobolev inequalities.

***Keywords** : Boltzmann Equation; Local Existence in Sobolev Spaces; Curved Lorentzian Manifold*

INTRODUCTION

The Boltzmann equation is one of the fundamental equations in kinetic theory. It describes the shock of particles by giving the expression of the collisions operator. We follow the theory of binary and elastic collisions due to Lichnerowicz and Chernikov (1940), according to which at a position and a given time, only two particles enter in collision, without destructing themselves, only the sum of their momenta being preserved (Noutchegueme, Dongo and Takou, 2005), (Noutchegueme and Takou, 2006) and (Noutchegueme and Dongo, 2005) had solutions in the sense of distributions. (Mucha, 1998) says that there exists a solution which is a function of class C^1, but without giving the elements of proof. In the present paper, we prove in details by giving the derivatives up to order three of the change of variables. We show why one must consider a weighted Sobolev space $H_d^3(\mathbb{R}^3)$ with $d > \frac{5}{2}$.

(Glassey, 1996) studies the Boltzmann equation in the Minkowski spacetime. He also gives the formulation of the relativistic Boltzmann equation we adopt here. We choose the Robertson-Walker spacetime, which is a simple curved space; notice that the Boltzmann equation is more complicated in a curved spacetime than in the flat spacetime, because of the Christoffel symbols which appear in the left hand side, and the determinant of the metric, which appears in the right hand side with the collision operator. This brings some obstructions in calculations. (Noutchegueme, Dongo and Takou, 2005), (Noutchegueme and Takou, 2006) and (Noutchegueme and Dongo, 2005) had function space in which the solutions were only distributions. In the present paper, we have as function space, the Sobolev space H_d^3 in which the solutions are regular functions of class C^1. The difference is very important. To obtain this result, we had to do calculations which are not trivial (fundamental inequality, linearized Boltzmann equation, $\cdots$) and we found solution in H_d^3 weak star. It was not easy. We leave for later the treatment for large time, since the short time is already long enough.

The work is organized as follows :

In section 2, we introduce the Boltzmann equation;
In section 3, we prove the fundamental inequality;
In section 4, we prove the existence theorem.

The Boltzmann Equation

The Equation

We consider the Robertson-Walker spacetime, which is a curved Lorentzian manifold, whose metric writes :

$$g = -dt^2 + a^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \qquad (2.1)$$

where $(x^\alpha) = (x^0, x^i) = (t, x^i)$ are the coordinates in $\mathbb{R}^4$.

We adopt the Einstein summation convention $A_\alpha B^\alpha = \sum_\alpha A_\alpha B^\alpha$. We suppose that $a > 0$ is sufficiently smooth and that $\frac{\dot{a}}{a}$ is bounded over each interval $[0, T], T > 0$. Hence there exists $C > 0$ such that $|\frac{\dot{a}}{a}| \leq C$, from where we deduce :

$$a(t) \leq a_0 e^{Ct} \,;\, (\frac{1}{a})(t) \leq \frac{1}{a_0} e^{Ct} \tag{2.2}$$

with $a_0 = a(0)$.

The only Christoffel symbols which are not zero are the Γ^0_{ii} and Γ^i_{i0} where :

$$\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = \dot{a}a \,;\, \Gamma^1_{10} = \Gamma^1_{20} = \Gamma^1_{30} = \frac{\dot{a}}{a} \tag{2.3}$$

The Boltzmann equation writes :

$$\mathcal{L}_X f = Q(f, f) \tag{2.4}$$

1. where f is the distribution function which mesures the probability of density of presence of the particles in a given domain. f is a real positive function of the position x and of the momentum $p = (p^\beta) = (p^0, \overline{p})$, $\overline{p} = (p^i)$. The function f is defined on the tangent bundle $T(\mathbb{R}^4)$:

$$\begin{aligned} f: \quad & T(\mathbb{R}^4) \to \mathbb{R}^+ \\ & (x^\alpha, p^\beta) \mapsto f(x^\alpha, p^\beta) \end{aligned}$$

2. $\mathcal{L}_X$ is the Lie derivative with respect to the vector X ; here :

$$(\mathcal{L}_X)^\alpha = -\Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu + e F^\alpha_\lambda p^\lambda \tag{2.5}$$

where e is the elementary charge of particles, $F = (F_{\lambda\mu})$ the Maxwell fields generated by the charged particles.

We normalize the rest mass of the particles to $m = 1$, and they are in the mass hyperboloid whose equation is : $g_{\alpha\beta} p^\alpha p^\beta = -1$ in the tangent bundle.

The expression (2.1) of the metric gives :

$$p^0 = \sqrt{1 + a^2((p^1)^2 + (p^2)^2 + (p^3)^2)} \tag{2.6}$$

where the choice $p^0 > 0$ symbolizes the fact that particles eject toward the future.

$(F^{\alpha\beta})$ is solution to the Maxwell equation :

$$\nabla_\alpha F^{\alpha\beta} = \int_{\mathbb{R}^3} \frac{p^\beta |g|^{\frac{1}{2}} f(\overline{p})}{p^0} d\overline{p} - e u^\beta \tag{2.7}$$

where we suppose that the particles are comoving which means $u=(u^{\beta})=(1,0,0,0)$. We have for $\beta=0$:

$\nabla_{\alpha}F^{\alpha 0}=\partial_{\alpha}F^{\alpha 0}+\Gamma^{\alpha}_{\alpha\lambda}F^{\lambda 0}+\Gamma^{0}_{\alpha\lambda}F^{\alpha\lambda}=0$

since F depends only on t and is antisymmetric. (2.7) then gives;

$$e=a^{3}(t)\int_{\mathbb{R}^3} f(\overline{p})d\overline{p} \tag{2.8}$$

3. Two particles have the momenta (p,q) before the collision and the momenta (p',q') after the collision and the conservation law implies :

$$p+q=p'+q' \tag{2.9}$$

4. Q is the collision operator which can be written, taking two functions f and g on $\mathbb{R}^3$:

$$Q(f,g)=Q^{+}(f,g)-Q^{-}(f,g) \tag{2.10}$$

Where

$$\begin{cases} Q^{+}(f,g)=\int_{\mathbb{R}^3}\frac{a^3(t)}{q^0}d\overline{q}\int_{\mathbf{S}^2} f(\overline{p}')g(\overline{q}')B\left(\overline{p},\overline{q},\overline{p}',\overline{q}'\right)d\omega & (2.11)\\ Q^{-}(f,g)=\int_{\mathbb{R}^3}\frac{a^3(t)}{q^0}d\overline{q}\int_{\mathbf{S}^2} f(\overline{p})g(\overline{q})B\left(\overline{p},\overline{q},\overline{p}',\overline{q}'\right)d\omega & (2.12)\end{cases}$$

with :

S^2 the unit sphere of $\mathbb{R}^3$ whose area element is denoted dw;

B is a positive regular function called the shock kernel or the cross section of collisions, on which we make the following assumptions :

$$(H_1)\quad \begin{cases}\exists C>0,\ 0\le B(\overline{p},\overline{q},\overline{p}',\overline{q}')\le C\\ (1+|\overline{p}|)^{l}\|\partial^{\beta}_{\overline{p}}B\|_{L^1(\mathbb{R}^3\times S^2)}\in L^{\infty}(\mathbb{R}^3),\ 0\le|\beta|\le 3,\ 0\le l\le 3\\ (1+|\overline{p}|)^{|\beta|-1}\partial^{\beta}_{\overline{p}}B\in L^{\infty}(\mathbb{R}^3\times\mathbb{R}^3\times S^2),\ 1\le|\beta|\le 3\end{cases}$$

5. The conservation law (2.9) split into :

$$\begin{cases} p^0+q^0=p'^0+q'^0 & (2.13)\\ \overline{p}+\overline{q}=\overline{p}'+\overline{q}' & (2.14)\end{cases}$$

(5) is the conservation of the elementary energy defined by :

$$\tilde{e} = \sqrt{1+a^2(t)[(p^1)^2+(p^2)^2+(p^3)^2]} + \sqrt{1+a^2(t)[(q^1)^2+(q^2)^2+(q^3)^2]} \tag{2.11}$$

we interpret (5) following R.T.GLASSEY by setting :

$$\begin{cases} \overline{p}' = \overline{p} + b(\overline{p},\overline{q},\omega)\omega \\ \overline{q}' = \overline{q} - b(\overline{p},\overline{q},\omega)\omega \\ (\omega \in S^2) \end{cases} \tag{2.12}$$

in which b is a regular function of its arguments; b is a real valued function which, following (2.6), (5), (5) is solution of a quadratic equation whose non trivial solution is :

$$b(\overline{p},\overline{q},\omega) = \frac{2p^0q^0\tilde{e}\,\omega\cdot(\overline{q}-\overline{p})}{(\tilde{e})^2 - [\omega\cdot(\overline{p}+\overline{q})]^2} \tag{2.13}$$

with $\overline{p} = \frac{p}{p^0}$, $\tilde{e}$ given by (2.11); the dot $(\cdot)$ is the scalar product defined on $\mathbb{R}^3$ by :

$$\overline{p}\cdot\overline{q} = a^2(t)[p^1q^1 + p^2q^2 + p^3q^3] \text{ and } \left\|p\right\|^2 = a^2(t)[(p^1)^2+(p^2)^2+(p^3)^2] \tag{2.14}$$

Note that the change of variables (2.12) give $\overline{p}'$, $\overline{q}'$ in function of $\overline{p}$, $\overline{q}$, ω, so that the integrals in (4), (4) which are taken with respect to $\overline{q}$ and ω leave a function of the only variable $\overline{p}$.
Now using the usual formulas, the Jacobian of the change of variables $(\overline{p},\overline{q}) \mapsto (\overline{p}',\overline{q}')$ given by (2.12) is :

$$\frac{\partial(\overline{p}',\overline{q}')}{\partial(\overline{p},\overline{q})} = -\frac{p'^0q'^0}{p^0q^0}. \tag{2.15}$$

Functional Space

Definition 2.1 *Let* $m \in \mathbb{N}$, $s \in \mathbb{R}^+$, $T>0$, $|\overline{p}| = \sqrt{(p^1)^2+(p^2)^2+(p^3)^2}$;

1. $\mathbf{H}_s^m(\mathbb{R}^3) = \{f : \mathbb{R}^3 \to \mathbb{R}\ ,\ (1+|\overline{p}|)^{s+|\beta|}\partial_{\overline{p}}^{\beta} f \in \mathbf{L}^2(\mathbb{R}^3), |\beta| \le m\}$

$\mathbf{H}_s^m(\mathbb{R}^3)$ is a seperable Hilbert space with the norm :

$$\|f\|_{\mathbb{H}_s^m(\mathbb{R}^3)} = \max_{0\le|\beta|\le 3} \left\|(1+|\overline{p}|)^{s+|\beta|}\partial_{\overline{p}}^{\beta} f\right\|_{\mathbb{L}^2(\mathbb{R}^3)}$$

2. $\mathbf{H}_s^m(0,T,\mathbb{R}^3) = \{f : [0,T]\times\mathbb{R}^3 \to \mathbb{R},\ f \text{ continuous}, f(t,\cdot) \in \mathbf{H}_s^m(\mathbb{R}^3), \forall t \in [0,T]\}$

Endowed with the norm :

$$\|f\|_{\mathbf{H}_s^m(0,T,\mathbb{R}^3)} = \sup_{t\in[0,T]} \max_{0\le|\beta|\le 3} \left\|(1+|\overline{p}|)^{s+|\beta|}\partial_{\overline{p}}^{\beta} f(t,\cdot)\right\|_{\mathbb{L}^2(\mathbb{R}^3)}$$

$\mathbf{H}_s^m(0,T,\mathbb{R}^3)$ is a Banach space.

Remark 2.1 *Since* $[0,T]$ *is compact, and the application* $f : [0,T]\times\mathbb{R}^3 \mapsto \mathbb{R}$,

$f(t,\cdot) \in \mathbf{H}_s^m(\mathbb{R}^3)$ continuous,

$\sup_{t\in[0,T]} \max_{0\le|\beta|\le3} \left\|(1+|\overline{p}|)^{s+|\beta|}\partial_{\overline{p}}^{\beta} f(t,\cdot)\right\|_{L^2(\mathbb{R}^3)}$ is reached, i.e, there exists $t_0 \in [0,T]$ such that :

$$\sup_{t\in[0,T]} \max_{0\le|\beta|\le3} \left\|(1+|\overline{p}|)^{s+|\beta|}\partial_{\overline{p}}^{\beta} f(t,\cdot)\right\|_{L^2(\mathbb{R}^3)} = \max_{0\le|\beta|\le3} \left\|(1+|\overline{p}|)^{s+|\beta|}\partial_{\overline{p}}^{\beta} f(t_0,\cdot)\right\|_{L^2(\mathbb{R}^3)}$$

this means that $\mathbf{H}_s^m(0,T,\mathbb{R}^3)$ is also a seperable Hilbert space.

For a fixed $r > 0$, we set :

$$\mathbf{H}_{s,r}^m(0,T,\mathbb{R}^3) = \{ f \in \mathbf{H}_s^m(0,T,\mathbb{R}^3),\ \|f\|_{\mathbf{H}_s^m(0,T,\mathbb{R}^3)} \le r \}$$

Endowed with the norm induced by $\mathbf{H}_s^m(0,T,\mathbb{R}^3)$, $\mathbf{H}_{s,r}^m(0,T,\mathbb{R}^3)$ is a completed metric space.

If $m=0$, $f \in \mathbf{H}_s^m(\mathbb{R}^3) \Leftrightarrow (1+|\overline{p}|)^s f \in \mathbf{L}^2(\mathbb{R}^3)$. So $\mathbf{H}_s^0(\mathbb{R}^3)$ will be denoted $\mathbf{L}_s^2(\mathbb{R}^3)$.

The choice of m and $d > \frac{5}{2}$

Since we are looking for regular solutions of the Boltzmann equation, we look for $f \in C_b^1(\mathbb{R}^3)$, the continuous and bounded functions of class C^1. So we look for m such that :

$$\mathbf{H}^m(\mathbb{R}^3) \hookrightarrow \mathcal{C}_b^1(\mathbb{R}^3)$$

By the Sobolev embbeding theorem, we know that :

$$\mathbf{W}_p^m(\mathbb{R}^n) \hookrightarrow \mathcal{E}^k(\mathbb{R}^n) \quad if \quad m > k + \frac{n}{p}$$

Here we have $n=3$, $p=2$, $k=1$ ($\mathbf{W}_2^m = \mathbf{H}^m$). We must take m such that :

$$m > 1 + \frac{3}{2} = \frac{5}{2}$$

the smallest enteger m such that $m > \frac{5}{2}$ is $m=3$, and we have :

$$\mathbf{H}_d^3(\mathbb{R}^3) \hookrightarrow \mathbf{H}^3(\mathbb{R}^3) \hookrightarrow \mathcal{C}_b^1(\mathbb{R}^3).$$

Now the authors of [2] , [3], [4] had solutions f such that : $f \in \mathbf{L}_2^1(0,T;\mathbb{R}^3)$ where :

$$\mathbf{L}_2^1(\mathbb{R}^3) = \{ f : \mathbb{R}^3 \to \mathbb{R} \,/\, (1+|\overline{p}|) f \in \mathbf{L}^1(\mathbb{R}^3) \}$$

and

$$\mathbf{L}_2^1(0,T,\mathbb{R}^3) = \{ f : [0,T]\times\mathbb{R}^3 \to \mathbb{R} \ / \ f(t,\cdot) \in \mathbf{L}_2^1(\mathbb{R}^3), \forall t \in [0,T] \}$$

$\mathbf{L}_2^1(0,T,\mathbb{R}^3)$ is endowed with the norm :

$$\|f\|_{\mathbf{L}^1(0,T,\mathbb{R}^3)} = \sup_{t\in[0,T]} \left\|(1+|\overline{p}|) f(t,\cdot)\right\|_{\mathbf{L}^1(\mathbb{R}^3)}$$

fromss which $\mathbf{L}_2^1(0,T,\mathbb{R}^3)$ is a Banach space.

Let us show that, if $d > \frac{5}{2}$, then :

$$\mathbf{H}_d^3(\mathbb{R}^3) \hookrightarrow \mathbf{L}_d^2(\mathbb{R}^3) \hookrightarrow \mathbf{L}_2^1(\mathbb{R}^3).$$

Let $f \in \mathbf{H}_d^3(\mathbb{R}^3)$. We have :

$$\|f\|_{\mathbf{L}^1_2(\mathsf{R}^3)} = \int_{\mathsf{R}^3}(1+|\overline{p}|)|f|\,d\overline{p} = \int_{\mathsf{R}^3}(1+|\overline{p}|)^{1-d}(1+|\overline{p}|)^d\,|f|\,d\overline{p}.$$

By Schwartz inequality, we have :

$$\|f\|_{\mathbf{L}^1_2(\mathsf{R}^3)} \le \left(\int_{\mathsf{R}^3}(1+|\overline{p}|)^{2-2d}\,d\overline{p}\right)^{\frac{1}{2}}\left(\int_{\mathsf{R}^3}(1+|\overline{p}|)^{2d}\,|f|^2\,d\overline{p}\right)^{\frac{1}{2}} = \left(\int_{\mathsf{R}^3}(1+|\overline{p}|)^{2-2d}\right)^{\frac{1}{2}}\|f\|_{\mathbf{L}^2_d}.$$

But $\int_{\mathsf{R}^3}(1+|\overline{p}|)^{2-2d}\,d\overline{p}$ has the same nature as $\int_0^{+\infty}(1+r)^{2-2d}r^2dr$ which is equivalent to $\int_0^{+\infty}r^{4-2d}dr$.

But this integral converges if $5-2d<0$, i.e $d>\frac{5}{2}$. Hence if $d>\frac{5}{2}$, then

$$\|f\|_{\mathbf{L}^1_2(\mathsf{R}^3)} \le C\|f\|_{\mathbf{L}^2_d(\mathsf{R}^3)} \le C\|f\|_{\mathbf{H}^3_d(\mathsf{R}^3)}.$$

2.3 Boundedness of $(\overline{p},\overline{q},\omega)\mapsto \partial_p^\beta b(\overline{p},\overline{q},\omega)$, $1\le\beta\le 3$

Proposition 2.1

1. There exists $T>0$ such that

$$(\tilde{e})^2-[\omega\cdot(\overline{p}+\overline{q})]^2 \ge 2,\ t\in[0,T] \tag{2.16}$$

2. We have :

$$\begin{cases}(\tilde{e})^2-[\omega\cdot(\overline{p}+\overline{q})]^2 \ge (1-a^2)(p^0)^2\\ (\tilde{e})^2-[\omega\cdot(\overline{p}+\overline{q})]^2 \ge (1-a^2)(q^0)^2\\ (\tilde{e})^2-[\omega\cdot(\overline{p}+\overline{q})]^2 \ge (1-a^2)\,p^0q^0\end{cases} \tag{2.17}$$

3. The function $(\overline{p},\overline{q},\omega)\mapsto \partial_p^\beta b(\overline{p},\overline{q},\omega)$, is bounded for $1\le|\beta|\le 3$.

Proof.

1. We have $|\omega\cdot(\overline{p}+\overline{q})|\le\|\omega\|\|\overline{p}+\overline{q}\|$, then $(\tilde{e})^2-[\omega\cdot(\overline{p}+\overline{q})]^2\ge(\tilde{e})^2-\|\omega\|^2\|\overline{p}+\overline{q}\|^2$.

But : $\|\omega\|^2 = a^2(\omega_1^2+\omega_2^2+\omega_3^2)\le a^2$. So :

$$(\tilde{e})^2-[\omega\cdot(\overline{p}+\overline{q})]^2 \ge (\tilde{e})^2-a^2\|\overline{p}+\overline{q}\|^2 = (\tilde{e})^2-a^2(\|\overline{p}\|^2+\|\overline{q}\|^2+2\|\overline{p}\|\|\overline{q}\|) \tag{a}$$

But by definition (2.11) of $\tilde{e}$ we have $\tilde{e}=p^0+q^0$. Then using (2.14) we have :

$$(\tilde{e})^2 = \left(\sqrt{1+\|\overline{p}\|^2}+\sqrt{1+\|\overline{q}\|^2}\right)^2 = 2+\|p\|^2+\|q\|^2+2\sqrt{(1+\|\overline{p}\|^2)(1+\|\overline{q}\|^2)} \ge 2+(\|p\|^2+\|q\|^2+2\|p\|\,\|q\|)$$

Hence, using (a), we have :

$$(\tilde{e})^2-[\omega\cdot(\overline{p}+\overline{q})]^2 \ge 2+(1-a^2)(\|p\|^2+\|q\|^2+2\|p\|\|q\|). \tag{b}$$

Now we had by (2.2) $0\le a(t)\le a_0e^{Ct}$, then $a^2(t)\le a_0^2e^{2Ct}\le a_0^2e^{2CT}$.

If T is such that : $a_0^2 e^{2CT} < 1$, then $1 - a^2 > 0$, and using (b), we obtain :
$(\tilde{e})^2 - [\omega \cdot (\overline{p} + \overline{q})]^2 \geq 2$; that is (2.16)

2. (b) implies that $(\tilde{e})^2 - [\omega \cdot (\overline{p} + \overline{q})]^2 \geq 1 + (1 - a^2)(\|p\|^2 + \|q\|^2 + 2\|p\|\,\|q\|)$

Since $1 - a^2 > 0$, we have :
$(\tilde{e})^2 - [\omega \cdot (\overline{p} + \overline{q})]^2 \geq 1 + (1 - a^2)\|p\|^2$; $(\tilde{e})^2 - [\omega \cdot (\overline{p} + \overline{q})]^2 \geq 1 + (1 - a^2)\|q\|^2$

But $1 > 1 - a^2$, so :

$$(\tilde{e})^2 - [\omega \cdot (\overline{p} + \overline{q})]^2 \geq (1 - a^2) + (1 - a^2)\|p\|^2 = (1 - a^2)(1 + \|p\|^2) = (1 - a^2)(p^0)^2 \quad \text{(c)}$$

$$(\tilde{e})^2 - [\omega \cdot (\overline{p} + \overline{q})]^2 \geq (1 - a^2) + (1 - a^2)\|q\|^2 = (1 - a^2)(1 + \|q\|^2) = (1 - a^2)(q^0)^2 \quad \text{(d)}$$

Now taking the product of the 2 inequalities, we obtain :
$((\tilde{e})^2 - [\omega \cdot (\overline{p} + \overline{q})]^2)^2 \geq (1 - a^2)^2 (p^0)^2 (q^0)^2$

And taking the root :

$$(\tilde{e})^2 - [\omega \cdot (\overline{p} + \overline{q})]^2 \geq (1 - a^2) p^0 q^0 \quad \text{(e)}$$

(c), (d), (e) are (2.17).

3. Let $\beta = 1$. We have :

$D_p^\beta b(\overline{p}, \overline{q}, \omega) = \dfrac{\partial b(\overline{p}, \overline{q}, \omega)}{\partial p^i}$, $i = 1,2,3$; $\omega \cdot \overline{p} = a^2(\omega^1 p^1 + \omega^2 p^2 + \omega^3 p^3)$. Set :

$$\begin{cases} A_1 = 2p^0 q^0 \tilde{e}\, \omega \cdot (\overline{q} - \overline{p}) = 2p^0 q^0 \tilde{e}\, \omega \cdot (\dfrac{\overline{q}}{q^0} - \dfrac{\overline{p}}{p^0}) = 2\tilde{e}\, \omega \cdot (p^0 \overline{q} - q^0 \overline{p}) \\ A_2 = (\tilde{e})^2 - [\omega \cdot (\overline{p} + \overline{q})]^2 \end{cases}$$

We have $b = \dfrac{A_1}{A_2}$. Then :

$$\frac{\partial b(\overline{p}, \overline{q}, \omega)}{\partial p^i} = \frac{A_2 \dfrac{\partial A_1}{\partial p^i} - A_1 \dfrac{\partial A_2}{\partial p^i}}{A_2^2} = \frac{\dfrac{\partial A_1}{\partial p^i}}{A_2} - \frac{A_1 \dfrac{\partial A_2}{\partial p^i}}{A_2^2} \quad \text{(f)}$$

But $\dfrac{\partial \tilde{e}}{\partial p^i} = \dfrac{a^2 p^i}{p^0}$; then :

$$\begin{cases} \dfrac{\partial A_1}{\partial p^i} = 2a^2 \dfrac{p^i}{p^0} \omega \cdot (p^0 \overline{q} - q^0 \overline{p}) + \dfrac{2a^2 \tilde{e} p^i}{p^0} \omega \cdot \overline{q} - 2a^2 q^0 \tilde{e}\, \omega_i & (g) \\ \dfrac{\partial A_2}{\partial p^i} = 2a^2 \tilde{e} \dfrac{p^i}{p^0} - 2a^2 \omega \cdot (\overline{p} + \overline{q}) \omega_i & (h) \end{cases}$$

We have :

$$A_1 \frac{\partial A_2}{\partial p^i} = 2\tilde{e}\,\omega\cdot(p^0\bar{q} - q^0\bar{p})[2a^2\tilde{e}\frac{p^i}{p^0} - 2a^2\omega\cdot(\bar{p}+\bar{q})\omega_i]$$

and :

$$|A_1 \frac{\partial A_2}{\partial p^i}| \le 2\tilde{e}\,|\omega|(p^0|\bar{q}| + q^0|\bar{p}|)[2a^2\tilde{e}\frac{|p^i|}{p^0} | + 2a^2|\omega|(|\bar{p}| + |\bar{q}|)|\omega_i|]$$
$$\le 4ap^0q^0(p^0+q^0)[2a(p^0+q^0) + 2(p^0+q^0)]$$
$$\le 8a(p^0q^0)(p^0+q^0)^2$$
$$|A_1 \frac{\partial A_2}{\partial p^i}| \le C(T)[(p^0q^0)^2 + p^o(q^0)^3 + q^0(p^0)^3]$$

Now we have :

$$|\frac{\partial A_1}{\partial p^i}| \le 2a^2\frac{|p^i|}{p^0}\|\omega\|(p^0\|\bar{q}\| + q^0\|\bar{p}\|) + 2a^2q^0\tilde{e}\,|\omega_i| + 2a^2\tilde{e}\frac{|p^i|}{p^0}\|\omega\|\,\|\bar{q}\|$$
$$\le 4ap^0q^0 + 2a^2q^0(p^0+q^0) + 2a^2q^0(p^0+q^0) = 4ap^0q^0 + 4a^2q^0(p^0+q^0)$$
$$|\frac{\partial A_1}{\partial p^i}| \le C(T)[p^0q^0 + (q^0)^2]$$

From the expression of $\frac{\partial b}{\partial p^i}$, we have :

$$|\frac{\partial b(\bar{p},\bar{q},\omega)}{\partial p^i}| \le \frac{|\frac{\partial A_1}{\partial p^i}|}{A_2} + \frac{|A_1\frac{\partial A_2}{\partial p^i}|}{A_2^2}.$$

we now use (2.17) which gives :

$$\begin{cases} A_2 \ge (1-a^2)(p^0)^2\,,\; A_2 \ge (1-a^2)(q^0)^2\,,\; A_2 \ge (1-a^2)p^0q^0 \\ A_2^2 \ge (1-a^2)^2(p^0)^4\,,\; A_2^2 \ge (1-a^2)^2(q^0)^4\,,\; A_2^2 \ge (1-a^2)^2(p^0q^0)^2 \\ A_2^2 \ge (1-a^2)^2p^0(q^0)^3\,,\; A_2^2 \ge (1-a^2)^2q^0(p^0)^3 \end{cases}$$

to obtain :

$$\begin{cases} \frac{|\frac{\partial A_1}{\partial p^i}|}{A_2} \le C(T)(\frac{p^0q^0 + (q^0)^2}{(q^0)^2 + p^0q^0}) \le C(T) \\ \frac{|\frac{A_1\partial A_2}{\partial p^i}|}{A_2} \le C(T)(\frac{(p^0q^0)^2 + p^0(q^0)^3 + q^0(p^0)^3}{(p^0q^0)^2 + p^0(q^0)^3 + q^0(p^0)^3}) \le C(T) \end{cases}$$

Hence

$$|\frac{\partial b(\bar{p},\bar{q},\omega)}{\partial p^i}| \le C(T)$$

Let $|\beta| = 2$, we have :

$$D_p^\beta b(\overline{p},\overline{q},\omega) = \frac{\partial^2 b(\overline{p},\overline{q},\omega)}{\partial p^i \partial p^j} = \partial^2_{p^i p^j} b(\overline{p},\overline{q},\omega),\ i,j = 1,2,3$$

We have, using (a) :

$$\partial^2_{p^i p^j} b(\overline{p},\overline{q},\omega) = \frac{\partial^2_{p^i p^j} A_1}{A_2} - \frac{\partial_{p^i} A_1 \partial_{p^j} A_2}{A_2^2} - \frac{\partial_{p^j} A_1 \partial_{p^i} A_2}{A_2^2} + \frac{A_1 \partial^2_{p^i p^j} A_2}{A_2^2} + 2\frac{A_1 \partial_{p^j} A_1 \partial_{p^i} A_2}{A_2^3} \qquad \text{(i)}$$

We have, using (b) and (c) :

$$\begin{cases} \partial^2_{p^i p^j} A_1 = 2a^2 \delta^i_j \omega \cdot \overline{q} - 2a^2 \delta^i_j (\omega \cdot \overline{p}) \dfrac{q^0}{p^0} - 2a^4 q^0 \omega_j \dfrac{p^i}{p^0} \dfrac{2a^4 q^0 p^i p^j \omega \cdot \overline{p}}{(p^0)^3} - 2a^4 q^0 \omega_j \dfrac{p^j}{p^0} + 2a^4 (\omega \cdot \overline{q}) \dfrac{p^i p^j}{(p^0)^2} \\ \qquad\qquad + \dfrac{2a^2 \delta^i_j \tilde{e} (\omega \cdot \overline{q})}{p^0} - \dfrac{2a^4 \tilde{e} (\omega \cdot \overline{q})}{p^0} \dfrac{p^i p^j}{(p^0)^2} \\ \partial^2_{p^i p^j} A_2 = 2a^4 \dfrac{p^i p^j}{(p^0)^2} + \dfrac{2a^2 \tilde{e} \delta^i_j}{p^0} - \dfrac{2a^4 \tilde{e} p^i p^j}{(p^0)^3} - 2a^4 \omega_i \omega_j \end{cases}$$

From the inequalities :

$$\begin{cases} |A_1| \le C(T)[(p^0)^2 q^0 + p^0 (q^0)^2] \\ |\partial^2_{p^i p^j} A_1| \le C(T)[(q^0)^2 + p^0 q^0] \\ |\partial_{p^i} A_1 \partial_{p^j} A_2| \le C(T)[q^0 (p^0)^2 + p^0 (q^0)^2 + (q^0)^3] \\ |\partial^2_{p^i p^j} A_2| \le C(T)[q^0 (p^0)^3 + (p^0 q^0)^2 + p^0 (q^0)^3] \\ |A_1 \partial^2_{p^i p^j} A_2| \le C(T)[q^0 (p^0)^3 + (p^0 q^0)^2 + p^0 (q^0)^3] \\ |\partial_{p^i} A_2 \partial_{p^j} A_2| \le C(T)[(p^0)^2 + p^0 q^0 + (q^0)^2] \end{cases} \qquad \text{(j)}$$

and given that from (i) we have :

$$|\partial^2_{p^i p^j} b(\overline{p},\overline{q},\omega)| \le \frac{|\partial^2_{p^i p^j} A_1|}{A_2} + 2\frac{|\partial_{p^i} A_1 \partial_{p^j} A_2|}{A_2^2} + \frac{|A_1| |\partial^2_{p^i p^j} A_2|}{A_2^2} + 2\frac{|A_1| |\partial_{p^i} A_2 \partial_{p^j} A_2|}{A_2^3}$$

we deduce that :

$$|\partial^2_{p^i p^j} b(\overline{p},\overline{q},\omega)| \le C(T)$$

Let $|\beta| = 3$. We have :

$$D_p^\beta b(\overline{p},\overline{q},\omega) = \frac{\partial^3 b(\overline{p},\overline{q},\omega)}{\partial p^i \partial p^j \partial p^k} = \partial^3_{p^i p^j p^k} b(\overline{p},\overline{q},\omega),\ i,j,k = 1,2,3$$

we derive the function $\dfrac{\partial^2 b(\overline{p},\overline{q},\omega)}{\partial p^i \partial p^j}$ given by (i), with respect to p^k, we use estimations analogous to (j) and we deduce that :

$$|\partial^3_{p^i p^j p^k} b(\overline{p},\overline{q},\omega)| \le C(T)$$

This ends to prove that $(\overline{p},\overline{q},\omega) \mapsto D_p^\beta b(\overline{p},\overline{q},\omega)$ is bounded for every β such that $1 \le |\beta| \le 3$.

This ends the proof of Proposition 2.1

The Fundamental Inequality

We want to prove that, if the collision kernel B satisfies the hypotheses (H_1) then, if $f, g \in \mathbf{H}_d^3(\mathbb{R}^3)$ with $d > \frac{5}{2}$, then $\frac{1}{p^0}Q(f,g)$ is also in $\mathbf{H}_d^3(\mathbb{R}^3)$, and that $(f,g) \mapsto \frac{1}{p^0}Q(f,g)$ is continuous from $\mathbf{H}_d^3(\mathbb{R}^3) \times \mathbf{H}_d^3(\mathbb{R}^3)$ to $\mathbf{H}_d^3(\mathbb{R}^3)$. We will use several propositions to prove this fundamental result. We will denote by $C(T)$ several constants depending only on T.

In what follows we suppose that B satisfies (H_1) and that $d > \frac{5}{2}$.

Proposition 3.1 *Let* $f, g \in \mathbf{H}_d^3(\mathbb{R}^3)$. *Then we have :*

$$\left\|(1+|\bar{p}|)^d \frac{1}{p^0} Q(f,g)\right\|_{\mathbf{L}^2(\mathbb{R}^3)} \leq C(T)\|f\|_{\mathbf{H}_d^3(\mathbb{R}^3)}\|g\|_{\mathbf{H}_d^3(\mathbb{R}^3)}$$

Proof :

Wess have :

$$(1+|\bar{p}|)^d \frac{1}{p^0} Q^+(f,g) = (1+|\bar{p}|)^d \int_{\mathbb{R}^3} \frac{a^3}{p^0 q^0} d\bar{q} \int_{S^2} f(\bar{p}')g(\bar{q}')B(\bar{p},\bar{q},\bar{p}',\bar{q}')d\omega.$$

thus

$$\left\|(1+|\bar{p}|)^d \frac{1}{p^0} Q^+(f,g)\right\|^2_{\mathbf{L}^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3}[(1+|\bar{p}|)^d \int_{\mathbb{R}^3} \frac{a^3}{p^0 q^0} d\bar{q} \int_{S^2} f(\bar{p}')g(\bar{q}')B(\bar{p},\bar{q},\bar{p}',\bar{q}')d\omega]^2 d\bar{p}$$

Or

$$\left\|(1+|\bar{p}|)^d \frac{1}{p^0} Q^+(f,g)\right\|^2_{\mathbf{L}^2(\mathbb{R}^3)} \leq C(T)\int_{\mathbb{R}^3} \frac{(1+|\bar{p}|)^{2d}}{(p^0)^2}[\iint_{\mathbb{R}^3 \times S^2} \frac{1}{q^0} f(\bar{p}')g(\bar{q}')B(\bar{p},\bar{q},\bar{p}',\bar{q}')d\bar{q}d\omega]^2 d\bar{p}$$

$$\leq C(T)\int_{\mathbb{R}^3} \frac{(1+|\bar{p}|)^{2d}}{(p^0)^2}[\iint_{\mathbb{R}^3 \times S^2} \frac{1}{q^0} f(\bar{p}')g(\bar{q}')B^{\frac{1}{2}}B^{\frac{1}{2}}(\bar{p},\bar{q},\bar{p}',\bar{q}')d\bar{q}d\omega]^2 d\bar{p}$$

We apply the Cauchy-Schwartz inequality for the integral on $\mathbb{R}^3 \times S^2$ to obtain :

$$\left\|(1+|\bar{p}|)^d \frac{1}{p^0} Q^+(f,g)\right\|^2_{\mathbf{L}^2(\mathbb{R}^3)} \leq C(T)\int_{\mathbb{R}^3} \frac{(1+|\bar{p}|)^{2d}}{(p^0)^2}[\iint_{\mathbb{R}^3 \times S^2} \frac{|f(\bar{p}')|^2 |g(\bar{q}')|^2 \, Bd\bar{q}d\omega}{(q^0)^2}][\iint_{\mathbb{R}^3 \times S^2} Bd\bar{q}d\omega]d\bar{p}$$

$$\leq C(T)\int_{\mathbb{R}^3} \frac{(1+|\bar{p}|)^{2d}}{(p^0)^2}\|B\|_{\mathbf{L}^1(\mathbb{R}^3 \times S^2)} d\bar{p}[\iint_{\mathbb{R}^3 \times S^2} \frac{|f(\bar{p}')|^2 |g(\bar{q}')|^2 \, Bd\bar{q}d\omega}{(q^0)^2}]d\bar{p}$$

But $\|B\|_{\mathbf{L}^1(\mathbb{R}^3 \times S^2)} \in \mathbf{L}^\infty(\mathbb{R}^3)$ and B is bounded, hence :

$$\left\|(1+|\bar{p}|)^d \frac{1}{p^0} Q^+(f,g)\right\|^2_{\mathbf{L}^2(\mathbb{R}^3)} \leq C(T)\int_{S^2} d\omega \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{(1+|\bar{p}|)^{2d}}{p^0 q^0} |f(\bar{p}')|^2 |g(\bar{q}')|^2 \frac{d\bar{p}d\bar{q}}{p^0 q^0} \quad (3.1)$$

We have : $(1+|\overline{p}|)^2 \leq C(T)(p^0)^2$, then :
$1+|\overline{p}| \leq C(T)p^0 \leq C(T)(p^0+q^0) \leq C(T)(p'^0+q'^0)$, since by (5), $p^0+q^0 = p'^0+q'^0$
and on the other hand :
$p'^0 = \sqrt{1+a^2[(p'^1)^2+(p'^2)^2+(p'^3)^2]} \leq 1+\sqrt{a^2[(p'^1)^2+(p'^2)^2+(p'^3)^2]} \leq C(T)(1+|\overline{p}'|)$
So :
$1+|\overline{p}| \leq C(T)(p'^0+q'^0) \leq C(T)[(1+|\overline{p}'|)+(1+|\overline{q}'|)] \leq C(T)(2+|\overline{p}'|+|\overline{q}'|) \leq C(T)(2+2|\overline{p}'|+2|\overline{q}'|) \leq C(T)(1+|\overline{p}'|$
. So we have :

$$(1+|\overline{p}|) \leq C(T)(1+|\overline{p}'|)(1+|\overline{q}'|) \tag{3.2}$$

We also know by (2.15) that :

$$\frac{d\overline{p}d\overline{q}}{p^0q^0} = \frac{d\overline{p}'d\overline{q}'}{p'^0q'^0} \tag{3.3}$$

Since $\frac{1}{p^0q^0} < 1$, (3.1) gives, using (3.2) and (3.3) :

$$\left\|(1+|\overline{p}|)^d \frac{1}{p^0}Q^+(f,g)\right\|^2_{L^2(\mathbb{R}^3)} \leq C(T)\int_{S^2}d\omega\int_{\mathbb{R}^3}(1+|\overline{p}'|)^{2d}\,|f(\overline{p}')|^2\,d\overline{p}'\int_{\mathbb{R}^3}(1+|\overline{q}'|)^{2d}\,|g(\overline{q}')|^2\,d\overline{q}'$$
$$\leq C(T)\|f\|^2_{L^2_d(\mathbb{R}^3)}\|g\|^2_{L^2(\mathbb{R}^3)}$$

Then :

$$\left\|(1+|\overline{p}|)^d \frac{1}{p^0}Q^+(f,g)\right\|_{L^2(\mathbb{R}^3)} \leq C(T)\|f\|_{L^2_d(\mathbb{R}^3)}\|g\|_{L^2(\mathbb{R}^3)} \leq C(T)\|f\|_{\mathbf{H}^3_d(\mathbb{R}^3)}\|g\|_{\mathbf{H}^3_d(\mathbb{R}^3)}$$

Proposition 3.2 *Let* $f,g \in \mathbf{H}^3_d(\mathbb{R}^3)$. *Then for* β *such that* $|\beta|=1$, *we have :*

$$\left\|(1+|\overline{p}|)^{d+1}\partial^{\beta}_{\overline{p}}\left(\frac{1}{p^0}Q^+(f,g)\right)\right\|_{\mathbf{L}^2(\mathbb{R}^3)} \leq C(T)\|f\|_{\mathbf{H}^3_d(\mathbb{R}^3)}\|g\|_{\mathbf{H}^3_d(\mathbb{R}^3)}$$

. Let $i=1,2,3$. We have $|\beta|=1$.

$$\partial^{\beta}_{\overline{p}}\left(\frac{1}{p^0}Q^+(f,g)\right) = \partial_{p^i}\left(\frac{1}{p^0}Q^+(f,g)\right) = \frac{1}{p^0}\partial_{p^i}Q^+(f,g) - \frac{a^2p^i}{(p^0)^3}Q^+(f,g)$$

and :

$$(1+|\overline{p}|)^{d+1}\partial_{p^i}\left(\frac{1}{p^0}Q^+(f,g)\right) = (1+|\overline{p}|)^{d+1}\frac{1}{p^0}\partial_{p^i}Q^+(f,g) - (1+|\overline{p}|)^{d+1}\frac{a^2p^i}{(p^0)^3}Q^+(f,g).$$

So :

$$\left\|(1+|\overline{p}|)^{d+1}\partial_{p^i}\left(\frac{1}{p^0}Q^+(f,g)\right)\right\|_{L^2(\mathbb{R}^3)} \leq \left\|(1+|\overline{p}|)^{d+1}\frac{1}{p^0}\partial_{p^i}Q^+(f,g)\right\|_{L^2(\mathbb{R}^3)} + \left\|(1+|\overline{p}|)^{d+1}\frac{a^2p^i}{(p^0)^3}Q^+(f,g)\right\|_{L^2(\mathbb{R}^3)}$$

and

$$\left\|(1+|\bar{p}|)^{d+1}\partial_{p^i}\left(\frac{Q^+(f,g)}{p^0}\right)\right\|_{L^2(\mathbb{R}^3)} \leq \left\|(1+|\bar{p}|)^{d+1}\frac{\partial_{p^i}Q^+(f,g)}{p^0}\right\|_{L^2(\mathbb{R}^3)} + C(T)\left\|(1+|\bar{p}|)^{d+1}\frac{Q^+(f,g)}{p^0}\right\|_{L^2(\mathbb{R}^3)}$$

(3.4)

Since $(1+|\bar{p}|)\leq C(T)p^0$ and $|a^2p^i|\leq C(T)p^0$.

The second term of (3.4) is estimated in proposotion 3.1; for the first term, we have :

$$\left\|(1+|\bar{p}|)^{d+1}\frac{1}{p^0}\partial_{p^i}Q^+(f,g)\right\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3}d\bar{p}\Big[\frac{1}{p^0}(1+|\bar{p}|)^{d+1}a^3\int_{\mathbb{R}^3}\frac{d\bar{q}}{q^0}\int_{S^2}d\omega\partial_{p^i}[f(\bar{p}')g(\bar{q}')B]\Big]^2 \quad (3.5)$$

in (3.5), we have :

$$\partial_{p^i}[f(\bar{p}')g(\bar{q}')B] = \partial_{p^i}\bar{p}'\partial_{p'^i}f(\bar{p}')g(\bar{q}')B + \partial_{p^i}\bar{q}'\partial_{q'^i}g(\bar{q}')f(\bar{p}')B + f(\bar{p}')g(\bar{q}')\partial_{p^i}B \quad (3.6)$$

We know by proposition 2.1, 3) that $D^\beta_p b(\bar{p},\bar{q},\omega)$ is bounded for $1\leq|\beta|\leq 3$

So, since $\bar{p}' = \bar{p}+b(\bar{p},\bar{q},\omega)$ and $\bar{q}' = \bar{p}-b(\bar{p},\bar{q},\omega)$, $\left|\frac{\partial\bar{p}'}{\partial\bar{p}}\right|$ and $\left|\frac{\partial\bar{q}'}{\partial\bar{p}}\right|$ are bounded. So, from (3.5) we have :

$$\left\|(1+|\bar{p}|)^{d+1}\frac{1}{p^0}\partial_{p^i}Q^+(f,g)\right\|^2_{L^2(\mathbb{R}^3)} \leq C(T)\int_{\mathbb{R}^3}d\bar{p}\Big(\frac{(1+|\bar{p}|)^{d+1}}{p^0}\int_{\mathbb{R}^3}\frac{d\bar{q}}{q^0}\int_{S^2}(|\partial_{p'^i}f(\bar{p}')g(\bar{q}')B|+|f(\bar{p}')\partial_{q'^i}g(\bar{q}')B|+|f(\bar{p}')g(\bar{q}')\partial_{p^i}B|)d\omega\Big)^2$$

$$\left\|(1+|\bar{p}|)^{d+1}\frac{1}{p^0}\partial_{p^i}Q^+(f,g)\right\|^2_{L^2(\mathbb{R}^3)} \leq 3C(T)\int_{\mathbb{R}^3}d\bar{p}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}\Big\{\Big(\iint_{\mathbb{R}^3\times S^2}\frac{|\partial_{p'^i}f(\bar{p}')g(\bar{q}')B|}{q^0}d\bar{q}d\omega\Big)^2 \quad (3.7)$$
$$+\Big(\int_{\mathbb{R}^3\times S^2}\frac{|f(\bar{p}')\partial_{p'^i}g(\bar{q}')B|}{q^0}d\bar{q}d\omega\Big)^2+\Big(\iint_{\mathbb{R}^3\times S^2}\frac{|f(\bar{p}')g(\bar{q}')\partial_{p'^i}B|}{q^0}d\bar{q}d\omega\Big)^2\Big\}$$

We have for the first term in the right of (3.7), using Cauchy Schwartz inequality :

$$\int_{\mathbb{R}^3}d\bar{p}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}\Big(\iint_{\mathbb{R}^3\times S^2}\frac{|\partial_{p'^i}f(\bar{p}')g(\bar{q}')B^{\frac{1}{2}}B^{\frac{1}{2}}|}{q^0}d\bar{q}d\omega\Big)^2$$
$$\leq\int_{\mathbb{R}^3}d\bar{p}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}\Big(\iint_{\mathbb{R}^3\times S^2}\frac{|\partial_{p'^i}f(\bar{p}')|^2|g(\bar{q}')|^2B}{(q^0)^2}\Big)\Big(\iint_{\mathbb{R}^3\times S^2}Bd\bar{q}d\omega\Big)$$

But $\iint_{\mathbb{R}^3\times S^2}Bd\bar{q}d\omega = \|B\|_{L^1(\mathbb{R}^3\times S^2)}\in L^\infty(\mathbb{R}^3)$ and B is bounded, so we have :

$$\int_{\mathbb{R}^3}d\bar{p}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}\Big(\iint_{\mathbb{R}^3\times S^2}\frac{|\partial_{p'^i}f(\bar{p}')g(\bar{q}')B|}{q^0}d\bar{q}d\omega\Big)^2\leq C\int_{\mathbb{R}^3}d\bar{p}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}\iint_{\mathbb{R}^3\times S^2}\frac{1}{(q^0)^2}|\partial_{p'^i}f(\bar{p}')|^2|g(\bar{q}')|^2\,d\bar{q}d\omega$$

Now we have :

$$\begin{cases} (1+|\bar{p}|)^2 = (1+|\bar{p}|)(1+|\bar{p}|) \le C(T)p^0(1+|\bar{p}'|)(1+|\bar{q}'|) \le C(T)p^0q^0p'^0q'^0; \ \dfrac{d\bar{p}d\bar{q}}{p^0q^0} = \dfrac{d\bar{p}'d\bar{q}'}{p'^0q'^0} \\ (1+|\bar{p}|) \le C(T)(1+|\bar{p}'|)(1+|\bar{q}'|) \end{cases}$$

(3.8)

Then :

$$\int_{\mathbb{R}^3} d\bar{p}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}\left(\iint_{\mathbb{R}^3\times S^2}\frac{|\partial_{p'^i}f(\bar{p}')g(\bar{q}')B|}{q^0}d\bar{q}d\omega\right)^2$$
$$\le \int_{S^2}d\omega\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(1+|\bar{p}|)^{2d+2}}{p^0q^0}|\partial_{p'^i}f(\bar{p}')|^2|g(\bar{q}')|^2\frac{d\bar{p}d\bar{q}}{p^0q^0}$$
$$\le \int_{S^2}d\omega\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(1+|\bar{p}|)^2}{p^0q^0}(1+|\bar{p}|)^{2d}|\partial_{p'^i}f(\bar{p}')|^2|g(\bar{q}')|^2\frac{d\bar{p}d\bar{q}}{p^0q^0}$$
$$\le \int_{S^2}d\omega\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(1+|\bar{p}|)^2}{p^0q^0}(1+|\bar{p}'|)^{2d}(1+|\bar{q}'|)^{2d}|\partial_{p'^i}f(\bar{p}')|^2\ g(\bar{p}')|^2\frac{d\bar{p}'d\bar{q}'}{p'^0q'^0}$$
$$\le C(T)\int_{S^2}d\omega\int_{\mathbb{R}^3}(1+|\bar{p}'|)^{2d+2}|\partial_{p'^i}f(\bar{p}')|^2\,d\bar{p}'\int_{\mathbb{R}^3}(1+|\bar{q}'|)^{2d}|g(\bar{p}')|^2\,d\bar{q}'$$

$$\int_{\mathbb{R}^3} d\bar{p}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}\left(\iint_{\mathbb{R}^3\times S^2}\frac{|\partial_{p'^i}f(\bar{p}')g(\bar{q}')B|}{q^0}d\bar{q}d\omega\right)^2 \le C(T)\|f\|^2_{\mathbf{H}^3_d(\mathbb{R}^3)}\|g\|^2_{\mathbf{H}^3_d(\mathbb{R}^3)} \tag{3.9}$$

Replacing $\partial_{p'^i}f$ by $\partial_{q'^i}g$ the same remarks gives

$$\int_{\mathbb{R}^3} d\bar{p}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}\left(\iint_{\mathbb{R}^3\times S^2}\frac{|f(\bar{p}')\partial_{q'^i}g(\bar{q}')B|}{q^0}d\bar{q}d\omega\right)^2 \le C(T)\|f\|^2_{\mathbf{H}^3_d(\mathbb{R}^3)}\|g\|^2_{\mathbf{H}^3_d(\mathbb{R}^3)} \tag{3.10}$$

Now we have, for the last term of (3.7), using Cauchy-Schwartz inequality :

$$\int_{\mathbb{R}^3} d\bar{p}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}\left(\iint_{\mathbb{R}^3\times S^2}\frac{|f(\bar{p}')\|g(\bar{q}')\|\partial_{p'^i}B|^{\frac{1}{2}}|\partial_{p'^i}B|^{\frac{1}{2}}}{q^0}d\bar{q}d\omega\right)^2$$
$$\le \int_{\mathbb{R}^3} d\bar{p}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}\left[\iint_{\mathbb{R}^3\times S^2}\frac{|f(\bar{p}')|^2|g(\bar{q}')|^2|\partial_{p^i}B|}{(q^0)^2}d\bar{q}d\omega\right]\left[\iint_{\mathbb{R}^3\times S^2}|\partial_{p^i}B|d\bar{q}d\omega\right]$$

But $\partial_{p^i}B \in L^\infty(\mathbb{R}^3\times\mathbb{R}^3\times S^2)$ and $\left\|\partial_{p^i}B\right\|_{L^1(\mathbb{R}^3\times S^2)} \in L^\infty(\mathbb{R}^3)$, then

$$\int_{\mathbb{R}^3} d\bar{p}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}\left(\iint_{\mathbb{R}^3\times S^2}\frac{|f(\bar{p}')\|g(\bar{q}')\|\partial_{p'^i}B|}{q^0}d\bar{q}d\omega\right)^2 \le C\int_{S^2}d\omega\iint_{\mathbb{R}^3\times S^2}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0q^0)^2}|f(\bar{p}')|^2|g(\bar{q}')|^2\,d\bar{p}d\bar{q}$$

We use (3.9) in which we replace $\partial_{p'^i}f$ by f to obtain :

$$\int_{\mathbb{R}^3} d\bar{p} \frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2} \left(\iint_{\mathbb{R}^3\times S^2} \frac{|f(\bar{p}')||g(\bar{q}')||\partial_{p'^i} B|}{q^0} d\bar{q}d\omega\right)^2$$
$$\leq C(T)\int_{S^2} d\omega \left(\int_{\mathbb{R}^3}(1+|\bar{p}'|)^{2d}|f(\bar{p}')|^2 d\bar{p}'\right)\left(\int_{\mathbb{R}^3}(1+|\bar{q}'|)^{2d}|g(\bar{q}')|^2 d\bar{q}'\right)$$

$$\int_{\mathbb{R}^3} d\bar{p} \frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2} \left(\iint_{\mathbb{R}^3\times S^2} \frac{|f(\bar{p}')||g(\bar{q}')||\partial_{p'^i} B|}{q^0} d\bar{q}d\omega\right)^2 \leq C(T)\|f\|^2_{\mathbf{H}^3_d(\mathbb{R}^3)}\|g\|^2_{\mathbf{H}^3_d(\mathbb{R}^3)} \quad (3.11)$$

From (3.9), (3.10), (3.11) have, given (3.5) :

$$\left\|(1+|\bar{p}|)^{d+1}\frac{1}{p^0}\partial_{p^i}\left(\frac{1}{p^0}Q(f,g)\right)\right\|^2_{\mathbf{L}^2(\mathbb{R}^3)} \leq C(T)\|f\|^2_{\mathbf{H}^3_d(\mathbb{R}^3)}\|g\|^2_{\mathbf{H}^3_d(\mathbb{R}^3)} \quad (3.12)$$

(3.12) and Proposition 3.1 applied to the last term of (3.4) give Proposition 3.2

Proposition 3.3 *Let* $f,g \in \mathbf{H}^3_d(\mathbb{R}^3)$. *Then for* β *such that* $|\beta| = 2$, *we have :*

$$\left\|(1+|\bar{p}|)^{d+2}\partial^{\beta}_{\bar{p}}\left(\frac{1}{p^0}Q(f,g)\right)\right\|_{\mathbf{L}^2(\mathbb{R}^3)} \leq C(T)\|f\|_{\mathbf{H}^3_d(\mathbb{R}^3)}\|g\|_{\mathbf{H}^3_d(\mathbb{R}^3)}$$

. We sow that :

$$\partial_{p^i}\left(\frac{1}{p^0}Q^+(f,g)\right) = \frac{1}{p^0}\partial_{p^i}Q^+(f,g) - \frac{a^2p^i}{(p^0)^3}Q^+(f,g)$$

Derivating this formula with respect to p^j, we have :

$$\partial^2_{p^ip^j}\left(\frac{1}{p^0}Q^+(f,g)\right) = \partial_{p^j}\left(\frac{1}{p^0}\partial_{p^i}Q^+(f,g)\right) - \partial_{p^j}\left(\frac{a^2p^i}{(p^0)^3}Q^+(f,g)\right)$$

$$\partial^2_{p^ip^j}\left(\frac{1}{p^0}Q^+(f,g)\right) = \frac{1}{p^0}\partial^2_{p^ip^j}Q^+(f,g) - \frac{a^2p^j}{(p^0)^3}\partial_{p^i}Q^+(f,g) - \frac{a^2p^i}{(p^0)^3}\partial_{p^j}Q^+(f,g) - \frac{a^2\delta^i_j}{(p^0)^3}Q^+(f,g) + 3\frac{a^4p^ip^j}{(p^0)^5}Q^+(f,g). \quad (3.13)$$

For the second and the thirth terms of (3.13), we will have, given (3.12) :

$$\left\|(1+|\bar{p}|)^{d+2}\frac{a^2p^j}{(p^0)^3}\partial_{p^i}Q^+(f,g)\right\|_{\mathbf{L}^2(\mathbb{R}^3)} = \left\|\frac{(1+|\bar{p}|)}{p^0}\frac{a^2p^j}{p^0}\frac{(1+|\bar{p}|)^{d+1}}{p^0}\partial_{p^i}Q^+(f,g)\right\|_{\mathbf{L}^2(\mathbb{R}^3)}$$
$$\leq C(T)\left\|(1+|\bar{p}|)^{d+1}\frac{1}{p^0}\partial_{p^i}Q^+(f,g)\right\|_{\mathbf{L}^2(\mathbb{R}^3)}$$
$$\leq C(T)\|f\|_{\mathbf{H}^3_d(\mathbb{R}^3)}\|g\|_{\mathbf{H}^3_d(\mathbb{R}^3)}$$

$$\begin{cases} \left\| (1+|\overline{p}|)^{d+2} \frac{a^2 p^i}{(p^0)^3} \partial_{p^j} Q^+(f,g) \right\|_{\mathbf{L}^2(\mathbb{R}^3)} \le C(T) \|f\|_{\mathbf{H}^3_d(\mathbb{R}^3)} \|g\|_{\mathbf{H}^3_d(\mathbb{R}^3)} \\ \left\| (1+|\overline{p}|)^{d+2} \frac{a^2 p^j}{(p^0)^3} \partial_{p^i} Q^+(f,g) \right\|_{\mathbf{L}^2(\mathbb{R}^3)} \le C(T) \|f\|_{\mathbf{H}^3_d(\mathbb{R}^3)} \|g\|_{\mathbf{H}^3_d(\mathbb{R}^3)} \end{cases} \quad (3.14)$$

For the fourth and fifth terms of (3.13) we will have, given Proposition 3.1 :

$$\left\| (1+|\overline{p}|)^{d+2} \frac{a^2 \delta^i_j}{(p^0)^3} Q^+(f,g) \right\|_{\mathbf{L}^2(\mathbb{R}^3)} = \left\| \frac{(1+|\overline{p}|)^2}{(p^0)^2} (a^2 \delta^i_j)(1+|\overline{p}|)^d \frac{1}{p^0} Q^+(f,g) \right\|_{\mathbf{L}^2(\mathbb{R}^3)}$$

$$\le C(T) \left\| (1+|\overline{p}|)^d \frac{1}{p^0} Q^+(f,g) \right\|_{\mathbf{L}^2(\mathbb{R}^3)}$$

$$\left\| (1+|\overline{p}|)^{d+2} \frac{a^2 \delta^i_j}{(p^0)^3} Q^+(f,g) \right\|_{\mathbf{L}^2(\mathbb{R}^3)} \le C(T) \|f\|_{\mathbf{H}^3_d(\mathbb{R}^3)} \|g\|_{\mathbf{H}^3_d(\mathbb{R}^3)} \quad (3.15)$$

$$\left\| (1+|\overline{p}|)^{d+2} \frac{a^4 p^i p^j}{(p^0)^5} Q^+(f,g) \right\|_{\mathbf{L}^2(\mathbb{R}^3)} = \left\| a^4 \frac{(1+|\overline{p}|)^2}{(p^0)^2} \frac{p^i p^j}{(p^0)^2} \frac{(1+|\overline{p}|)^d}{p^0} Q^+(f,g) \right\|_{\mathbf{L}^2(\mathbb{R}^3)}$$

$$\le C(T) \left\| (1+|\overline{p}|)^d \frac{1}{p^0} Q^+(f,g) \right\|_{\mathbf{L}^2(\mathbb{R}^3)}$$

$$\left\| (1+|\overline{p}|)^{d+2} \frac{a^4 p^i p^j}{(p^0)^5} Q^+(f,g) \right\|_{\mathbf{L}^2(\mathbb{R}^3)} \le C(T) \|f\|_{\mathbf{H}^3_d(\mathbb{R}^3)} \|g\|_{\mathbf{H}^3_d(\mathbb{R}^3)} \quad (3.16)$$

We then have to study only the term $\left\| (1+|\overline{p}|)^{d+2} \frac{1}{p^0} \partial^2_{p^i p^j} Q^+(f,g) \right\|_{\mathbf{L}^2(\mathbb{R}^3)}$

We have :

$$\left\| \frac{(1+|\overline{p}|)^{d+2} \partial^2_{p^i p^j} Q^+(f,g)}{p^0} \right\|^2_{\mathbf{L}^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} d\overline{p} [\frac{1}{p^0} (1+|\overline{p}|)^{d+2} a^3 \int_{\mathbb{R}^3} \frac{d\overline{q}}{q^0} \int_{S^2} \partial^2_{p^i p^j} (f(\overline{p}') g(\overline{q}') B) d\omega]^2 \quad (3.17)$$

If we derive $\partial_{p^i} (f(\overline{p}') g(\overline{q}') B)$ given by (3.6) in the proof of Proposition 3.2, with respect to p^j, we obtain :

$$\partial^2_{p^i p^j} (f(\overline{p}' g(\overline{q}') B)) = \partial_{p^j} [\partial_{p^i} \overline{p}' \partial_{p'^i} f(\overline{p}') g(\overline{q}') B + f(\overline{p}') \partial_{p^i} \overline{q}' \partial_{q'^i} g(\overline{q}') B + f(\overline{p}') g(\overline{q}') \partial_{p^i} B]$$

$$= \partial_{p^j} [\partial_{p^i} \overline{p}' \partial_{p'^i} f(\overline{p}') g(\overline{q}') B] + \partial_{p^j} [f(\overline{p}') \partial_{p^i} \overline{q}' \partial_{q'^i} g(\overline{q}') B] + \partial_{p^j} [f(\overline{p}') g(\overline{q}') \partial_{p^i} B]$$

We then have
:

$$\left\|(1+|\bar{p}|)^{d+2}\frac{1}{p^0}\partial^2_{p^ip^j}Q^+(f,g)\right\|^2_{L^2(R^3)} \leq 3\{\int_{R^3}d\bar{p}[\frac{1}{p^0}(1+|\bar{p}|)^{d+2}a^3\int_{R^3}\frac{d\bar{q}}{q^0}\int_{S^2}d\omega|\partial_{p^j}(\partial_{p^i}\bar{p}'\partial_{p'^i}f(\bar{p}')g(\bar{q}')B)|]^2$$

$$+\int_{R^3}d\bar{p}[\frac{1}{p^0}(1+|\bar{p}|)^{d+2}a^3\int_{R^3}\frac{d\bar{q}}{q^0}\int_{S^2}d\omega|\partial_{p^j}(f(\bar{p}')\partial_{p^i}\bar{q}'\partial_{q'^i}g(\bar{q}')B)|]^2$$

$$+\int_{R^3}d\bar{p}[\frac{1}{p^0}(1+|\bar{p}|)^{d+2}a^3\int_{R^3}\frac{d\bar{q}}{q^0}\int_{S^2}d\omega|\partial_{p^j}(f(\bar{p}')g(\bar{q}')\partial_{p^i}B)|]^2\}$$

$$\left\|(1+|\bar{p}|)^{d+2}\frac{1}{p^0}\partial^2_{p^ip^j}Q^+(f,g)\right\|^2_{L^2(R^3)} \leq 3(\alpha+\beta+\gamma) \qquad (3.18)$$

where

$$\begin{cases}\alpha=\int_{R^3}d\bar{p}[\frac{1}{p^0}(1+|\bar{p}|)^{d+2}a^3\int_{R^3}\frac{d\bar{q}}{q^0}\int_{S^2}d\omega|\partial_{p^j}(\partial_{p^i}\bar{p}'\partial_{p'^i}f(\bar{p}')g(\bar{q}')B)|]^2\\ \beta=\int_{R^3}d\bar{p}[\frac{1}{p^0}(1+|\bar{p}|)^{d+2}a^3\int_{R^3}\frac{d\bar{q}}{q^0}\int_{S^2}d\omega|\partial_{p^j}(f(\bar{p}')\partial_{p^i}\bar{q}'\partial_{q'^i}g(\bar{q}')B)|]^2\\ \gamma=\int_{R^3}d\bar{p}[\frac{1}{p^0}(1+|\bar{p}|)^{d+2}a^3\int_{R^3}\frac{d\bar{q}}{q^0}\int_{S^2}d\omega|\partial_{p^j}(f(\bar{p}')g(\bar{q}')\partial_{p^i}B)|]^2\end{cases} \qquad (3.19)$$

Estimation of α

We have :

$$\partial_{p^j}(\partial_{p^i}\bar{p}'\partial_{p'^i}f(\bar{p}')g(\bar{q}')B)=\partial^2_{p^ip^j}\bar{p}'\partial_{p'^i}f(\bar{p}')g(\bar{q}')B+\partial_{p^i}\bar{p}'\partial^2_{p^jp'^i}f(\bar{p}')g(\bar{q}')B$$
$$+\partial_{p^i}\bar{p}'\partial_{p'^i}f(\bar{p}')\partial_{p^j}\bar{q}'\partial_{q'^j}g(\bar{q}')B+\partial_{p^i}\bar{p}'\partial_{p'^i}f(\bar{p}')g(\bar{q}')\partial_{p^j}B$$

Set

$$\begin{cases}I=\int_{R^3}d\bar{p}[\frac{1}{p^0}(1+|\bar{p}|)^{d+2}a^3\int_{R^3}\frac{1}{q^0}d\bar{q}\int_{S^2}|\partial^2_{p^ip^j}\bar{p}'\partial_{p'^i}f(\bar{p}')g(\bar{q}')B|d\omega]^2\\ J=\int_{R^3}d\bar{p}[\frac{1}{p^0}(1+|\bar{p}|)^{d+2}a^3\int_{R^3}\frac{1}{q^0}d\bar{q}\int_{S^2}|\partial_{p^i}\bar{p}'\partial^2_{p^jp'^i}f(\bar{p}')g(\bar{q}')B|d\omega]^2\\ K=\int_{R^3}d\bar{p}[\frac{1}{p^0}(1+|\bar{p}|)^{d+2}a^3\int_{R^3}\frac{1}{q^0}d\bar{q}\int_{S^2}|\partial_{p^i}\bar{p}'\partial_{p'^i}f(\bar{p}')\partial_{p^j}\bar{q}'\partial_{q'^j}g(\bar{q}')B|d\omega]^2\\ L=\int_{R^3}d\bar{p}[\frac{1}{p^0}(1+|\bar{p}|)^{d+2}a^3\int_{R^3}\frac{1}{q^0}d\bar{q}\int_{S^2}|\partial_{p^i}\bar{p}'\partial_{p'^i}f(\bar{p}')g(\bar{q}')\partial_{p^j}B|d\omega]^2\end{cases}$$

$\partial^2_{p^ip^j}\bar{p}'$ is bounded since $D^{\beta}_{\bar{p}}b(\bar{p},\bar{q},\omega)$ is bounded for $|\beta|=2$.

We can then write :

$$I \le C(T)\int_{\mathbb{R}^3} \frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}(1+|\bar{p}|)^2[\iint_{\mathbb{R}^3\times S^2} \frac{|\partial_{p'^i} f(\bar{p}')g(\bar{q}')B^{\frac{1}{2}}|B^{\frac{1}{2}}}{q^0} d\bar{q}d\omega]^2 d\bar{p}$$

and using the Cauchy-Schwartz inequality for the integral on $\mathbb{R}^3\times S^2$, we obtain :

$$I \le C(T)\int_{\mathbb{R}^3}(1+|\bar{p}|)^{2d+2}\frac{(1+\bar{p}|)^2}{(p^0)^2}[\iint_{\mathbb{R}^3\times S^2}\frac{|\partial_{p'^i} f(\bar{p}')|^2|g(\bar{q}')|^2 B}{(q^0)^2}d\bar{q}d\omega][\iint_{\mathbb{R}^3\times S^2}Bd\bar{q}d\omega]d\bar{p}$$

We know that $(1+|\bar{p}|)^2 \| B \|_{L^1(\mathbb{R}^3\times S^2)} \in L^\infty(\mathbb{R}^3)$, that B is bounded, so :

$$I \le C(T)\iint_{\mathbb{R}^3\times\mathbb{R}^3}(1+|\bar{p}|)^{2d}|\partial_{p'^i} f(\bar{p}')|^2|g(\bar{q}')|^2\frac{(1+|\bar{p}|)^2}{p^0q^0}\frac{d\bar{p}d\bar{q}}{p^0q^0}$$

We know that $\frac{d\bar{p}d\bar{q}}{p^0q^0}=\frac{d\bar{p}'d\bar{q}'}{p'^0q'^0}$ and that $(1+|\bar{p}|)^2 \le C(T)p^0q^0p'^0q'^0$. So we have :

$$I \le C(T)\iint_{\mathbb{R}^3\times\mathbb{R}^3}(1+|\bar{p}'|)^{2d}(1+|\bar{q}'|)^{2d}|\partial_{p'^i} f(\bar{p}')|^2|g(\bar{q}')|^2\frac{(1+|\bar{p}|)^2}{p^0q^0}\frac{d\bar{p}'d\bar{q}'}{p'^0q'^0}$$
$$\le C(T)(\int_{\mathbb{R}^3}(1+|\bar{p}'|)^{2d+2}|\partial_{p'^i} f(\bar{p}')|^2 d\bar{p}')(\int_{\mathbb{R}^3}(1+|\bar{q}'|)^{2d}|g(\bar{q}')|^2 d\bar{q}')$$

$$I \le C(T)\|f\|^2_{\mathbf{H}^3_d(\mathbb{R}^3)}\|g\|^2_{\mathbf{H}^3_d(\mathbb{R}^3)} \qquad (3.20)$$

We now have for J, since
$\partial^2_{p^jp'^i} f(\bar{p}') = \partial_{p'^i}(\partial_{p^j}\bar{p}'\partial_{p'^j} f(\bar{p}')) = \partial_{p^j}\bar{p}'\partial^2_{p'^ip'^j} f(\bar{p}') + \partial^2_{p'^ip^j}\bar{p}'\partial_{p'^j} f(\bar{p}')$

$$J \le C(T)\int_{\mathbb{R}^3}\frac{(1+|\bar{p}|)^{2d+4}}{(p^0)^2}[\iint_{\mathbb{R}^3\times S^2}\frac{|\partial^2_{p'^ip^j} f(\bar{p}')g(\bar{q}')B|}{q^0}d\bar{q}d\omega]^2 d\bar{p}$$

$$\le C(T)\int_{\mathbb{R}^3}\frac{(1+|\bar{p}|)^{2d+4}}{(p^0)^2}[\iint_{\mathbb{R}^3\times S^2}\frac{|\partial_{p^j}\bar{p}'\partial^2_{p'^ip'^j} f(\bar{p}')+\partial^2_{p'^ip^j}\bar{p}'\partial_{p'^j} f(\bar{p}')\|g(\bar{q}')|B}{q^0}d\bar{q}d\omega]^2 d\bar{p}$$

$$\le C(T)\int_{\mathbb{R}^3}\frac{(1+|\bar{p}|)^{2d+4}}{(p^0)^2}[\iint_{\mathbb{R}^3\times S^2}\frac{(|\partial^2_{p'^ip'^j} f(\bar{p}')|+|\partial_{p'^j} f(\bar{p}')|)|g(\bar{q}')|B^{\frac{1}{2}}}{q^0}B^{\frac{1}{2}}d\bar{q}d\omega]^2 d\bar{p}$$

Using Cauchy-Schwartz inequality we obtain :

$$J \le C(T)\int_{\mathbb{R}^3}d\bar{p}\frac{(1+|\bar{p}|)^{2d+4}}{(p^0)^2}(\iint_{\mathbb{R}^3\times S^2}\frac{(|\partial^2_{p'^ip'^j} f(\bar{p}')+\partial_{p'^j} f(\bar{p}')|)^2|g(\bar{q}')|^2 B}{(q^0)^2}d\bar{q}d\omega)\|B\|_{L^1(\mathbb{R}^3\times S^2)}$$

B is bounded and $(1+|\bar{p}|)^2\|B\|_{L^1(\mathbb{R}^3\times S^2)} \in L^\infty(\mathbb{R}^3\times S^2)$, this gives :

$$J \le C(T)\int_{\mathbb{R}^3} d\bar{p}\,\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}\iint_{\mathbb{R}^3\times S^2}\frac{[|\partial^2_{p'^i p'^j} f(\bar{p}')|^2 + |\partial_{p'^j} f(\bar{p}')|^2]}{(q^0)^2}|g(\bar{q}')|^2\, d\bar{q}d\omega$$

Since $\frac{d\bar{p}d\bar{q}}{p^0q^0} = \frac{d\bar{p}'d\bar{q}'}{p'^0q'^0}$, $(1+|\bar{p}|)^2 \le C(T)p^0q^0p'^0q'^0$, $1+|\bar{p}| \le C(T)(1+|\bar{p}'|)(1+|\bar{q}'|)$

$$J \le C(T)\iint_{\mathbb{R}^3\times\mathbb{R}^3}(1+|\bar{p}|)^{2d}\frac{(1+|\bar{p}|)^2}{p^0q^0}(|\partial^2_{p'^i p'^j} f(\bar{p}')|^2 + |\partial_{p'^j} f(\bar{p}')|^2)|g(\bar{q}')|^2\frac{d\bar{p}d\bar{q}}{p^0q^0}$$

$$\le C(T)\iint_{\mathbb{R}^3\times\mathbb{R}^3}(1+|\bar{p}'|)^{2d}[|\partial^2_{p'^i p'^j} f(\bar{p}')|^2 + |\partial_{p'^j} f(\bar{p}')|^2](1+|\bar{q}'|)^{2d}|g(\bar{q}')|^2\frac{(1+|\bar{p}|)^2}{p^0q^0p'^0q'^0}d\bar{p}'d\bar{q}'$$

$$\le C(T)\left(\int_{\times\mathbb{R}^3}(1+|\bar{p}'|)^{2d}[|\partial^2_{p'^i p'^j} f(\bar{p}')|^2 + |\partial_{p'^j} f(\bar{p}')|^2]d\bar{p}'\right)\left(\int_{\mathbb{R}^3}(1+|\bar{q}'|)^{2d}|g(\bar{q}')|^2\, d\bar{q}'\right)$$

$$\le C(T)\left(\int_{\mathbb{R}^3}(1+|\bar{p}'|)^{2d+4}|\partial^2_{p'^i p'^j} f(\bar{p}')|^2\, d\bar{p}' + \int_{\mathbb{R}^3}(1+|\bar{p}'|)^{2d+2}|\partial_{p'^j} f(\bar{p}')|^2\, d\bar{p}'\right)\left(\int_{\mathbb{R}^3}(1+|\bar{q}'|)^{2d}|g(\bar{q}')|^2\, d\bar{q}'\right)$$

$$J \le C(T)\|f\|^2_{H^3_d(\mathbb{R}^3)}\|g\|^2_{H^3_d(\mathbb{R}^3)} \tag{3.21}$$

By a calculation identical to I we obtain :

$$\begin{cases} K \le C(T)\|f\|^2_{H^3_d(\mathbb{R}^3)}\|g\|^2_{H^3_d(\mathbb{R}^3)} \\ L \le C(T)\|f\|^2_{H^3_d(\mathbb{R}^3)}\|g\|^2_{H^3_d(\mathbb{R}^3)} \end{cases} \tag{3.22}$$

Finally (3.20), (3.21), (3.22) imply for α defined in (3.19)

$$\alpha \le C(T)\|f\|^2_{H^3_d(\mathbb{R}^3)}\|g\|^2_{H^3_d(\mathbb{R}^3)} \tag{3.23}$$

Estimation of β and γ given in (3.19)

$$\begin{cases} \beta = \int_{\mathbb{R}^3} d\bar{p}[\frac{1}{p^0}(1+|\bar{p}|)^{d+2}a^3\int_{\mathbb{R}^3}\frac{d\bar{q}}{q^0}\int_{S^2}d\omega|\partial_{p^j}(f(\bar{p}')\partial_{p^i}\bar{q}'\partial_{q'^i}g(\bar{q}')B)|]^2 \\ \gamma = \int_{\mathbb{R}^3} d\bar{p}[\frac{1}{p^0}(1+|\bar{p}|)^{d+2}a^3\int_{\mathbb{R}^3}\frac{d\bar{q}}{q^0}\int_{S^2}d\omega|\partial_{p^j}(f(\bar{p}')g(\bar{q}')\partial_{p^i}B)|]^2 \end{cases}$$

For β, we treat the term $\partial^2_{q'^i q'^j}g(\bar{q}')$ as we treated the term $\partial^2_{p'^i p'^j}f(\bar{p}')$, and, using the same arguments, we obtain :

$$\beta \le C(T)\|f\|^2_{H^3_d(\mathbb{R}^3)}\|g\|^2_{H^3_d(\mathbb{R}^3)} \tag{3.24}$$

For γ, we use the fact that $(1+|\bar{p}|)^2\left\|\partial^2_{p^i p^j}B\right\|_{\mathbf{L}^1(\mathbb{R}^3\times S^2)} \in \mathbf{L}^\infty(\mathbb{R}^3)$ and $\partial^2_{p^i p^j}B \in \mathbf{L}^\infty(\mathbb{R}^3\times\mathbb{R}^3\times S^2)$.

The same arguments as in the other cases give :

$$\gamma \leq C(T)\|f\|^2_{H^3_d(\mathbb{R}^3)}\|g\|^2_{H^3_d(\mathbb{R}^3)} \tag{3.25}$$

using (3.23), (3.24), (3.25) we obtain for (3.17) :

$$\left\|\frac{(1+|\overline{p}|)^{d+2}\partial^2_{p^ip^j}Q^+(f,g)}{p^0}\right\|^2_{\mathbf{L}^2(\mathbb{R}^3)} \leq C(T)\|f\|^2_{H^3_d(\mathbb{R}^3)}\|g\|^2_{H^3_d(\mathbb{R}^3)} \tag{3.26}$$

(3.26) gives with (3.15) and (3.16) :

$$\left\|(1+|\overline{p}|)^{d+2}\partial^\beta_p(\frac{1}{p^0}Q^+(f,g))\right\|_{L^2(\mathbb{R}^3)} \leq C(T)\|f\|_{H^3_d(\mathbb{R}^3)}\|g\|_{H^3_d(\mathbb{R}^3)},\ |\beta|=2\,.$$

This ends the proof of proposition 3.3

Proposition 3.4 *Let* $f,g\in\mathbf{H}^3_d(\mathbb{R}^3)$. *Then for* β *such that* $|\beta|=3$, *we have :*

$$\left\|(1+|\overline{p}|)^{d+3}\partial^\beta_p(\frac{1}{p^0}Q^+(f,g))\right\|_{\mathbf{L}^2(\mathbb{R}^3)} \leq C(T)\|f\|_{\mathbf{H}^3_d(\mathbb{R}^3)}\|g\|_{\mathbf{H}^3_d(\mathbb{R}^3)}$$

Proof.

If we derive $\partial^2_{p^ip^j}[\frac{1}{p^0}Q^+(f,g)]$ given by (3.13) with respect to p^k, we obtain :

$$(1+|\overline{p}|)^{d+3}\partial^3_{p^ip^jp^k}\left[\frac{1}{p^0}Q^+(f,g)\right] = \frac{(1+|\overline{p}|)^{d+3}\partial^3_{p^ip^jp^k}Q^+(f,g)}{p^0} - \frac{a^2p^k(1+|\overline{p}|)}{(p^0)^2}\frac{(1+|\overline{p}|)^{d+2}\partial^2_{p^ip^j}Q^+(f,g)}{p^0}$$
$$-\frac{a^2p^j(1+|\overline{p}|)}{(p^0)^2}\frac{(1+|\overline{p}|)^{d+2}\partial^2_{p^ip^k}Q^+(f,g)}{p^0} - a^2\left[\frac{(\delta^i_k(p^0)^3-3a^2p^0p^jp^k)}{(p^0)^5}(1+|\overline{p}|)^2\right]\frac{(1+|\overline{p}|)^{d+1}\partial_{p^i}Q^+(f,g)}{p^0}$$
$$-a^2\left[\frac{(\delta^i_k(p^0)^3-3a^2p^0p^ip^k)}{(p^0)^5}(1+|\overline{p}|)^2\right]\frac{(1+|\overline{p}|)^{d+1}\partial_{p^j}Q^+(f,g)}{p^0} - \frac{a^2p^i(1+|\overline{p}|)}{(p^0)^2}\frac{(1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k}Q^+(f,g)}{p^0} - \frac{a^2\delta^i_j(1+|\overline{p}|)}{(p^0)^2}\frac{(1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k}Q^+(f,g)}{p^0}$$
$$+\frac{a^4\delta^i_jp^k(1+|\overline{p}|)^3}{(p^0)^6}\frac{(1+|\overline{p}|)^dQ^+(f,g)}{p^0} + 3\frac{a^4p^ip^j(1+|\overline{p}|)^2}{(p^0)^4}\frac{(1+|\overline{p}|)^{d+1}\partial_{p^k}Q^+(f,g)}{p^0} + 3a^4\left[\frac{(\delta^i_k\delta^j_k(p^0)^5p^ip^j-5a^2(p^0)^3p^k)}{(p^0)^9}(1+|\overline{p}|)^3\right]\frac{(1+|\overline{p}|)^d\partial_{p^k}Q^+(f,g)}{p^0}$$

We remark that :

$$\left|\frac{a^2p^k(1+|\overline{p}|)}{(p^0)^2}\right|\leq C(T)\quad,\quad \left|\frac{a^2\delta^i_j(1+|\overline{p}|)}{(p^0)^2}\right|\leq C(T)\quad,\quad \left|\frac{a^4p^ip^j(1+|\overline{p}|)^2}{(p^0)^4}\right|\leq C(T)\quad,$$

$$\left|\frac{a^4\delta^i_jp^k(1+|\overline{p}|)^3}{(p^0)^6}\right|\leq C(T),$$

$$\left|a^2\left[\frac{(\delta^i_k(p^0)^3-3a^2p^0p^jp^k)}{(p^0)^5}\right|\leq C(T), \text{ and } \left|a^4\left[\frac{(\delta^i_k\delta^j_k(p^0)^5p^ip^j-5a^2(p^0)^3p^k)}{(p^0)^9}\right]\right|\leq C(T).$$

Given the results obtained in Proposition 3.1, Proposition 3.2 and Proposition 3.3, we can say that :

$$\left\|(1+|\overline{p}|)^{d+3}\partial^3_{p^ip^jp^k}(\frac{1}{p^0}Q^+(f,g))\right\|_{\mathbf{L}^2(\mathbb{R}^3)} \leq \left\|\frac{(1+|\overline{p}|)^{d+3}\partial^3_{p^ip^jp^k}Q^+(f,g)}{p^0}\right\|_{\mathbf{L}^2(\mathbb{R}^3)} + C(T)\|f\|_{\mathbf{H}^3(\mathbb{R}^3)}\|g\|_{\mathbf{H}^3(\mathbb{R}^3)}$$

(3.27)

For the first term of (3.27), we observe that :

$$\begin{cases}(1+|\bar{p}|)^2\partial^3_{p^i p^j p^k}B\in\mathbf{L}^\infty(\mathsf{R}^3\times\mathsf{R}^3\times S^2),\ (1+|\bar{p}|)^2\left\|\partial^3_{p^i p^j p^k}B\right\|_{\mathbf{L}^1(\mathsf{R}^3\times S^2)}\in\mathbf{L}^\infty(\mathsf{R}^3)\\ D^\beta_{\bar{p}}\bar{p}',D^\beta_{\bar{p}}\bar{q}' \text{ are bounded},\ (1+|\bar{p}|)^2\le C(T)p^0q^0p'^0q'^0\end{cases}$$

then, with calculations similar to those we did in previous steps, we conclude that :

$$\left\|\frac{(1+|\bar{p}|)^{d+3}\partial^3_{p^i p^j p^k}Q^+(f,g)}{p^0}\right\|_{\mathbf{L}^2(\mathsf{R}^3)}\le C(T)\|f\|_{H^3(\mathsf{R}^3)}\|g\|_{H^3(\mathsf{R}^3)}.$$

This inequality applied to the first term of (3.27) ends the proof of Proposition 3.4

Remark 3.1 *By Proposition 3.1, Proposition 3.2, Proposition 3.3 and Proposition 3.4, we have :*

$$\max_{0\le|\beta|\le3}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}}[\frac{1}{p^0}Q^+(f,g)]\right\|_{\mathsf{L}^2(\mathsf{R}^3)}\le C(T)\|f\|_{H^3(\mathsf{R}^3)}\|g\|_{H^3(\mathsf{R}^3)} \qquad (3.28)$$

Proposition 3.5 *Let* $f,g\in\mathbf{H}^3_d(\mathsf{R}^3)$. *Then we have :*

$$\max_{0\le|\beta|\le3}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial^\beta_{\bar{p}}[\frac{1}{p^0}Q^-(f,g)]\right\|_{\mathbf{L}^2(\mathsf{R}^3)}\le C(T)\|f\|_{\mathbf{H}^3(\mathsf{R}^3)}\|g\|_{\mathbf{H}^3(\mathsf{R}^3)}$$

. Recall that :

$$Q^-(f,g)=\int_{\mathsf{R}^3}\frac{a^3}{q^0}d\bar{q}\int_{S^2}f(\bar{p})g(\bar{q})Bd\omega \qquad (3.29)$$

<u>For $|\beta|=0$</u> : we have

$$\left\|(1+|\bar{p}|)^d\frac{1}{p^0}Q^-(f,g)\right\|^2_{\mathsf{L}^2(\mathsf{R}^3)}=\int_{\mathsf{R}^3}[\frac{1}{p^0}(1+|\bar{p}|)^d a^3\int_{\mathsf{R}^3}\frac{1}{q^0}d\bar{q}\int_{\mathsf{S}^2}f(\bar{p})g(\bar{q})B(\bar{p},\bar{q},\bar{p}',\bar{q}')d\omega]^2d\bar{p}$$

$$\le C(T)\int_{\mathsf{R}^3}\frac{(1+|\bar{p}|)^{2d}}{(p^0)^2}[\iint_{\mathsf{R}^3\times\mathsf{S}^2}\frac{f(\bar{p})g(\bar{q})B^{\frac{1}{2}}}{q^0}B^{\frac{1}{2}}d\bar{q}d\omega]^2d\bar{p}$$

$$\le C(T)\int_{\mathsf{R}^3}\frac{(1+|\bar{p}|)^{2d}}{(p^0)^2}[\iint_{\mathsf{R}^3\times\mathsf{S}^2}\frac{|f(\bar{p})|^2|g(\bar{q})|^2\ B}{(q^0)^2}d\bar{q}d\omega][\iint_{\mathsf{R}^3\times\mathsf{S}^2}Bd\bar{q}d\omega]d\bar{p}$$

Now B is bounded, $\|B\|_{L^1(\mathsf{R}^3\times S^2)}\in L^\infty(\mathsf{R}^3)$, $1+|\bar{p}|\le(1+|\bar{p}|)(1+|\bar{q}|)$ and $p^0q^0\ge1$ we then have :

$$\left\|(1+|\bar{p}|)^d\frac{1}{p^0}Q^-(f,g)\right\|^2_{\mathsf{L}^2(\mathsf{R}^3)}\le C(T)(\int_{\mathsf{R}^3}(1+|\bar{p}|)^{2d}|f(\bar{p})|^2d\bar{p})(\int_{\mathsf{R}^3}(1+|\bar{p}|)^{2d}|g(\bar{q})|^2d\bar{q})$$

$$\left\|(1+|\bar{p}|)^d \frac{1}{p^0} Q^-(f,g)\right\|^2_{L^2(\mathbb{R}^3)} \leq C(T)\|f\|^2_{\mathbf{H}^3_d(\mathbb{R}^3)}\|g\|^2_{\mathbf{H}^3_d(\mathbb{R}^3)} \quad (3.30)$$

For $|\beta|=1$ Let $i=1,2,3$. We have to compute $\left\|(1+|\bar{p}|)^{d+1}\partial_{p^i}(\frac{1}{p^0}Q^-(f,g))\right\|_{L^2(\mathbb{R}^3)}$

We have :

$$\partial_{p^i}(\frac{1}{p^0}Q^-(f,g)) = \frac{1}{p^0}\partial_{p^i}Q^-(f,g) - \frac{a^2 p^i}{(p^0)^3}Q^-(f,g) \quad (3.31)$$

and :

$$(1+|\bar{p}|)^{d+1}\partial_{p^i}(\frac{1}{p^0}Q^+(f,g)) = (1+|\bar{p}|)^{d+1}\frac{1}{p^0}\partial_{p^i}Q^+(f,g) - (1+|\bar{p}|)^{d+1}\frac{a^2 p^i}{(p^0)^3}Q^+(f,g).$$

Then :

$$\left\|(1+|\bar{p}|)^{d+1}\partial_{p^i}(\frac{Q^-(f,g)}{p^0})\right\|_{L^2(\mathbb{R}^3)} \leq \left\|(1+|\bar{p}|)^{d+1}\frac{\partial_{p^i}Q^-(f,g)}{p^0}\right\|_{L^2(\mathbb{R}^3)} + \left\|(1+|\bar{p}|)^{d+1}\frac{a^2p^i}{(p^0)^3}Q^-(f,g)\right\|_{L^2(\mathbb{R}^3)}$$

$$\leq \left\|(1+|\bar{p}|)^{d+1}\frac{\partial_{p^i}Q^-(f,g)}{p^0}\right\|_{L^2(\mathbb{R}^3)} + C(T)\left\|(1+|\bar{p}|)^{d}\frac{Q^-(f,g)}{p^0}\right\|_{L^2(\mathbb{R}^3)}$$

$$\left\|(1+|\bar{p}|)^{d+1}\partial_{p^i}(\frac{Q^-(f,g)}{p^0})\right\|_{L^2(\mathbb{R}^3)} \leq \left\|(1+|\bar{p}|)^{d+1}\frac{\partial_{p^i}Q^-(f,g)}{p^0}\right\|_{L^2(\mathbb{R}^3)} + C(T)\|f\|_{H^3(\mathbb{R}^3)}\|g\|_{H^3(\mathbb{R}^3)} \quad (3.32)$$

Since $1+|\bar{p}| \leq C(T)p^0$ et $a^2p^i \leq C(T)p^0$, and applying (3.30).
Let us compute the first term in (3.22). We have :

$$\left\|\frac{(1+|\bar{p}|)^{d+1}}{p^0}\partial_{p^i}Q^-(f,g)\right\|^2_{L^2(\mathbb{R}^3)} \leq \int_{\mathbb{R}^3}[\frac{(1+|\bar{p}|)^{d+1}}{p^0}a^3\int_{\mathbb{R}^3}\frac{1}{q^0}d\bar{q}\int_{S^2}\partial_{p^i}(f(\bar{p})g(\bar{q})B)d\Omega]^2 d\bar{p}$$

$$\leq C(T)\int_{\mathbb{R}^3}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}[\iint_{\mathbb{R}^3\times S^2}\frac{(|\partial_{p^i}f(\bar{p})g(\bar{q})B| + |f(\bar{p})g(\bar{q})\partial_{p^i}B|}{q^0}d\bar{q}d\omega]^2 d\bar{p}$$

Now we use Cauchy-Schwartz inequality to obtain :

$$\left\|\frac{(1+|\bar{p}|)^{d+1}}{p^0}\partial_{p^i}Q^-(f,g)\right\|^2_{L^2(\mathbb{R}^3)} \leq C(T)\int_{\mathbb{R}^3}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}[(\iint_{\mathbb{R}^3\times S^2}\frac{|\partial_{p^i}f(\bar{p})g(\bar{q})B|}{q^0}d\bar{q}d\omega)^2$$

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$$+ (\iint_{\mathbb{R}^3\times S^2} \frac{|f(\bar{p})g(\bar{q})\partial_{p^i}B|}{q^0} d\bar{q}d\omega)^2]d\bar{p}$$

$$\leq C(T)\int_{\mathbb{R}^3} d\bar{p}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}[(\iint_{\mathbb{R}^3\times S^2} \frac{|\partial_{p^i} f(\bar{p})||g(\bar{q})|B^{\frac{1}{2}}}{q^0} B^{\frac{1}{2}} d\bar{q}d\omega)^2$$

$$+ (\iint_{\mathbb{R}^3\times S^2} \frac{|f(\bar{p})||g(\bar{q})||\partial_{p^i}B|^{\frac{1}{2}}}{q^0} |\partial_{p^i}B|^{\frac{1}{2}} d\bar{q}d\omega)^2]$$

$$\leq C(T)\int_{\mathbb{R}^3} d\bar{p}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}[(\iint_{\mathbb{R}^3\times S^2} \frac{|\partial_{p^i} f(\bar{p})|^2|g(\bar{q})|^2 B}{(q^0)^2} d\bar{q}d\omega)(\iint_{\mathbb{R}^3\times S^2} Bd\bar{q}d\omega)$$

$$+ \int_{\mathbb{R}^3} d\bar{p}\frac{(1+|\bar{p}|)^{2d+2}}{(p^0)^2}(\iint_{\mathbb{R}^3\times S^2} \frac{|f(\bar{p})|^2|g(\bar{q})|^2|\partial_{p^i}B|}{(q^0)^2} d\bar{q}d\omega)(\iint_{\mathbb{R}^3\times S^2} |\partial_{p^i}B|d\bar{q}d\omega)]$$

and since

$\|B\|_{L^1(\mathbb{R}^3\times S^2)} \in L^\infty(\mathbb{R}^3)$, $\|\partial_{p^i}B\|_{L^1(\mathbb{R}^3\times S^2)} \in L^\infty(\mathbb{R}^3)$, B and $\partial_{p^i}B$ are bounded, $(1+|\bar{p}|)\leq C(T)p^0$, and $(1+|\bar{p}|)\leq(1+|\bar{p}|)(1+|\bar{q}|)$, we have :

$$\left\|\frac{(1+|\bar{p}|)^{d+1}}{p^0}\partial_{p^i}Q^-(f,g)\right\|^2_{L^2(\mathbb{R}^3)} \leq C(T)\int_{\mathbb{R}^3}(1+|\bar{p}|)^{2d}[\iint_{\mathbb{R}^3\times S^2}|\partial_{p^i} f(\bar{p})|^2|g(\bar{q})|^2 d\bar{q}d\omega$$

$$+ \iint_{\mathbb{R}^3\times S^2}|f(\bar{p})|^2|g(\bar{q})|^2 d\bar{q}d\omega]d\bar{p}$$

$$\leq C(T)\iint_{\mathbb{R}^3\times\mathbb{R}^3}(1+|\bar{p}|)^{2d}(1+|\bar{q}|)^{2d}(|\partial_{p^i} f(\bar{p})|^2 g(\bar{q})^2+|f(\bar{p})|^2|g(\bar{q})|^2)d\bar{p}d\bar{q}$$

$$\leq C(T)(\int_{\mathbb{R}^3}(1+|\bar{p}|)^{2d+2}|\partial_{p^i} f(\bar{p})|^2 d\bar{p})(\int_{\mathbb{R}^3}(1+|\bar{q}|)^{2d}|g(\bar{q})|^2 d\bar{q})$$

$$+ C(T)(\int_{\mathbb{R}^3}(1+|\bar{p}|)^{2d}|f(\bar{p})|^2 d\bar{p})(\int_{\mathbb{R}^3}(1+|\bar{q}|)^{2d}|g(\bar{q})|^2 d\bar{q})$$

then

$$\left\|\frac{(1+|\bar{p}|)^{d+1}}{p^0}\partial_{p^i}Q^-(f,g)\right\|^2_{L^2(\mathbb{R}^3)} \leq C(T)\|f\|^2_{H^3_d(\mathbb{R}^3)}\|g\|^2_{H^3_d(\mathbb{R}^3)}$$

If we apply this inequality to the first term of (3.32), we obtain :

$$\left\|(1+|\bar{p}|)^{d+1}\partial_{p^i}[\frac{1}{p^0}Q^-(f,g)]\right\|_{L^2(\mathbb{R}^3)} \leq C(T)\|f\|_{H^3_d(\mathbb{R}^3)}\|g\|_{H^3_d(\mathbb{R}^3)} \quad (3.33)$$

which give the required inequality for $|\beta|=1$.

As for $Q^+(f,g)$ we establish, using the same method that for $|\beta|=2$ and $|\beta|=3$, we have :

$$\begin{cases} \left\| (1+|\overline{p}|)^{d+2}\partial^2_{p^i p^j}[\frac{1}{p^0}Q^-(f,g)] \right\|_{L^2(\mathbb{R}^3)} \leq C(T)\|f\|_{H^3_d(\mathbb{R}^3)}\|g\|_{H^3_d(\mathbb{R}^3)} \\ \left\| (1+|\overline{p}|)^{d+3}\partial^3_{p^i p^j p^k}[\frac{1}{p^0}Q^-(f,g)] \right\|_{L^2(\mathbb{R}^3)} \leq C(T)\|f\|_{H^3_d(\mathbb{R}^3)}\|g\|_{H^3_d(\mathbb{R}^3)} \end{cases} \quad (3.34)$$

(3.30), (3.33) and (3.34) end the proof of Proposition 3.5

Proposition 3.6 *(The fundamental inequality)*

Let $f, g \in \mathbf{H}^3_d(\mathbb{R}^3)$, where $d > \frac{5}{2}$. Then $\frac{1}{p^0}Q(f,g) \in \mathbf{H}^3_d(\mathbb{R}^3)$ and :

$$\left\| \frac{1}{p^0}Q(f,g) \right\|_{\mathbf{H}^3_d(\mathbb{R}^3)} \leq C(T)\|f\|_{H^3_d(\mathbb{R}^3)}\|g\|_{H^3_d(\mathbb{R}^3)}$$

. Write : $Q(f,g) = Q^+(f,g) - Q^-(f,g)$ and apply Remark 3.1 and Proposition 3.5

The Existence Theorem

4.1 The Linearized Boltzmann Equation

We will prove that the Boltzmann equation which writes, since f depend only on t and $\overline{p}$:

$$\begin{cases} \frac{\partial f}{\partial t} + \frac{\mathcal{P}^i}{p^0}\frac{\partial f}{\partial p^i} = \frac{1}{p^0}Q(f,f) & (4.1) \\ \mathcal{P}^i = -\Gamma^i_{\lambda\mu}p^\lambda p^\mu + a^3 p^\beta F^i_\beta \int_{\mathbb{R}^3} f d\overline{p} & (4.2) \\ f(0,\overline{p}) = f_0(\overline{p}) & (4.3) \end{cases}$$

where $f_0 \in H^3_{d,r}(\mathbb{R}^3)$, has a unique solution in $H^3_d(\mathbb{R}^3)$ weak star å.

We will need that $\partial_{\overline{p}}\mathrm{P}^i$ which depends on f, be bounded.

We suppose that : $f_0(N\overline{p}) = f_0(\overline{p})$, $\forall N \in SO_3$. Then f satisfies $f(N\overline{p}) = f(\overline{p})$, see [3].

Let us set :

$$\tilde{\mathrm{P}}^i = -\Gamma^i_{\lambda\mu}p^\lambda p^\mu + a^3 p^\beta F^i_\beta \int_{\mathbb{R}^3} \tilde{f} d\overline{p} \quad (4.1)$$

Lemma 4.1 *Let i,j,k=1,2,3. Then :*

F^{0i}, $\partial_{p^i}(\frac{\tilde{\mathrm{P}}^i}{p^0})$, $\partial_{p^j}(\frac{\tilde{\mathrm{P}}^i}{p^0})$ are bounded and $\partial^2_{p^i p^j}(\frac{\tilde{\mathrm{P}}^i}{p^0}) = 0$

.

1. For F^{0i}

Since $u^i = 0$, $i = 1,2,3$, equation (2.7) in which we put $\beta = i$ implies :

$$\nabla_\alpha F^{\alpha i} = \int_{\mathbb{R}^3} \frac{p^i f |g|^{\frac{1}{2}} d\bar{p}}{p^0} \tag{4.2}$$

We prove that :

$$\int_{\mathbb{R}^3} \frac{p^i f |g|^{\frac{1}{2}} d\bar{p}}{p^0} = 0 \tag{4.3}$$

Take $N = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO_3$; put $\bar{p} = N\bar{q} = (-q^1, -q^2, q^3)$,

then since $f(N\bar{q}) = f(\bar{q})$, $p^0(N\bar{q}) = q^0(\bar{q})$:

$$\int_{\mathbb{R}^3} \frac{p^1 f(\bar{p}) |g|^{\frac{1}{2}}}{p^0} d\bar{p} = -\int_{\mathbb{R}^3} \frac{q^1 f(N\bar{q}) |g|^{\frac{1}{2}} d\bar{p}}{p^0(N\bar{q})} = -\int_{\mathbb{R}^3} \frac{q^1 f(\bar{q}) |g|^{\frac{1}{2}} d\bar{q}}{q^0} = -\int_{\mathbb{R}^3} \frac{p^1 f(\bar{p}) |g|^{\frac{1}{2}} d\bar{p}}{p^0}$$

Hence $\int_{\mathbb{R}^3} \frac{p^1 f(\bar{p}) |g|^{\frac{1}{2}} d\bar{p}}{p^0} = 0$.

Taking $N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $N = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

we prove that $\int_{\mathbb{R}^3} \frac{p^2 f(\bar{p}) |g|^{\frac{1}{2}} d\bar{p}}{p^0} = \int_{\mathbb{R}^3} \frac{p^3 f(\bar{p}) |g|^{\frac{1}{2}} d\bar{p}}{p^0} = 0$

So we have (4.3). Then applying (4.3), (4.2) shows that $F^{\alpha i}$ satisfies :

$$\nabla_\alpha F^{\alpha i} = 0 \tag{4.4}$$

since $\Gamma^\lambda_{\alpha\beta} = \Gamma^\lambda_{\beta\alpha}$, $F^{\alpha\beta} = -F^{\beta\alpha}$ and $\Gamma^i_{i0} = \frac{\dot{a}}{a}$ (see (2.3)), we have :

$$\partial_\alpha F^{\alpha i} + \Gamma^\alpha_{\alpha\lambda} F^{\lambda i} + \Gamma^i_{\alpha\lambda} F^{\alpha\lambda} = \partial_0 F^{0i} + 3\frac{\dot{a}}{a} F^{0i} = 0$$

Hence :

$$F^{0i} = \left(\frac{a(0)}{a(t)}\right)^3 F^{0i}(0)$$

So, by (2.2) :

$$|F^{0i}| = \left(\frac{a(0)}{a(t)}\right)^3 |F^{0i}(0)| \le |F^{0i}(0)| e^{Ct} \le |F^{0i}(0)| C(T)$$

2. For $\partial_{p^i}(\frac{\widetilde{\mathrm{P}}^i}{p^0})$ we have :

$$\frac{\widetilde{\mathrm{P}}^i}{p^0} = -\frac{\Gamma^i_{\lambda\mu} p^\lambda p^\mu + a^3 p^\beta F^i_\beta \int_{\mathbb{R}^3} f d\overline{p}}{p^0} = -2\Gamma^i_{i0} p^i + (a^3 F^i_0 + a^3 \frac{p^j}{p^0} F^i_j)\int_{\mathbb{R}^3} f d\overline{p}$$

But $\mathbb{R}^4$ is simply connected; so there exists a potential (A_λ) such that :

$F_{\lambda\mu} = \nabla_\lambda A_\mu - \nabla_\mu A_\lambda$

Then : $F_{ij} = \nabla_i A_j - \nabla_j A_i = 0$ since A depends only on t.

And $F^i_j = g^{i\lambda} F_{i\lambda} = g^{ii} F_{ji} = 0$. Hence :

$$\frac{\widetilde{\mathrm{P}}^i}{p^0} = -2\Gamma^i_{i0} p^i + a^3 F^i_0 \int_{\mathbb{R}^3} f d\overline{p}$$

$$|\partial_{p^i}(\frac{\widetilde{\mathrm{P}}^i}{p^0})| = |-2\Gamma^i_{i0}| = 2|\frac{\dot{a}}{a}| \le 2CT \quad (\text{since } |\frac{\dot{a}}{a}| \le C)$$

3. For $\partial_{p^j}(\frac{\widetilde{\mathrm{P}}^i}{p^0})$

$$|\partial_{p^j}(\frac{\widetilde{\mathrm{P}}^i}{p^0})| = |2\Gamma^i_{i0}\delta^i_j| \le 2|\Gamma^i_{i0}| \le 2CT$$

4. For $\partial^2_{p^i p^j}(\frac{\widetilde{\mathrm{P}}^i}{p^0})$, since $\partial_{p^j}(\frac{\widetilde{\mathrm{P}}^i}{p^0})$ does not depend on p^k, $\partial^2_{p^i p^j}(\frac{\widetilde{\mathrm{P}}^i}{p^0}) = 0$.

This ends the proof of Lemma 4.1

Let $\widetilde{f} \in H^3_{d,r}(\mathbb{R}^3)$. The linearized Boltzmann equation in f is the equation :

$$\frac{\partial f}{\partial t} + \frac{\widetilde{\mathrm{P}}^i}{p^0}\frac{\partial f}{\partial p^i} = \frac{1}{p^0} Q(\widetilde{f}, \widetilde{f}) \tag{4.5}$$

where $\widetilde{\mathrm{P}}^i(\widetilde{f}) = -\Gamma^i_{\lambda\mu} p^\lambda p^\mu + a^3 p^\beta F^i_\beta \int_{\mathbb{R}^3} \widetilde{f} d\overline{p}$; with $f(0, \overline{p}) = f_0(\overline{p})$.

We will prove that (4.5) has a unique solution in $\mathbf{H}^3_d(\mathbb{R}^3)* -$ weak star. We use the Faedo-Galerking method by constructing a sequence. We prove that the sequence is bounded in $\mathbf{H}^3_d(\mathbb{R}^3)$.

4.2 *Construction of the Sequence*

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Let (w_k) be a Hilbertian basis on $\mathbf{H}_d^3(\mathsf{R}^3)$. We look for a solution f of (4.5) as limit of a sequence (f^N) where :

$$f^N = \sum_{k=1}^{N} c_k(t) w_k, N \in \mathsf{N}^* \tag{4.6}$$

where the coefficients $c_k(t)$ which are supposed to be derivables, are the solutions of the N first differential equations of the system :

$$(\partial_t f^N / w_k) + (\frac{\tilde{\mathsf{P}}^i}{p^0} \partial_{p^i} f^N / w_k) = (\frac{1}{p^0} Q(\tilde{f}, \tilde{f}) / w_k), k \in \mathsf{N} \tag{4.7}$$

where $(/)$ is the scalar product in $\mathbf{H}_d^3(\mathsf{R}^3)$; the initial data are $c_k(0) = (f_0 / w_k)$.

Let us multiply the N first differential equation by c_k. Taking the sum from 1 to N, we obtain :

$$(\partial_t f^N / f^N) + (\frac{\tilde{\mathsf{P}}^i}{p^0} \partial_{p^i} f^N / f^N) = (\frac{1}{p^0} Q(\tilde{f}, \tilde{f}) / f^N) \tag{4.8}$$

From (4.7), we have $(\partial_t f^N + \frac{\tilde{\mathsf{P}}^i}{p^0} \partial_{p^i} f^N - \frac{1}{p^0} Q(\tilde{f}, \tilde{f}) / w_k) = 0$, $\forall k \in \mathsf{N}$

Hence $V = \partial_t f^N + \frac{\tilde{\mathsf{P}}^i}{p^0} \partial_{p^i} f^N - \frac{1}{p^0} Q(\tilde{f}, \tilde{f})$ is orthogonal in $\mathbf{H}_d^3(\mathsf{R}^3)$ to the subspace generated by (w_k), $k \in \mathsf{N}^*$. But this subspace is dense in $\mathbf{H}_d^3(\mathsf{R}^3)$; hence V is orthogonal to the whole space $\mathbf{H}_d^3(\mathsf{R}^3)$, then to V itself. So $V = 0$. We then have :

$$\partial_t f^N + \frac{\tilde{\mathsf{P}}^i}{p^0} \partial_{p^i} f^N = \frac{1}{p^0} Q(\tilde{f}, \tilde{f}) \tag{4.9}$$

So f^N is solution of (4.5), with the initial data :

$$f^N(0) = \sum_{k=1}^{N} c_k(0) w_k = \sum_{k=1}^{N} (f_0 / w_k) w_k$$

4.3 *Estimations*

We will prove that : $\forall \beta \in \mathsf{N}^3, |\beta| \leq 3$

$$|((1+|\bar{p}|)^{d+|\beta|} \partial_{\bar{p}}^{\beta} (\frac{\tilde{\mathsf{P}}^i}{p^0} \frac{\partial f^N}{\partial p^i}) / (1+|\bar{p}|)^{d+|\beta|} \partial_{\bar{p}}^{\beta} f^N)| \leq C(\tilde{f}, F)(\sum_{|\alpha| \leq |\beta|} \left\| (1+|\bar{p}|)^{d+|\alpha|} \partial_{\bar{p}}^{\alpha} f^N \right\|_{L^2(\mathsf{R}^3)}) \left\| (1+|\bar{p}|)^{d+|\beta|} \partial_{\bar{p}}^{\beta} f^N \right\|_{L^2(\mathsf{R}^3)}. \tag{4.10}$$

where $(/)$ is the scalar product of $L^2(\mathsf{R}^3)$.

***Lemma 4.2** we have :*

$$|((1+|\overline{p}|)^d(\frac{\tilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i})/(1+|\overline{p}|)^d f^N)| \leq C(\tilde{f},F)\left\|(1+|\overline{p}|)^d f^N\right\|^2_{L^2(\mathbb{R}^3)} \quad (4.11)$$

Proof.
We have :

$$\partial_{p^i}((1+|\overline{p}|)^d f^N) = \partial_{p^i}((1+|\overline{p}|)^d) f^N + (1+|\overline{p}|)^d \partial_{p^i} f^N$$

Multiplying by $\frac{\tilde{P}^i}{p^0}$,

$$(1+|\overline{p}|)^d \frac{\tilde{P}^i}{p^0}\partial^i_p f^N = \frac{\tilde{P}^i}{p^0}\partial_{p^i}((1+|\overline{p}|)^d f^N) - \frac{\tilde{P}^i}{p^0}\partial_{p^i}((1+|\overline{p}|)^d) f^N$$

Taking the scalar product with $(1+|\overline{p}|)^d f^N$:

$$|((1+|\overline{p}|)^d(\frac{\tilde{P}^i}{p^0}\partial_{p^i} f^N)/(1+|\overline{p}|)^d f^N)| \leq |(\frac{\tilde{P}^i}{p^0}\partial_{p^i}((1+|\overline{p}|)^d f^N)/(1+|\overline{p}|)^d f^N)| + |(\frac{\tilde{P}^i}{p^0}\partial_{p^i}((1+|\overline{p}|)^d) f^N/(1+|\overline{p}|)^d f^N)| \quad (4.12)$$

For the first term of the right of (4.12) :

$$\begin{aligned}
(\frac{\tilde{P}^i}{p^0}\partial_{p^i}((1+|\overline{p}|)^d f^N)/(1+|\overline{p}|)^d f^N) &= (\partial_{p^i}[\frac{\tilde{P}^i}{p^0}(1+|\overline{p}|)^d f^N]/(1+|\overline{p}|)^d f^N) \\
&- (\partial_{p^i}[\frac{\tilde{P}^i}{p^0}](1+|\overline{p}|)^d f^N/(1+|\overline{p}|)^d f^N) \\
&= -(\frac{\tilde{P}^i}{p^0}(1+|\overline{p}|)^d f^N/\partial_{p^i}[(1+|\overline{p}|)^d f^N]) \\
&- (\partial_{p^i}[\frac{\tilde{P}^i}{p^0}](1+|\overline{p}|)^d f^N/(1+|\overline{p}|)^d f^N) \\
&= -((1+|\overline{p}|)^d f^N/\frac{\tilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^d f^N]) \\
&- (\partial_{p^i}[\frac{\tilde{P}^i}{p^0}](1+|\overline{p}|)^d f^N/(1+|\overline{p}|)^d f^N)
\end{aligned}$$

From there :

$$2(\frac{\tilde{P}^i}{p^0}\partial_{p^i}((1+|\overline{p}|)^d f^N)/(1+|\overline{p}|)^d f^N) = -(\partial_{p^i}[\frac{\tilde{P}^i}{p^0}](1+|\overline{p}|)^d f^N/(1+|\overline{p}|)^d f^N) \quad (4.13)$$

then, since $\partial_{p^i}(\frac{\tilde{P}}{p^0})$ is bounded, we have for the first term of (4.12)

$$|(\frac{\tilde{P}^i}{p^0}\partial_{p^i}((1+|\overline{p}|)^d f^N)/(1+|\overline{p}|)^d f^N)| \leq C(\tilde{f},F)\left\|(1+|\overline{p}|)^d f^N\right\|^2_{L^2(\mathbb{R}^3)} \quad (4.14)$$

For the second term of the right of (4.12) we have :

$$\partial_{p^i}(1+|\overline{p}|)^d = \frac{d(1+|\overline{p}|)^{d-1}p^i}{|\overline{p}|} = \frac{dp^i(1+|\overline{p}|)^d}{|\overline{p}|(1+|\overline{p}|)}. \quad (4.15)$$

and :

$$|(\frac{\widetilde{P}^i}{p^0}\partial_{p^i}((1+|\overline{p}|)^d)f^N/(1+|\overline{p}|)^d f^N)| = |(\frac{\widetilde{P}^i}{p^0}\frac{p^i d}{|\overline{p}|(1+|\overline{p}|)}((1+|\overline{p}|)^d)f^N/(1+|\overline{p}|)^d f^N)|$$

$$|(\frac{\widetilde{P}^i}{p^0}\partial_{p^i}((1+|\overline{p}|)^d)f^N/(1+|\overline{p}|)^d f^N)| \le C(\widetilde{f},F)\left\|(1+|\overline{p}|)f^N\right\|^2_{L^2(\mathbb{R}^3)} \quad (4.16)$$

Since $|\frac{\widetilde{P}^i}{p^0}\frac{p^i}{|\overline{p}|(1+|\overline{p}|)}| \le C(\widetilde{f},F)$. Then (4.12) give, using (4.14) and (4.16)

$$|((1+|\overline{p}|)^d(\frac{\widetilde{P}^i}{p^0}\partial^i_p f^N)/(1+|\overline{p}|)^d f^N)| \le C(\widetilde{f},F)\left\|(1+|\overline{p}|)^d f^N\right\|^2_{L^2(\mathbb{R}^3)}$$

that is (4.11).

***Lemma 4.3** we have for* $|\beta|=1$*:*

$$|((1+|\overline{p}|)^{d+1}\partial_{p^j}(\frac{\widetilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i})/(1+|\overline{p}|)^{d+1}\partial^\beta_{\overline{p}} f^N)| \le C(\widetilde{f},F)\{\sum_{i=1}^{3}\left\|(1+|\overline{p}|)^{d+1}\partial_{p^i}f^N\right\|_{L^2(\mathbb{R}^3)}\}\left\|(1+|\overline{p}|)^{d+1}\partial^\beta_{\overline{p}}f^N\right\|_{L^2(\mathbb{R}^3)}$$
(4.17)

. We have to estimate $((1+|\overline{p}|)^{d+1}\partial_{p^j}(\frac{\widetilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i})/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N),\ j=1,2,3.$

We have :

$$\partial_{p^j}(\frac{\widetilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i}) = \partial_{p^j}(\frac{\widetilde{P}^i}{p^0})\partial_{p^i}f^N + \frac{\widetilde{P}^i}{p^0}\partial^2_{p^j p^i}f^N$$

So :

$$|((1+|\overline{p}|)^{d+1}\partial_{p^j}(\frac{\widetilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i})/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N)| \quad (4.18)$$

$$\le |((1+|\overline{p}|)^{d+1}\partial_{p^j}(\frac{\widetilde{P}^i}{p^0})\partial_{p^i}f^N/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N)| + |((1+|\overline{p}|)^{d+1}(\frac{\widetilde{P}^i}{p^0})\partial^2_{p^j p^i}f^N/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N)|$$

But $\partial_{p^j}(\frac{\widetilde{P}^i}{p^0})$ is bounded, so we have, for the first term of (4.18) :

$$|((1+|\overline{p}|)^{d+1}\partial_{p^j}(\frac{\tilde{P}^i}{p^0})\partial_{p^i}f^N/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N)|\leq C(\tilde{f},F)\sum_{i=1}^{3}\left\|(1+|\overline{p}|)^{d+1}\partial_{p^i}f^N\right\|_{L^2(\mathbb{R}^3)}\left\|(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N\right\|_{L^2(\mathbb{R}^3)} \quad (4.19)$$

Now we have : $\partial_{p^i}[(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N]=(1+|\overline{p}|)^{d+1}\partial^2_{p^ip^j}f^N+\partial_{p^i}[(1+|\overline{p}|)^{d+1}]\partial_{p^j}f^N$

So we have for the second term of (4.18) :

$$|((1+|\overline{p}|)^{d+1}(\frac{\tilde{P}^i}{p^0})\partial^2_{p^jp^i}f^N/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N)| \quad (4.20)$$

$$\leq|(\frac{\tilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N]/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N)|+|(\frac{\tilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^{d+1}]\partial_{p^j}f^N/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N)|$$

But $|\frac{\tilde{P}^i}{p^0}\partial_{p^i}(1+|\overline{p}|)^{d+1}|=|\frac{\tilde{P}^i}{p^0}(d+1)(1+|\overline{p}|)^d\frac{p^i}{|\overline{p}|}|=|\frac{\tilde{P}^i}{p^0}\frac{(d+1)}{(1+|\overline{p}|)}\frac{p^i}{|\overline{p}|}(1+|\overline{p}|)^{d+1}|$, and $|\frac{\tilde{P}^i}{p^0}\frac{(d+1)}{(1+|\overline{p}|)}\frac{p^i}{|\overline{p}|}|$ is bounded, then, for the second term of (4.20), we have :

$$|(\frac{\tilde{P}^i}{p^0}\partial_{p^i}(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N)|\leq C(\tilde{f},F)\left\|(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N\right\|^2_{L^2(\mathbb{R}^3)} \quad (4.21)$$

Now we proced for $|(\frac{\tilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N]/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N)|$ as we did in (4.13) and we obtain, similarly to (4.13) :

$$2(\frac{\tilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N]/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N)=-(\partial_{p^i}(\frac{\tilde{P}^i}{p^0})(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N/(1+|\overline{p}|)^d\partial_{p^j}f^N)$$

Since $\partial_{p^i}(\frac{\tilde{P}^i}{p^0})$ is bounded, we obtain, for the first term of (4.20)

$$|(\frac{\tilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N]/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N)|\leq C(\tilde{f},F)\left\|(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N\right\|^2_{L^2(\mathbb{R}^3)} \quad (4.22)$$

(4.21) and (4.22) give, using (4.20) :

$$|((1+|\overline{p}|)^{d+1}(\frac{\tilde{P}^i}{p^0})\partial^2_{p^ip^j}f^N]/(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N)|\leq C(\tilde{f},F)\left\|(1+|\overline{p}|)^{d+1}\partial_{p^j}f^N\right\|^2_{L^2(\mathbb{R}^3)}$$

This inequality added to (4.19) give (4.17).

Lemma 4.4 *we have :*

$$|((1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k}(\frac{\tilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i})/(1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k}f^N)|\leq \quad (4.23)$$

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$$C(\tilde{f},F)\sum_{i,j,k}[\left\|(1+|\overline{p}|)^{d+2}\partial^2_{p^ip^k} f^N\right\|_{L^2(\mathbb{R}^3)}+\left\|(1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k} f^N\right\|_{L^2(\mathbb{R}^3)}]\left\|(1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k} f^N\right\|_{L^2(\mathbb{R}^3)}$$

. We have to estimate $((1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k}(\frac{\tilde{\mathrm{P}}^i}{p^0}\frac{\partial f^N}{\partial p^i})/(1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k} f^N)$.

We have :

$$\begin{cases}\partial_{p^j}(\frac{\tilde{\mathrm{P}}^i}{p^0}\frac{\partial f^N}{\partial p^i})=\partial_{p^j}(\frac{\tilde{\mathrm{P}}^i}{p^0})\partial_{p^i} f^N+\frac{\tilde{\mathrm{P}}^i}{p^0}\partial^2_{p^jp^i} f^N\\ \partial^2_{p^jp^k}(\frac{\tilde{\mathrm{P}}^i}{p^0}\frac{\partial f^N}{\partial p^i})=\partial^2_{p^jp^k}(\frac{\tilde{\mathrm{P}}^i}{p^0})\partial_{p^i} f^N+\partial_{p^j}[\frac{\tilde{\mathrm{P}}^i}{p^0}]\partial^2_{p^ip^k} f^N+\partial_{p^k}[\frac{\tilde{\mathrm{P}}^i}{p^0}]\partial^2_{p^jp^i} f^N+\frac{\tilde{\mathrm{P}}^i}{p^0}\partial^3_{p^kp^jp^i} f^N\end{cases}$$

Since $\partial^2_{p^jp^k}(\frac{\tilde{\mathrm{P}}^i}{p^0})=0$, we have :

$$\partial^2_{p^jp^k}(\frac{\tilde{\mathrm{P}}^i}{p^0}\frac{\partial f^N}{\partial p^i})=\partial_{p^j}(\frac{\tilde{\mathrm{P}}^i}{p^0})\partial^2_{p^ip^k} f^N+\partial_{p^k}(\frac{\tilde{\mathrm{P}}^i}{p^0})\partial^2_{p^jp^i} f^N+\frac{\tilde{\mathrm{P}}^i}{p^0}\partial^3_{p^kp^jp^i} f^N$$

Now, since $\partial_{p^j}(\frac{\tilde{\mathrm{P}}^i}{p^0})$ and $\partial_{p^k}(\frac{\tilde{\mathrm{P}}^i}{p^0})$ are bounded, we have :

$$\begin{cases}|((1+|\overline{p}|)^{d+2}\partial_{p^j}(\frac{\tilde{\mathrm{P}}^i}{p^0})\partial^2_{p^ip^k} f^N]/(1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k} f^N)|\leq C(\tilde{f},F)\left\|(1+|\overline{p}|)^{d+2}\partial^2_{p^ip^k} f^N\right\|_{L^2(\mathbb{R}^3)}\times\left\|(1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k} f^N\right\|_{L^2(\mathbb{R}^3)}\\ |((1+|\overline{p}|)^{d+2}\partial_{p^k}(\frac{\tilde{\mathrm{P}}^i}{p^0})\partial^2_{p^ip^j} f^N]/(1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k} f^N)|\leq C(\tilde{f},F)\left\|(1+|\overline{p}|)^{d+2}\partial^2_{p^ip^j} f^N\right\|_{L^2(\mathbb{R}^3)}\times\left\|(1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k} f^N\right\|_{L^2(\mathbb{R}^3)}\end{cases} \tag{4.24}$$

Now about $\frac{\tilde{\mathrm{P}}^i}{p^0}\partial^3_{p^kp^jp^i} f^N$, we remark that :

$$\partial_{p^i}[(1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k} f^N]=(1+|\overline{p}|)^{d+2}\partial^3_{p^kp^jp^i} f^N+\partial_{p^i}[(1+|\overline{p}|)^{d+2}]\partial^2_{p^jp^k} f^N$$

So :

$$(1+|\overline{p}|)^{d+2}\partial^3_{p^kp^jp^i} f^N=\partial_{p^i}[(1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k} f^N]-\partial_{p^i}[(1+|\overline{p}|)^{d+2}]\partial^2_{p^jp^k} f^N$$

and :

$$\frac{\tilde{\mathrm{P}}^i}{p^0}(1+|\overline{p}|)^{d+2}\partial^3_{p^kp^jp^i} f^N=\frac{\tilde{\mathrm{P}}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k} f^N]-\frac{\tilde{\mathrm{P}}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^{d+2}]\partial^2_{p^jp^k} f^N \tag{4.25}$$

For the first term of (4.25) we proceed as we did to obtain (4.13) and similarly :

$$2(\frac{\tilde{\mathrm{P}}^i}{p^0}\partial_{p^i}((1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k} f^N)/(1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k} f^N)=-(\partial_{p^i}(\frac{\tilde{\mathrm{P}}^i}{p^0})(1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k} f^N/(1+|\overline{p}|)^{d+2}\partial^2_{p^jp^k} f^N)$$

So, since $\partial_{p^i}(\frac{\tilde{\mathrm{P}}^i}{p^0})$ is bounded, we obtain :

$$|(\frac{\tilde{\mathsf{P}}^i}{p^0}\partial_{p^i}((1+|\bar{p}|)^{d+2}\partial^2_{p^j p^k} f^N)\,/\,(1+|\bar{p}|)^{d+2}\partial^2_{p^j p^k} f^N)| \le C(\tilde{f},F)\left\|(1+|\bar{p}|)^{d+2}\partial^2_{p^j p^k} f^N\right\|^2_{L^2(\mathbb{R}^3)} \quad (4.26)$$

In the second term of (4.25), we have :

$$\frac{\tilde{\mathsf{P}}^i}{p^0}\partial_{p^i}((1+|\bar{p}|)^{d+2}) = \frac{\tilde{\mathsf{P}}^i}{p^0}(d+2)p^i\frac{(1+|\bar{p}|)^{d+1}}{|\bar{p}|} = \frac{\tilde{\mathsf{P}}^i}{p^0}\frac{(d+2)p^i(1+|\bar{p}|)^{d+2}}{|\bar{p}|(1+|\bar{p}|)}\ ; \text{ but } \frac{\tilde{\mathsf{P}}^i}{p^0}\frac{(d+2)p^i}{|\bar{p}|(1+|\bar{p}|)}$$

is bounded, so, for the second term of (4.25), we obtain :

$$|(\frac{\tilde{\mathsf{P}}^i}{p^0}\partial_{p^i}((1+|\bar{p}|)^{d+2})\partial^2_{p^j p^k} f^N\,/\,(1+|\bar{p}|)^{d+2}\partial^2_{p^j p^k} f^N)| \le C(\tilde{f},F)\left\|(1+|\bar{p}|)^{d+2}\partial^2_{p^j p^k} f^N\right\|^2_{L^2(\mathbb{R}^3)} \quad (4.27)$$

(4.26) and (4.27) give, using (4.25) :

$$|(\frac{\tilde{\mathsf{P}}^i}{p^0}((1+|\bar{p}|)^{d+2})\partial^3_{p^i p^j p^k} f^N\,/\,(1+|\bar{p}|)^{d+2}\partial^2_{p^j p^k} f^N)| \le C(\tilde{f},F)\left\|(1+|\bar{p}|)^{d+2}\partial^2_{p^j p^k} f^N\right\|^2_{L^2(\mathbb{R}^3)}$$

This last formula and (4.24) give (4.23).

Lemma 4.5 *we have :*

$$|((1+|\bar{p}|)^{d+3}\partial^3_{p^l p^j p^k}(\frac{\tilde{\mathsf{P}}^i}{p^0}\frac{\partial f^N}{\partial p^i})\,/\,(1+|\bar{p}|)^{d+3}\partial^3_{p^l p^j p^k} f^N)| \le \quad (4.28)$$

$$C(\tilde{f},F)\{[\sum_{i=1}^{3}\left\|(1+|\bar{p}|)^{d+3}\partial^3_{p^l p^i p^k} f^N\right\|_{L^2(\mathbb{R}^3)} \times \left\|(1+|\bar{p}|)^{d+3}\partial^3_{p^l p^j p^k} f^N\right\|_{L^2(\mathbb{R}^3)}] + \left\|(1+|\bar{p}|)^{d+3}\partial^3_{p^l p^j p^k} f^N\right\|^2_{L^2(\mathbb{R}^3)}\}$$

. We have to estimate $((1+|\bar{p}|)^{d+3}\partial^3_{p^l p^k p^j}(\frac{\tilde{\mathsf{P}}^i}{p^0}\frac{\partial f^N}{\partial p^i})\,/\,(1+|\bar{p}|)^{d+3}\partial^3_{p^l p^k p^j} f^N)$

Recall that : $\partial^2_{p^j p^k}(\frac{\tilde{\mathsf{P}}^i}{p^0}) = \partial_{p^i}(\frac{\tilde{\mathsf{P}}^i}{p^0})\partial^2_{p^i p^k} f^N + \partial_{p^k}(\frac{\tilde{\mathsf{P}}^i}{p^0})\partial^2_{p^i p^j} f^N + (\frac{\tilde{\mathsf{P}}^i}{p^0})\partial^3_{p^k p^j p^i} f^N$

then, since $\partial^2_{p^l p^j}(\frac{\tilde{\mathsf{P}}^i}{p^0}) = 0$:

$$\partial^3_{p^l p^k p^j} f^N = \partial_{p^i}(\frac{\tilde{\mathsf{P}}^i}{p^0})\partial^3_{p^l p^i p^k} f^N + \partial_{p^k}(\frac{\tilde{\mathsf{P}}^i}{p^0})\partial^3_{p^l p^i p^j} f^N + \partial_{p^l}(\frac{\tilde{\mathsf{P}}^i}{p^0})\partial^3_{p^k p^i p^j} f^N + (\frac{\tilde{\mathsf{P}}^i}{p^0})\partial^4_{p^l p^k p^j p^i} f^N \quad (4.29)$$

We use the fact that $\partial_{p^i}(\frac{\tilde{\mathsf{P}}^i}{p^0})$ is bounded to obtain :

$$|(1+|\bar{p}|)^{d+3}\partial_{p^i}(\frac{\tilde{\mathsf{P}}^i}{p^0})\partial^3_{p^l p^i p^k} f^N)/(1+|\bar{p}|)^{d+3}\partial^3_{p^l p^i p^k} f^N| \le C(\tilde{f},F)\left\|(1+|\bar{p}|)^{d+3}\partial^3_{p^l p^i p^k} f^N\right\|_{L^2(\mathbb{R}^3)} \times \left\|(1+|\bar{p}|)^{d+3}\partial^3_{p^l p^j p^k} f^N\right\|_{L^2(\mathbb{R}^3)}$$

(4.30)

We then have only to estimate the last term of (4.29). We have :

$$\partial_{p^i}[(1+|\bar{p}|)^{d+3}\partial^3_{p^l p^k p^j} f^N] = \partial_{p^i}[(1+|\bar{p}|)^{d+3}]\partial^3_{p^l p^k p^j} f^N + (1+|\bar{p}|)^{d+3}\partial^4_{p^l p^k p^j p^i} f^N$$

We then have :

$$\frac{\widetilde{P}^i}{p^0}(1+|\overline{p}|)^{d+3}\partial^4_{p^l p^k p^j p^i} f^N = \frac{\widetilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^{d+3}\partial^3_{p^l p^k p^j} f^N] - \frac{\widetilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^{d+3}]\partial^3_{p^l p^k p^j} f^N \quad (4.31)$$

We proceed for the first term of (4.31) as we did to obtain (4.13); and similarly :

$$2(\frac{\widetilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^{d+3}\partial^3_{p^l p^k p^j} f^N] / (1+|\overline{p}|)^{d+3}\partial^3_{p^l p^k p^j} f^N) = -(\partial_{p^i}(\frac{\widetilde{P}^i}{p^0})(1+|\overline{p}|)^{d+3}\partial^3_{p^l p^k p^j} f^N / (1+|\overline{p}|)^{d+3}\partial^3_{p^l p^j p^k} f^N)$$

since $\partial_{p^i}(\frac{\widetilde{P}^i}{p^0})$ is bounded, we have :

$$|(\frac{\widetilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^{d+3}\partial^3_{p^l p^k p^j} f^N] / (1+|\overline{p}|)^{d+3}\partial^3_{p^l p^k p^j} f^N)| \leq C(\widetilde{f},F)\left\|(1+|\overline{p}|)^{d+3}\partial^3_{p^l p^i p^k} f^N\right\|^2_{L^2(\mathbb{R}^3)} \quad (4.32)$$

In the second term of (4.31), we have :

$$\frac{\widetilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^{d+3}] = \frac{\widetilde{P}^i}{p^0}(d+3)\frac{p^i}{|\overline{p}|}(1+|\overline{p}|)^{d+2} = \frac{\widetilde{P}^i}{p^0}\frac{(d+3)p^i}{|\overline{p}|(1+|\overline{p}|)}(1+|\overline{p}|)^{d+3}$$

and $\frac{\widetilde{P}^i}{p^0}\frac{(d+3)p^i}{|\overline{p}|(1+|\overline{p}|)}$ is bounded. So we have for the second term of (4.31) :

$$|(\frac{\widetilde{P}^i}{p^0}\partial_{p^i}[(1+|\overline{p}|)^{d+3}]\partial^3_{p^l p^k p^j} f^N / (1+|\overline{p}|)^{d+3}\partial^3_{p^l p^k p^j} f^N)| \leq C(\widetilde{f},F)\left\|(1+|\overline{p}|)^{d+3}\partial^3_{p^l p^i p^k} f^N\right\|^2_{L^2(\mathbb{R}^3)} \quad (4.33)$$

(4.32) and (4.33) give for the first term of (4.31) :

$$|(\frac{\widetilde{P}^i}{p^0}(1+|\overline{p}|)^{d+3}\partial^4_{p^l p^k p^j p^i} f^N / (1+|\overline{p}|)^{d+3}\partial^3_{p^l p^k p^j} f^N)| \leq C(\widetilde{f},F)\left\|(1+|\overline{p}|)^{d+3}\partial^3_{p^l p^i p^k} f^N\right\|^2_{L^2(\mathbb{R}^3)} \quad (4.34)$$

(4.34) and (4.30) give (4.28).

Remark 4.1 *Lemmas 4.2, 4.3, 4.4 and 4.5 prove formula (4.10).*

4.4 *The Existence Theorem*

Theorem 4.1

The Linearized Boltzmann equation (4.5) has a unique solution in $\mathbf{H}^3_d(\mathbb{R}^3)-$weak star.

Proof.

1. *Existence*

We prove that the sequence (f^N) is bounded in $\mathbf{H}^3_d(\mathbb{R}^3)$.

Since $\widetilde{f} \in \mathbf{H}^3_d(\mathbb{R}^3)$ and $f^N \in \mathbf{H}^3_d(\mathbb{R}^3)$, from Proposition (3.6) and formula (4.10), we have : $(1+|\overline{p}|)^{d+|\beta|}\partial^\beta_p(\frac{1}{p^0}Q(\widetilde{f},\widetilde{f})) \in L^2(\mathbb{R}^3)$ and $(1+|\overline{p}|)^{d+|\beta|}\partial^\beta_p f^N \in L^2(\mathbb{R}^3)$, for β such that $|\beta| \leq 3$.

From (4.9), we have :

$$((1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}[\frac{1}{p^0}Q(\tilde{f},\tilde{f})]/(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N)=((1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}[\frac{\partial f^N}{\partial t}+\frac{\tilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i}]/(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N)$$

then

$$((1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}[\frac{1}{p^0}Q(\tilde{f},\tilde{f})]/(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N) \tag{4.35}$$

$$=((1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}[\frac{\partial f^N}{\partial t}]/(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N)+((1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}[\frac{\tilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i}]/(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N)$$

But $(1+|\overline{p}|)^{d+|\beta|}$ does not depend on t, so :

$$(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}[\frac{\partial f^N}{\partial t}]=\partial_t[(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N]$$

So that :

$$((1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}[\frac{\partial f^N}{\partial t}]/(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N)=(\partial_t[(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N]/(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N)$$

$$=\frac{1}{2}\frac{d}{dt}\|(1+|\overline{p})|^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N\|_{L^2(\mathbb{R}^3)}^2$$

From (4.35), we then have :

$$((1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}[\frac{1}{p^0}Q(\tilde{f},\tilde{f})]/(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N) \tag{4.36}$$

$$=((1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}[\frac{\tilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i}]/(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N)+\frac{1}{2}\frac{d}{dt}\left\|(1+|\overline{p})|^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N\right\|_{L^2(\mathbb{R}^3)}^2$$

Now, we always have :

$$|((1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}[\frac{1}{p^0}Q(\tilde{f},\tilde{f})]/(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N)|\leq\left\|(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}(\frac{1}{p^0}Q(\tilde{f},\tilde{f}))\right\|_{L_d^2(\mathbb{R}^3)}\left\|(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N\right\|_{L^2(\mathbb{R}^3)}$$

$$\leq\left\|\frac{1}{p^0}Q(\tilde{f},\tilde{f})\right\|_{H_d^3(\mathbb{R}^3)}\left\|(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N\right\|_{L^2(\mathbb{R}^3)}$$

So, from (4.36) we have :

$$\frac{1}{2}\frac{d}{dt}\left\|(1+|\overline{p})|^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N\right\|_{L^2(\mathbb{R}^3)}^2\leq-((1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}[\frac{\tilde{P}^i}{p^0}\frac{\partial f^N}{\partial p^i}]/(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N)+$$

$$\left\|\frac{1}{p^0}Q(\tilde{f},\tilde{f})\right\|_{H_d^3(\mathbb{R}^3)}\left\|(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N\right\|_{L^2(\mathbb{R}^3)}$$

Now, by (4.10) we have :

$$\frac{1}{2}\frac{d}{dt}\left\|(1+|\overline{p}|)^{d+|\beta|}\partial_{\overline{p}}^{\beta}f^N\right\|_{L^2(\mathbb{R}^3)}^2 \tag{4.37}$$

$$\leq C(\tilde{f},F)(\sum_{|\alpha|\leq|\beta|}\left\|(1+|\bar{p}|)^{d+|\alpha|}\partial_p^{\alpha}f^N\right\|_{L^2(\mathbb{R}^3)})\left\|(1+|\bar{p}|)^{d+|\beta|}\partial_p^{\beta}f^N\right\|_{L^2(\mathbb{R}^3)}+\left\|\frac{1}{p^0}Q(\tilde{f},\tilde{f})\right\|_{H_d^3(\mathbb{R}^3)}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial_p^{\beta}f^N\right\|_{L^2(\mathbb{R}^3)}$$

We have $\frac{1}{2}\frac{d}{dt}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial_p^{\beta}f^N\right\|^2_{L^2(\mathbb{R}^3)}=\left\|(1+|\bar{p}|)^{d+|\beta|}\partial_p^{\beta}f^N\right\|_{L^2(\mathbb{R}^3)}\frac{d}{dt}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial_p^{\beta}f^N\right\|_{L^2(\mathbb{R}^3)}$

We then deduce from (4.37) that :

$$\frac{d}{dt}\left\|(1+|\bar{p})|^{d+|\beta|}\partial_p^{\beta}f^N\right\|_{L^2(\mathbb{R}^3)}\leq C(\tilde{f},F)(\sum_{|\alpha|\leq|\beta|}\left\|(1+|\bar{p}|)^{d+|\alpha|}\partial_p^{\alpha}f^N\right\|_{L^2(\mathbb{R}^3)})+\left\|\frac{1}{p^0}Q(\tilde{f},\tilde{f})\right\|_{H_d^3(\mathbb{R}^3)}$$

Taking the sum of (4.37) over $|\beta|=0,1,2,3$, we obtain :

$$\frac{d}{dt}(\sum_{|\beta|\leq 3}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial_p^{\beta}f^N\right\|_{L^2(\mathbb{R}^3)})\leq C(\tilde{f},F)(\sum_{|\beta|\leq 3}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial_p^{\beta}f^N\right\|_{L^2(\mathbb{R}^3)}+\left\|\frac{1}{p^0}Q(\tilde{f},\tilde{f})\right\|_{H_d^3(\mathbb{R}^3)})\quad(4.38)$$

Integrating and applying Gronwall Lemma, we obtain :

$$\sum_{|\beta|\leq 3}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial_p^{\beta}f^N\right\|_{L^2(\mathbb{R}^3)}\leq C(\tilde{f},F)(\sum_{|\beta|\leq 3}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial_p^{\beta}f^N(0)\right\|_{L^2(\mathbb{R}^3)}+\int_0^t\left\|\frac{1}{p^0}Q(\tilde{f},\tilde{f})(s)\right\|_{H_d^3(\mathbb{R}^3)}ds)$$

By Proposition 3.6 (the fundamental inequality) we have :

$$\sum_{|\beta|\leq 3}\left\|(1+|\bar{p}|)^{d+|\beta|}\partial_p^{\beta}f^N\right\|_{L^2(\mathbb{R}^3)}\leq C(\tilde{f},F)(\|f_0\|_{H_d^3(\mathbb{R}^3)}+T\left\|\tilde{f}\right\|^2_{H_d^3(0,T,\mathbb{R}^3)})\quad(4.39)$$

Since $\tilde{f},f_0\in H^3_{d,r}(\mathbb{R}^3)$, we have

$$\left\|f^N\right\|_{H_d^3(\mathbb{R}^3)}\leq C(\tilde{f},F,r)\quad(4.40)$$

(4.40) shows that (f^N) is bounded in $H_d^3(\mathbb{R}^3)$ which is reflexive. So (f^N) admits a subsequence we denote again (f^N) which converges to a solution f of the linearized Botzmann equation (4.5).

2. *Uniqueness*

Let us suppose that, for the same function $\tilde{f}$, there exists two solutions f_1, f_2 of the problem with the initial data : $f_1(0)=f_2(0)=f_0$.

Then f_1 and f_2 satisfy : $\begin{cases}\dfrac{\partial f_1}{\partial t}+\dfrac{\tilde{P}^i}{p^0}\dfrac{\partial f_1}{\partial p^i}=\dfrac{1}{p^0}Q(\tilde{f},\tilde{f})\\ \dfrac{\partial f_2}{\partial t}+\dfrac{\tilde{P}^i}{p^0}\dfrac{\partial f_2}{\partial p^i}=\dfrac{1}{p^0}Q(\tilde{f},\tilde{f})\end{cases}$ By substraction, we obtain :

$$\frac{\partial V}{\partial t}+\frac{\tilde{P}^i}{p^0}\frac{\partial V}{\partial p^i}=0\quad with\quad V=f_1-f_2$$

We have :

$$\frac{d}{dt}\left\|(1+|\bar{p}|)^d V(t)\right\|^2_{L^2(\mathbb{R}^3)} = (\partial_t[(1+|\bar{p}|)^d V(t)]/(1+|\bar{p}|)^d V(t)) + ((1+|\bar{p}|)^d V(t)/\partial_t[(1+|\bar{p}|)^d V(t)])$$

$$= -((1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0}\partial_{p^i} V(t)/(1+|\bar{p}|)^d V(t)) - ((1+|\bar{p}|)^d V(t)/(1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0}\partial_{p^i} V(t))$$

$$= -((1+|\bar{p}|)^d \partial_{p^i} V(t)/(1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0} V(t)) - ((1+|\bar{p}|)^d V(t)/(1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0}\partial_{p^i} V(t))$$

$$= -(\partial_{p^i}[(1+|\bar{p}|)^d V(t)] - (\partial_{p^i}(1+|\bar{p}|)^d)V(t)/(1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0} V(t))$$

$$- ((1+|\bar{p}|)^d V(t)/(1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0}\partial_{p^i} V(t))$$

$$= -(\partial_{p^i}[(1+|\bar{p}|)^d V(t)]/(1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0} V(t)) + ((\partial_{p^i}(1+|\bar{p}|)^d)V(t)/(1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0} V(t))$$

$$- ((1+|\bar{p}|)^d V(t)/(1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0}\partial_{p^i} V(t))$$

$$= ((1+|\bar{p}|)^d V(t)/\partial_{p^i}[(1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0} V(t)]) + ((\partial_{p^i}(1+|\bar{p}|)^d)V(t)/(1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0} V(t))$$

$$- ((1+|\bar{p}|)^d V(t)/(1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0}\partial_{p^i} V(t))$$

$$= ((1+|\bar{p}|)^d V(t)/\partial_{p^i}[(1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0}]V(t)) + ((\partial_{p^i}(1+|\bar{p}|)^d)V(t)/(1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0} V(t))$$

$$\frac{d}{dt}\left\|(1+|\bar{p}|)^d V(t)\right\|^2_{L^2(\mathbb{R}^3)} = ((1+|\bar{p}|)^d V(t)/\partial_{p^i}[(1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0}]V(t)) + (\frac{\tilde{P}^i}{p^0}(\partial_{p^i}(1+|\bar{p}|)^d)V(t)/(1+|\bar{p}|)^d V(t)) \quad (4.41)$$

We know that :

$$\partial_{p^i}[(1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0}] = \frac{d(1+|\bar{p}|)^{d-1} p^i}{|\bar{p}|}\frac{\tilde{P}^i}{p^0} + (1+|\bar{p}|)^d \partial_{p^i}(\frac{\tilde{P}^i}{p^0}) = [\frac{dp^i}{|\bar{p}|(1+|\bar{p}|)}\frac{\tilde{P}^i}{p^0} + \partial_{p^i}(\frac{\tilde{P}^i}{p^0})](1+|\bar{p}|)^d$$

But $\frac{dp^i}{|\bar{p}|(1+|\bar{p}|)}\frac{\tilde{P}^i}{p^0}$ and $\partial_{p^i}(\frac{\tilde{P}^i}{p^0})$ are bounded. On the other hand :

$$\frac{\tilde{P}^i}{p^0}\partial_{p^i}(1+|\bar{p}|)^d = \frac{\tilde{P}^i}{p^0}\frac{dp^i}{|\bar{p}|}(1+|\bar{p}|)^{d-1} = \frac{\tilde{P}^i}{p^0}\frac{dp^i}{|\bar{p}|(1+|\bar{p}|)}(1+|\bar{p}|)^d$$

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and $\frac{\tilde{P}^i}{p^0}\frac{p^i}{|\bar{p}|(1+|\bar{p}|)}$ is bounded.

Hence in (4.41) we have :

$$\begin{cases} |((1+|\bar{p}|)^d V(t)/\partial_{p^i}[(1+|\bar{p}|)^d \frac{\tilde{P}^i}{p^0}]V(t))| \leq C(\tilde{f},F)\|(1+|\bar{p}|)^d V(t)\|^2_{L^2(\mathbb{R}^3)} \\ |(\frac{\tilde{P}^i}{p^0}[\partial_{p^i}(1+|\bar{p}|)^d]V(t)/(1+|\bar{p}|)^d V(t))| \leq C(\tilde{f},F)\|(1+|\bar{p}|)^d V(t)\|^2_{L^2(\mathbb{R}^3)} \end{cases}$$

From (4.41), we deduce that :

$$\frac{d}{dt}\|(1+|\bar{p}|)^d V(t)\|^2_{L^2(\mathbb{R}^3)} \leq C(\tilde{f},F)\|(1+|\bar{p}|)^d V(t)\|^2_{L^2(\mathbb{R}^3)} \quad (4.42)$$

But $\frac{d}{dt}\|(1+|\bar{p}|)^d V(t)\|^2_{L^2(\mathbb{R}^3)} = 2\|(1+|\bar{p}|)^d V(t)\|_{L^2(\mathbb{R}^3)}\frac{d}{dt}\|(1+|\bar{p}|)^d V(t)\|_{L^2(\mathbb{R}^3)}$

then (4.42) implies :

$$\frac{d}{dt}\|(1+|\bar{p}|)^d V(t)\|_{L^2(\mathbb{R}^3)} \leq C(\tilde{f},F)\|(1+|\bar{p}|)^d V(t)\|_{L^2(\mathbb{R}^3)} \quad (4.43)$$

Since $V(0)=0$, (4.43) implies, using Gronwall Lemma, that $V=0$.

Then $f_1 = f_2$ and the solution is unique.

This ends the proof of theorem 4.1

Theorem 4.2

The Boltzmann equation (4.1) has a unique solution f in $H^3_d(0,T,\mathbb{R}^3)$ such that $f(0,\bar{p}) = f_0(\bar{p})$.

. Consider the application

$$\begin{aligned} \Theta:\ H^3_d(0,T,\mathbb{R}^3) &\to H^3_d(0,T,\mathbb{R}^3) \\ \tilde{f} &\mapsto \theta(\tilde{f}) = f \end{aligned}$$

where f is the unique solution of (4.5).

1. We will show that one can choose $\|f_0\|_{H^3_{d,r}(0,T,\mathbb{R}^3)}$ and $T>0$ such that Θ sends $H^3_{d,r}(0,T,\mathbb{R}^3)$ in $H^3_{d,r}(0,T,\mathbb{R}^3)$. In fact, this is to show that one can choose $\|f_0\|_{H^3_{d,r}(0,T,\mathbb{R}^3)}$ and $T>$ such that :

$$(\|\tilde{f}\|_{H^3_{d,r}(0,T,\mathbb{R}^3)} \leq r) \Rightarrow (\|f\|_{H^3_{d,r}(0,T,\mathbb{R}^3)} \leq r)$$

Let us take the bound (4.39) in (4.38), we obtain :

$$\frac{d}{dt}(\sum_{|\beta|\leq 3}\|(1+|\bar{p}|)^{d+|\beta|}\partial^\beta_p f^N\|_{L^2(\mathbb{R}^3)}) \leq C(\tilde{f},F)(\|f_0\|_{H^3_d(\mathbb{R}^3)} + T\|\tilde{f}\|^2_{H^3_d(\mathbb{R}^3)} + \left\|\frac{1}{p^0}Q(\tilde{f},\tilde{f})\right\|_{H^3_d(\mathbb{R}^3)})$$

Integrating this inequality on $[0,t]$, $0 \leq t \leq T$, we obtain, using $\left\|\frac{1}{p^0}Q(\tilde{f},\tilde{f})\right\|_{H^3_d(\mathbb{R}^3)} \leq r^2$:

$$\sum_{|\beta|\le 3}\left\|(1+|\overline{p}|)^{d+|\beta|}\partial_p^\beta f^N\right\|_{L^2(\mathbb{R}^3)} \le \sum_{|\beta|\le 3}\left\|(1+|\overline{p}|)^{d+|\beta|}\partial_p^\beta f^N(0)\right\|_{L^2(\mathbb{R}^3)} + C(\tilde{f},F)(\|f_0\|_{H_d^3(\mathbb{R}^3)}T + r^2T^2 + r^2T)$$

or

$$\|f^N\|_{H_d^3(\mathbb{R}^3)} \le \|f_0\|_{H_d^3(\mathbb{R}^3)} + C(\tilde{f},F)(T\|f_0\|_{H_d^3(\mathbb{R}^3)}T + r^2T^2 + r^2T) \qquad (4.44)$$

since $r>0$ is given, if we take $\|f_0\|_{H_d^3(\mathbb{R}^3)}$ and $T>0$ such that :

$$\begin{cases}\|f_0\|_{H_d^3(\mathbb{R}^3)} \le \dfrac{r}{2} \\ C(\tilde{f},F)(\|f_0\|_{H_d^3(\mathbb{R}^3)}T + r^2T^2 + r^2T) \le \dfrac{r}{2}\end{cases} \qquad (4.44) \quad \text{implies :}$$

$$\|f^N\|_{H_d^3(\mathbb{R}^3)} \le r$$

Since the sequence (f^N) converges to f in $H_d^3(\mathbb{R}^3)-$weak star, we have :

$$\|f\|_{H_d^3(\mathbb{R}^3)} \le r$$

this shows that

$$(\|\tilde{f}\|_{H_{d,r}^3(0,T,\mathbb{R}^3)} \le r) \Rightarrow (\|f\|_{H_{d,r}^3(0,T,\mathbb{R}^3)} \le r)$$

2. We show that Θ is a contraction in the space $H_{d,r}^3(0,T,\mathbb{R}^3)$.

Let $\tilde{f}_1,\tilde{f}_2 \in H_{d,r}^3(0,T,\mathbb{R}^3)$, $f_1,f_2 \in H_{d,r}^3(0,T,\mathbb{R}^3)$ solutions of :

$$\begin{cases}\dfrac{\partial f_1}{\partial t} + \dfrac{\tilde{P}^i}{p^0}\dfrac{\partial f_1}{\partial p^i} = \dfrac{1}{p^0}Q(\tilde{f}_1,\tilde{f}_1) \\ \dfrac{\partial f_2}{\partial t} + \dfrac{\tilde{P}^i}{p^0}\dfrac{\partial f_2}{\partial p^i} = \dfrac{1}{p^0}Q(\tilde{f}_2,\tilde{f}_2) \\ f_1(0) = f_2(0); \tilde{P}_j^i = \tilde{P}_j^i(\tilde{f}_j) \;\; j=1,2\end{cases}$$

Integrating the equations on $[0,t]$, $0\le t\le T$, we have :

$$\begin{cases} f_1(t) = \int_0^t \left(\dfrac{1}{p^0}Q(\tilde{f}_1,\tilde{f}_1) - \dfrac{\tilde{P}_1^i}{p^0}\partial_{p^i} f_1\right)(s)ds + f_1(0) \\ f_2(t) = \int_0^t \left(\dfrac{1}{p^0}Q(\tilde{f}_2,\tilde{f}_2) - \dfrac{\tilde{P}_2^i}{p^0}\partial_{p^i} f_2\right)(s)ds + f_2(0)\end{cases}$$

Then, subtracting the second equation to the first one, we obtain :

$$f_1(t) - f_2(t) = \int_0^t \left(\frac{1}{p^0}Q(\tilde{f}_1,\tilde{f}_1) - \frac{1}{p^0}Q(\tilde{f}_2,\tilde{f}_2)\right)(s)ds + \int_0^t \left(\frac{\tilde{P}_1^i}{p^0}\partial_{p^i} f_1 - \frac{\tilde{P}_2^i}{p^0}\partial_{p^i} f_2\right)(s)ds \qquad (4.45)$$

For the first term of (4.45) we have, using $Q = Q^+ - Q^-$ and the fact that Q^+, Q^- are bilinear :

$$X = \int_0^t \left(\frac{1}{p^0}Q(\tilde{f}_1,\tilde{f}_1) - \frac{1}{p^0}Q(\tilde{f}_2,\tilde{f}_2)\right)(s)ds$$

$$= \int_0^t \{[\frac{Q^+(\tilde{f}_1, \tilde{f}_1)}{p^0} - \frac{Q^+(\tilde{f}_2, \tilde{f}_2)}{p^0}] + [\frac{Q^-(\tilde{f}_2, \tilde{f}_2)}{p^0} - \frac{Q^-(\tilde{f}_1, \tilde{f}_1)}{p^0}]\}(s)ds$$

$$= \int_0^t \{[\frac{Q^+(\tilde{f}_1, \tilde{f}_1 - \tilde{f}_2)}{p^0} + \frac{Q^+(\tilde{f}_1 - \tilde{f}_2, \tilde{f}_2)}{p^0}] + [\frac{Q^-(\tilde{f}_2, \tilde{f}_1 - \tilde{f}_2)}{p^0} + \frac{Q^-(\tilde{f}_1, \tilde{f}_1 - \tilde{f}_2)}{p^0}]\}(s)ds$$

then, using $\left\| \frac{P(f,g)}{p^0} \right\|_{H_d^3(\mathbb{R}^3)} \leq \|f\|_{H_d^3(\mathbb{R}^3)} \|g\|_{H_d^3(\mathbb{R}^3)}$, for $P = Q^+$ or $P = Q^-$; given by (3.28) and Proposition 3.5, we have :

$$\|X\|_{H_d^3(0,T,\mathbb{R}^3)} \leq CT(\|\tilde{f}_1\|_{H_d^3(0,T,\mathbb{R}^3)} + \|\tilde{f}_2\|_{H_d^3(0,T,\mathbb{R}^3)}) \|\tilde{f}_1 - \tilde{f}_2\|_{H_d^3(0,T,\mathbb{R}^3)} \leq CTr \|\tilde{f}_1 - \tilde{f}_2\|_{H_d^3(0,T,\mathbb{R}^3)} \quad (4.46)$$

For the second term of (4.45), we have :

$$\frac{\tilde{\mathsf{P}}_1^i}{p^0} \partial_{p^i} f_1 - \frac{\tilde{\mathsf{P}}_2^i}{p^0} \partial_{p^i} f_2 = \frac{\Gamma^i_{\lambda\mu} p^\lambda p^\mu}{p^0} (\frac{\partial f_1}{\partial p^i} - \frac{\partial f_2}{\partial p^i}) + \sum_i F_0^i \int_{\mathbb{R}^3} (\tilde{f}_1 - \tilde{f}_2) d\bar{p}$$

So : $Y = \int_0^t \frac{\Gamma^i_{\lambda\mu} p^\lambda p^\mu}{p^0} (\frac{\partial f_1}{\partial p^i} - \frac{\partial f_2}{\partial p^i}) ds + \sum_i \int_0^t F_0^i [\int_{\mathbb{R}^3} (\tilde{f}_1 - \tilde{f}_2) d\bar{p}] ds$

and $|Y| \leq C(\int_0^t |p^i (\frac{\partial f_1}{\partial p^i} - \frac{\partial f_2}{\partial p^i})| ds + \int_0^t ds \int_{\mathbb{R}^3} |\tilde{f}_1 - \tilde{f}_2| d\bar{p})$

So :

$$\|Y\|_{H_d^3(0,T,\mathbb{R}^3)} \leq CT(\|f_1 - f_2\|_{H_d^3(0,T,\mathbb{R}^3)} + \|\tilde{f}_1 - \tilde{f}_2\|_{H_d^3(0,T,\mathbb{R}^3)}) \quad (4.47)$$

From (4.45), (4.46) and (4.47), we deduce that :

$$\|f_1 - f_2\|_{H_d^3(0,T,\mathbb{R}^3)} \leq CT(r+1)(\|f_1 - f_2\|_{H_d^3(0,T,\mathbb{R}^3)} + \|\tilde{f}_1 - \tilde{f}_2\|_{H_d^3(0,T,\mathbb{R}^3)}) \quad (4.48)$$

If in (4.48) we take T such that : $CT(r+1) < \frac{1}{3}$, then we have :

$$\|f_1 - f\|_{2H_d^3(0,T,\mathbb{R}^3)} \leq \frac{1}{2} \|\tilde{f}_1 - \tilde{f}\|_{2H_d^3(0,T,\mathbb{R}^3)} \quad (4.49)$$

(4.49) shows that Θ is a contraction in the space $H_d^3(0,T,\mathbb{R}^3)$.

Then Θ has a fixed point f solution of de Boltzmann equation.

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