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Research Article

ON THE GEOMETRY OF THE LEVEL SURFACES

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ABSTRACT
In this paper we study the geometry of the level surfaces of Riemannian submersions. It is proved that the level surfaces of Riemannian submersions generate a foliation whose leaves are manifolds of constant Gaussian curvature.

***Keywords**: Riemannian Manifold, Connection, Foliation, Leaf, Hessian, Submersion, Gauss Curvature*

INTRODUCTION
Let M be a smooth Riemannian manifold of dimension n with Riemannian metric g, F -foliation of dimension k on M, where $0<k<n$ (Tondeur, 1988). We denote by L_p - a leaf of the foliation F passing through the point $p \in M$, by T_qF - the tangent space of leaf L_p at the point $q \in L_p$, and by $H(q)$ - orthogonal complement of T_qF. As a result, we get two sub-bundles $TF=\{T_qF\}$, $TH=\{H(q)\}$ of the tangent bundle TM, and we have the orthogonal decomposition $TM=TF \oplus H$. Thus every vector field X can be written in the form $X=X^v+X^h$, where $X^v \in TF$, $X^h \in TH$. If $X^h=0$ (respectively $X^v=0$), the field is called the vertical (horizontal) vector field.

The Riemannian metric g on a manifold M induces Riemannian metric $\tilde{g}$ on the leaf L_p. Canonical injection $i:L_p \to M$ is an isometric immersion with respect to these metrics. A connection ∇ (Levi-Civita connection) defined by the Riemannian metric g induces a connection $\tilde{\nabla}$ on L_p which coincides with the connection defined by the Riemannian metric $\tilde{\nabla}$ (Gromoll *et al.*, 1968).

Let Z be a horizontal vector field. For each vertical vector field X we define the vertical vector field
$S(X,Z)=(\nabla_X Z)^v$
and we get the tensor field of type (1,1) $X,Z \to S_Z X=S(X,Z)$.

This tensor field gives the bilinear form $l_Z(X,Y)=\langle S_Z X,Y\rangle$, where $\langle X,Y\rangle$ - the inner product defined by the Riemannian metric g.

Tensor field S_Z is called the second fundamental tensor, and the form $l_Z(X,Y)$ is called the second main form with respect to the horizontal field Z.

Mapping $S_Z:T_qF \to T_qF$ defined by the formula $X_q \to S(X,Z)_q$ is self-endomorphism with respect to the inner product defined by the Riemannian metric $\tilde{g}$. If the vector field Z is a vector of unit length, the eigenvalues of this endomorphism called the principal curvatures of L_p at the point q, and the corresponding eigenvectors are called the principal directions.

The mean curvature H_Z and the Gauss-Kronecker curvature K_Z of the leaf L_p at the point q are determined by the principal curvatures (Gromoll *et al.*, 1968):

$$H_Z=\frac{1}{k}trS_Z \ , K_Z=detS_Z .$$

Let $f: M \to B$ be a differentiable map of maximal rank, where M, B a smooth manifolds of dimensions n, m respectively, $n > m.$ Such maps are called submersions. By the theorem on the rank of a differentiable function for each point $p \in B$ inverse image $f^{-1}(p)$ is a submanifold of dimension $k = n - m$. Thus submersion $f: M \to B$ generates a foliation on a manifold of dimension $k = n - m$ whose leaves are submanifolds $L_p = f^{-1}(p), p \in B$.

The geometry and topology of the foliations generated by submersions were the subject of numerous studies (Tondeur, 1988); (Hermann, 1960); (Narmanov and Kaipnazarova, 2010); (O'Neil, 1966); (Sharipov, 2014).

If the differential df of mapping $f: M \to B$ preserves the length of horizontal vectors, the submersion $f: M \to B$ is called the Riemannian submersion. Geometry of Riemannian submersions investigated in numerous papers, in particular in (O'Neil, 1966) derived the fundamental equations of the Riemannian submersion. In (Hermann, 1960) it is proved that Riemannian submersion generates Riemannian foliation. Recall that if geodesics of foliated manifold (M, g) orthogonal to a leaf of F at one point are orthogonal to the leaves of F everywhere we call F Riemannian (metric) foliation (or foliation with bundle-like metric g).

The fibers of classical Hopf map $f: S^3 \to S^2$ are integral lines of Killing vector field (the fibers of Hopf map are great circles) and by known fact there exists a Riemannian metric ρ on S^2 such that Hopf map will be Riemannian submersion with respect ρ (Gromoll and Walschap, 2009, p.6.Theorem-1.2.1).

We consider the case where the manifold B is one-dimensional manifold, to be more precise we consider smooth function $f: M \to R$. If $Crit\{f\}$ - the set of critical points of the function f, then on the manifold M, $Crit\{f\}$ arises foliation F of dimension $n-1$ (or codimension one foliation), leaves of which are level surfaces of function f.

In (Tondeur, 1988) it is studied the geometry of the level surfaces of the functions $f: M \to R$ for which $X(|gradf|^2) = 0$ for each vertical vector field X. In particular, it is proved that level surfaces of function $f: M \to R$ generates Riemannian foliation if and only if $X(|gradf|^2) = 0$ for each vertical vector field X (Tondeur, 1988, p.107,Theorem-8.9).

From other side if a submersion $f: M \to B$ generates Riemannian foliation, the Lie derivative $L_X g$ vanishes in any vertical direction i.e. $L_X g = 0$ for each vertical vector field X. By the theorem -1.2 from Gromoll and Walschap, (2009), there exists a Riemannian metric ρ on B such that $f: M \to B$ will be a Riemannian submersion with respect ρ.

Main Part

In this paper, we show that the level surfaces of the function of this class are the surfaces of constant Gaussian curvature.

Theorem *Let $M-$ complete Riemannian manifold of constant curvature, $f: M \to R$ be a smooth function. Suppose that $Critf = \varnothing$ and $X(|gradf|^2)) = 0$ for each vertical vector field X. Then every leaf of foliation F is a manifold of constant Gaussian curvature.*

Proof As is known the Hessian is given by

$$h_f(X,Y) = l_Z(X,Y) = \langle \nabla_X Z, Y \rangle$$

where $Z = gradf$, ∇ - the Levi-Civita connection defined by Riemannian metric g.

The map $X \to h_f\left(X\right) = \nabla_X Z$ (Hesse tensor) is a linear operator and is given by a symmetric matrix $A : h_f\left(X\right) = \nabla_X Z = AX$

We denote by $\chi(\lambda)$ the characteristic polynomial of the matrix A with a free term $(-1)^n \det A$ and define a new polynomial $\rho(\lambda)$ by the equation

$\lambda\rho(\lambda) = \det A - (-1)^n \chi(\lambda)$.

Since $\chi\left(A\right) = 0$ we have that $A\rho\left(A\right) = \det A \cdot E$, where E - is the identity matrix. The elements of the matrix $\rho\left(A\right)$ are cofactors of the matrix A. This matrix is denoted by H_f^c.

It is well known that the Gaussian curvature of the surface is calculated by the formula (Gromoll *et al.*, 1968, p.110)

$$K = \det S = \frac{1}{|gradf|^{n+1}}\left\langle H_f^c\left(gradf\right), gradf\right\rangle.$$

To prove the theorem, it suffices to show that $X\left(K\right) = 0$ for each vertical vector field X at any point q of a leaf L_p.

By hypothesis of the theorem we have $X\left(|gradf|^2\right) = 0$ and so $X\left(\frac{1}{|gradf|^{n+1}}\right) = 0$ therefore we need to show that $\left\langle \nabla_X H_f^c Z, Z\right\rangle + \left\langle H_f^c Z, \nabla_X Z\right\rangle = 0$.

We know that if $X(|gradf|^2) = 0$ for each vertical vector field X, each gradient line of f is a geodesic line of Riemannian manifold (Narmanov and Kaipnazarova, 2010). By definition, the gradient line is a geodesic if and only if $\nabla_N N = 0,$ where $N = \frac{Z}{|Z|}$.

We calculate the covariant differential

$$\nabla_N N = \frac{1}{|Z|}\nabla_Z N = \frac{1}{|Z|}\left(\frac{1}{|Z|}\nabla_Z Z + Z\left(\frac{1}{|Z|}\right)Z\right) = 0$$

and get $\nabla_Z Z = \lambda Z$, where $\lambda = -|Z|Z\left(\frac{1}{|Z|}\right)$. This means that the gradient vector Z is the eigenvector of matrix A.

Let $X_1^0, X_2^0, \cdots, X_{n-1}^0, Z^0$-be mutually orthogonal eigenvectors of A at the point $q \in L_p$ such that $X_1^0, X_2^0, \cdots, X_{n-1}^0$ the unit vectors, Z^0 - the value of the gradient field at a point q. Locally, they can be extended to the vector fields $X_1, X_2, \cdots, X_{n-1}, Z$ to a neighborhood of (say U) point q so that they formed at each point of an orthogonal basis consisting of eigenvectors. We construct the Riemannian normal system of coordinates $\left(x_1, x_2, \cdots, x_n\right)$ in a neighborhood U via vectors $X_1^0, X_2^0, \cdots, X_{n-1}^0, Z^0$ (Gromoll *et al.*,1968, p.112).

The components g_{ij} of the metric g and the connection components Γ_{ij}^k in the normal coordinate system satisfies the conditions of Gromoll *et al.*, (1968, p. 132):

$g_{ij}(q) = \delta_{ij}, \Gamma_{ij}^k(q) = 0.$

We show that $X(\lambda) = 0$ for each vertical field X. From the equality

$$X(\lambda) = -X(|Z|)Z\left(\frac{1}{|Z|}\right) - |Z|X\left(Z\left(\frac{1}{|Z|}\right)\right)$$

and from the condition $X(|Z|) = 0$ follows equality

$$X\left(Z\left(\frac{1}{|Z|}\right)\right) = X(Z(\phi)) = [X,Z](\phi) - Z(X(\phi,))$$

where $\phi = \frac{1}{|Z|}$, $[X,Z]$-Lie bracket of vector fields X,Z.

From the condition of the theorem follows $X(Z(\phi)) = 0$. In (Tondeur, 1988) it is shown that $X\left(|gradf|^2\right) = 0$ for each of the vertical vector field X if and only if $[X,Z](\phi) = 0$ a vertical field. Therefore $[X,Z](\phi) = 0$. Thus, λ is a constant function on the leaf L.

Now we denote by $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ the eigenvalues of the matrix A corresponding to the eigenvectors $X_1, X_2, \ldots, X_{n-1}$. Then in the basis $X_1, X_2, \ldots, X_{n-1}, Z$ matrix A has the form:

$$A = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

By hypothesis of the theorem, the vector field $\nabla_X Z$ is vertical field. It follows Codazzi equations have the form (Kobayashi and Nomizu, 1969, p.29)

$$(\nabla_X A)Y = (\nabla_Y A)X$$

From this equation we get

$$\nabla_{X_i} AX_j = \nabla_{X_j} AX_i, \nabla_{X_i} AZ = \nabla_Z AX_i \qquad (1)$$

at any point of U for each vector field X_i. From first equation of (1) we take following equality

$$X_i(\lambda_j)X_j + \lambda_j \nabla_{X_i} X_j = X_j(\lambda_i)X_j + \lambda_i \nabla_{X_j} X_i. \qquad (2)$$

Since $\nabla_{X_i} X_j = \Gamma_{ij}^k X_k = 0$ at the point q by properties of normal coordinate system, from (2) follows equality

$$X_i(\lambda_j)X_j = X_j(\lambda_i)X_j. \qquad (3)$$

By the linear independence $X_1, X_2, \cdots, X_{n-1}$, we have that

$X_i(\lambda_j) = 0$ for $i \neq j$ for all i.

From second equation of (1) we take following

$$X_i(\lambda)Z + \lambda \nabla_{X_i} Z = Z(\lambda_i)X_i + \lambda_i \nabla_Z X_i. \qquad (4)$$

Since $\nabla_{X_i} Z = \nabla_Z X_i = 0$ at the point q

from the linear independence of vectors X_i, Z we have that

$X_i(\lambda) = 0, Z(\lambda_i) = 0$ for all i.

On the other hand

$\nabla_Z AX_i = Z(\lambda_i)X_i + \lambda_i \nabla_Z X_i, \nabla_{X_i} AZ = \nabla_Z AX_i, \nabla_{X_i} Z = \lambda_i X_i.$ (5)

From (5) we get that

$$\lambda_i^2 X_i + Z(\lambda_i)X_i = X_i(\lambda)Z + \lambda\lambda_i X_i. \qquad (6)$$

Since $Z(\lambda_i)=0$, $X_i(\lambda)=0$ at the point q from the (6)follows that $\lambda_i^2 = \lambda\lambda_i$. In particular, this implies that if $\lambda_i \neq 0$ then $\lambda_i = \lambda$ and $X(\lambda_i) = X(\lambda) = 0$, $Z(\lambda) = Z(\lambda_i) = 0$ for all i. Thus, in the neighborhood U of the point q non-zero eigenvalues of the matrix A are constant and equal λ.

Given this fact we compute $X(K)$. We denote by m the number of zero eigenvalues of A. If $m=0$, all the eigenvalues are equal to the number λ. In this case, by the definition of the matrix H_f^c we get that $H_f^c Z = \lambda^{n-1} Z$.

Consider the case when $m>0$. If $m>1$, then $H_f^c = 0$. If $m=1$ than $\lambda_i = 0$ for some i and $AX_i = \nabla_{X_i} Z = 0.$ This means that the vector field Z is parallel along the integral curve of a vector field X_i (along i - coordinate line). If $i=n$ we have $\lambda = \lambda_i = 0$ for all i and $H_f^c = 0.$

Without loss of generality we assume that $i<n.$ In this case vector $H_f^c Z$ have only one nonzero component b_i and $H_f^c Z = b_i \frac{\partial}{\partial x_i}.$ In this case we get $\nabla_X H_f^c Z = X(b_i)\frac{\partial}{\partial x_i} + b_i \nabla_X \frac{\partial}{\partial x_i}.$ As we know that $X_i = \frac{\partial}{\partial x_i}$ vertical and $\nabla_X \frac{\partial}{\partial x_i} = 0.$ Thus in the case $m=1$ we have $\left\langle \nabla_X H_f^c(gradf), gradf \right\rangle = 0$

Let us consider the case $m=0.$ In this case we have equalities $H_f^c Z = \lambda^{n-1} Z$ and $\nabla_X H_f^c Z = X\left(\lambda^{n-1}\right)Z + \lambda^{n-1}\nabla_X Z.$ As mentioned above field $\nabla_X gradf$ is a vertical vector field for each vertical vector field X (the field AX is vertical). From this equalities follows $\left\langle \nabla_X H_f^c(gradf), gradf \right\rangle = 0$ at the point q. The theorem is proved.

Examples:

1. $M = R^3$, $\{(x,y,z): x=0, y=0\}$, $f(x,y,z) = x^2 + y^2$. Level surfaces of this submersion are manifolds of zero Gaussian curvature.
2. $M = R^3$, $\{(0,0,0)\}$, $f(x,y,z) = x^2 + y^2 + z^2$. Level surfaces of this submersion are manifolds of constant positive Gaussian curvature.

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