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Research Article

A NOTE ON THE TWO APPROACHES TO DEFINE A RING

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ABSTRACT
In the basic ring theory there are two approaches to define a ring. However both the approaches are not equivalent as several examples of rings and subrings which are valid in one approach become invalid in the other. This general note sheds some light on this issue.

Keywords: *Algebra, Basic Ring Theory, Binary Operation, Ring, Subring*

INTRODUCTION
There are two approaches which are commonly used to define a ring in abstract (modern) algebra text books. As per the first approach (Artin, 2000; Wickless, 2004) a non empty set R together with two binary operations addition ($+$) and multiplication ($\cdot$) is a ring if
(1) $(R,+)$ is an Abelian group.
(2) $(R,.)$ is a semigroup.
(3) $\exists$ an element 1 in R such that $1a = a1 = a, \forall a \in R$.
(4) $a.(b+c) = a.b + a.c$ and $(b+c).a = b.a + c.a$, $\forall a,b,c \in R$. 1 is known as the multiplicative identity of the ring.
In the second approach (Herstein, 2011; Hungerford, 2005) a non empty set R together with two binary operations '$+$' and '$\cdot$' is a ring if
(1) $(R,+)$ is an Abelian group.
(2) $(R,.)$ is a semigroup.
(3) $a.(b+c) = a.b + a.c$ and $(b+c).a = b.a + c.a$, $\forall a,b,c \in R$.
In the first approach described above a non empty subset S of a ring R is called a subring of R if the following properties hold:
1. $(S,+,.)$ is a ring.
2. The multiplicative identity of R is in S.
However in the second approach described above a non empty subset S of a ring R is called a subring of R if the following property holds:
1. $(S,+,.)$ is a ring.
These are the two approaches to define a ring and a subring of a given ring. This much detail suffices for the purpose of this note. Now, in the next section, we shall provide our comments and some examples to explore the differences between the two approaches.

Comments on the Two Approaches and Some Examples
It is obvious that in the second approach one does not need the third condition of the first approach to define a ring. However if in addition this condition holds then the ring R is known as a ring with identity element.
Similarly in order to define a subring of a ring R, in the second approach, it is not required that the set S should possesses the multiplicative identity of R however it is required in the first approach. The second approach is more general than the first. Let us consider the set Z of integers. Then Z is a ring under ordinary addition and multiplication with respect to both the approaches. However if we consider the set E of all even integers then E is an example of a ring in the second approach. But it is not a ring

as per the first approach. Similarly for $m > 1 \in Z$, $mZ = \{mx : x \in Z\}$ is a ring in the second approach but it is not a ring in the first approach. From this we conclude that the number of examples is quite reduced in the first approach.

Let us consider the set M of all square matrices of order two over the integers. Then M is a ring under the usual addition and multiplication of matrices in both the approaches.

Let $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Z \right\}$. Then S is a ring under usual addition and multiplication of matrices with identity $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ as per both the approaches. Further S is a subring of M as per the second approach however it is not a subring of M as per the first approach. This is because $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity of M and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is the identity of S and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. In the same line as above if we take $S = \left\{ m\begin{pmatrix} a & b \\ c & d \end{pmatrix} : m > 1, a \in Z \right\}$ then S is a subring of M as per the second approach but it is not a subring of M as per the first approach. We shall conclude this section by giving one more example.

Let $M_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Q \right\}$ where Q denotes the set of all rational numbers. It is easy to see that M_1 is a ring under usual addition and multiplication of matrices and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the multiplicative identity of this ring.

Let $S = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in Q \right\}$. Then S is a ring under usual addition and multiplication of matrices with identity $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. This ring is a subring of M_1 as per the second approach. We see that as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ therefore S is not a subring of M_1 as per the first approach.

The ring of matrices provides various interesting examples of rings. Let $S_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in Q \right\}$, $D_1 = \left\{ \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} : c, d \in Q \right\}$ then S_1 and D_1 both are subrings of M_1 (as per both the approaches) such that $S_1 \subseteq D_1 \subseteq M_1$. Moreover S_1 is a commutative ring whose every non-zero element has multiplicative inverse and hence it provides an example of a ring which is a field of matrices under usual addition and multiplication of matrices with the usual identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

It is seen that a larger ring with identity may contain a ring with identity which is different from the identity of the larger ring and hence it fails to be a subring of the larger ring according to the first approach. Also there are some rings (e. g. mZ given above) which do not have multiplicative identity and due to this reason only they fail to be a subring of the larger ring (in which they are supposed to be contained) according to the first approach.

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