International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jpms.htm
2014 Vol. 4 (4) October-December, pp. 19-33/Kalaimaran
Research Article

SOME NEW GEOMETRICAL THEOREMS ON VARIOUS PARAMETERS OF AN ELLIPSE

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ABSTRACT
Ellipse is one of the conic sections. It is an elongated circle. It is the locus of a point that moves in such a way that the ratio of its distance from a fixed point (called Focus) to its distance from a fixed line (called Directrix) equals to constant ‘e’ which is less than or equal to unity. According to the Keplar’s lawof Planetary Motion, the ellipse is very important in geometry and the field of Astronomy, since every planet is orbiting its star in an elliptical path and its star is as one of the foci. The objective of the research article is to establish few new theorems for mathematical properties related to various parameters of an ellipse. These five new properties have been defined with necessary derivations of equations and appropriate drawings. The mathematical expressions of each theorem have also been given. These theorems may be very useful to the research scholars for reference to the higher level research works.

Keywords: *Ellipse, Conic Sections, Keplar’s Law of Planetary Motion, Eccentricity of Ellipse, Foci, Directrix, Semi-major Axis, Semi-minor Axis*

INTRODUTION
An *ellipse* (Eric, 2003) is the set of all points in a plane such that the sum of the distances from two fixed points called *foci* (Eric, 2003) is a given constant. Things that are in the shape of an ellipse are said to be elliptical. In the 17th century, a mathematician Mr. Johannes Kepler discovered that the orbits along which the planets travel around the Sun are ellipses with the Sun at one of the foci, in his First law of planetary motion. Later, Isaac Newton explained that this as a corollary of his law of universal gravitation. One of the physical properties of ellipse is that sound or light rays emanating from one focus will reflect back to the other focus. This property can be used, for instance, in medicine. A point inside the ellipse which is the midpoint of the *line* (Eric, 2003) linking the two foci is called centre. The longest and shortest diameters of an ellipse are called *Major axis* (Eric, 2003) and *Minor axis* (Eric, 2003) respectively. The two points that define the ellipse is called foci. The *eccentricity* (Eric, 2003), of an ellipse, usually denoted by ε or e, is the ratio of the distance between the two foci to the length of the major axis. A line segment linking any two points on an ellipse is called *chord* (Eric, 2003). A straight line passing an ellipse and touching it at just one point is called *tangent* (Eric, 2003).
A straight line which passing through the centre of ellipse is called *diameter* (Eric, 2003). A straight line that passing through the centre of the parallel lines to the diameter ellipse is called *conjugate diameter* (Clapham and Nicholson 2009). The distances between foci and a point are called *focal distances* (Eric, 2003) of that point.
There are some existing properties of ellipse such as "sum of focal distances is a constant" (called focal constant), reflection property of focus of the ellipse, etc. Now an attempt has been made by the Author to develop mathematical theorems regarding the various parameters such as focal distance, semi-diameters, semi-conjugate diameters, semi-major axis, semi-minor axis, tangent and intercepts of the tangent. The new properties have been derived mathematically.
In this article, step by step derivations have been presented. The geometrical properties, which have been defined in this research article is very useful for those doing research work or further study in the field of Astronomy, Conics and Euclidean geometry, since this is also one of the important properties of an ellipse.
This may also be very important to scientists who work in the field of *Optics* (John, 2003).

MATERIALS AND METHODS

Theorem- 1

Suppose a tangent is drawn at point 'P' anywhere on an ellipse (Ref. figure1), points 'T' & 'S' are the intersection of tangent with x-axis and y-axis. Point 'O' is the centre, 'OA' & 'OB' are the semi-major axis & semi-minor axis of the ellipse respectively and $\angle OTS = \alpha°, \angle TOP = \beta°$ and $\angle CBO = \gamma°$ then $\tan(\alpha°) \times \tan(\beta°) = \tan^2(\gamma°)$ is an arbitary constant and its value is equal to $\frac{b^2}{a^2}$

Derivation of Equations and Proof of the Theorem- 1

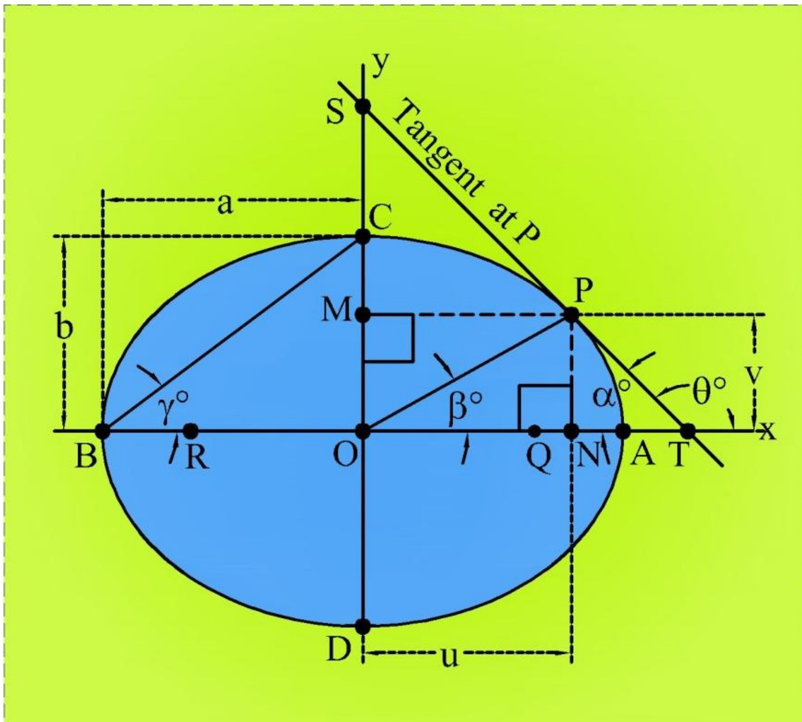

Figure 1: An Ellipse with tangent and semi diameter

In figure 1, 'P' is the point anywhere on an ellipse. Point 'O' is the geometric centre of the ellipse. 'Q' and 'R' are the foci of the ellipse. Line 'TS' is the tangent to the ellipse at 'P', 'T' and 'S' are the meeting point of the tangent with 'x' and 'y' axes respectively, 'N' and 'M' are the projection of point 'P' on 'x' and 'y' axes respectively, therefore 'PN' ⊥ 'OT', 'PM' ⊥ 'OS'. Let, ∠OTS be $\alpha°$. The parameters 'a' and 'b' are the semi-major axis and semi-minor axis of the ellipse respectively.

We know that

(i) The equation of ellipse (Hazewinkel, 1987)

$$\Rightarrow \left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1 \quad ---------------------------------- [1.1]$$

(ii) The equation of tangent of an ellipse (Bali, 2005) at P (x_1, y_1)

$$\Rightarrow \left(\frac{xx_1}{a^2}\right) + \left(\frac{yy_1}{b^2}\right) = 1 \quad ---------------------------------- [1.2]$$

(ii) The equation for slope of tangent of an ellipse (Bali, 2005) at P (x_1, y_1)

$$\Rightarrow\ m = \tan\theta° - \left(\frac{b^2 x_1}{a^2 y_1}\right) ---------------------------- [1.3]$$

In figure 1, points A, B, C and D are extremities of an ellipse. Point O is the centre of ellipse. $\mathrm{P}(x_1,\ x_1)$ is the point anywhere on ellipse. Points 'T' and 'S' are the intersection of tangent drawn at point P and x-axis and y-axis respectively. N is the projection of point P on x-axis.

Let, $ON = x_1 = u, \quad PN = y_1 = v, \quad \angle OTP = \alpha°, \quad \angle TOP = \beta°$ and $\angle CBO = \gamma°$

Let, $\angle XTP = \theta°$ therefore $\angle OTP = \alpha° = 180 - \theta°$

Therefore, $\tan\alpha° = \tan(180 - \theta°) = -\tan\theta°$

$$\therefore \tan\alpha° = \left(\frac{b^2 x_1}{a^2 y_1}\right)$$

$$\therefore \tan\alpha° = \frac{b^2}{a^2}\left(\frac{u}{v}\right) ---------------------- [1.4]$$

In right-angled triangle ONP,

$$\tan\beta° = \frac{PN}{ON} = \frac{v}{u}$$

$$\therefore \tan\beta° = \frac{v}{u} ------------------------[1.5]$$

Substituting [1.5] in [1.4], we get

$$\tan\alpha° = \frac{b^2}{a^2}\left(\frac{1}{\tan\beta°}\right)$$

$$\text{Therefore, } (\tan\alpha°)\times(\tan\beta°) = \left(\frac{b}{a}\right)^2 ------------ [1.6]$$

In right-angled triangle BOC,

$$\tan\gamma° = \frac{OC}{OB}$$

$$\therefore \tan\gamma° = \frac{b}{a} --------------------[1.7]$$

Substituting [1.7] in [1.6], we get

$$\tan\alpha° \times \tan\beta° = \tan^2\gamma° \text{ is an arbitrary constant} ----------------- [1.8]$$

The above equation 1.8 is the mathematical form of the property.

Theorem- 2

If set of conjugate focal distances are drawn from any point 'P' on ellipse and another line is drawn from geometric centre of ellipse 'O' to point 'P' [Ref. figure 2], then $\cot\beta° - \cot\alpha° = 2\cot\varphi°$

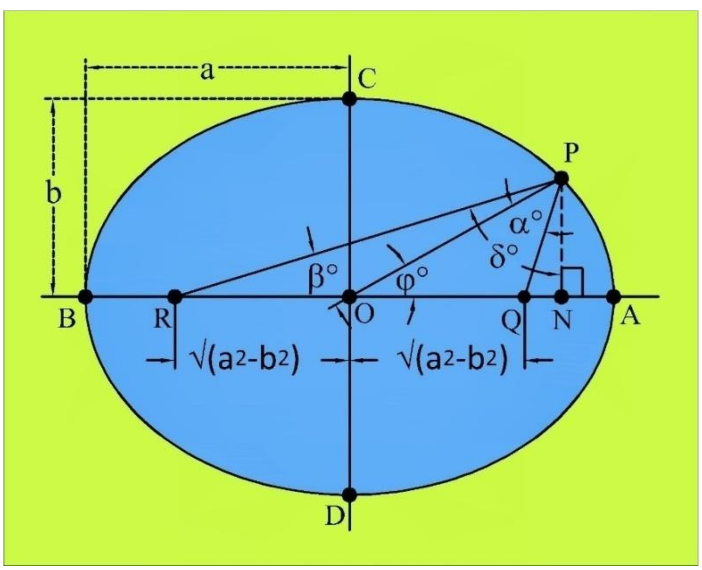

Figure 2: An Ellipse with focal distances and semi diameter

Derivation of Equations and Proof of the Theorem- 2

In figure 2, points A, B, C and D are extremities of an ellipse. O is the centre of ellipse. $P(x_1,\ x_1)$is the point anywhere on ellipse. Points Q and R are the foci of the ellipse. N is the projection of point P on x-axis.

Let, $ON = x_1 = u, \quad PN = y_1 = v, \quad \angle OPQ = \alpha°, \quad \angle OPR = \beta°$ and $\angle OPQ = \varphi°$

Referring Figure 2, O is the geometric centre of the ellipse, Q is one of the foci, P is one of the points on ellipse and N is the projection of point P on OA.

Let, $\angle QPN = \mu°;\ \angle OPQ = \alpha°$

We know, $\angle OPN + \angle NOP = 90°$

$\angle OPN = \angle OPQ + \angle QPN$

Therefore, $\mu° + \alpha° + \varphi° = 90° - [2.1]$

$$\text{In right} - \text{angled triangle ONP}, \qquad \tan\mu° = \frac{r\cos\varphi° - \sqrt{a^2 - b^2}}{r\sin\varphi°}$$

$$\therefore \mu° = \tan^{-1}\left(\frac{r\cos\varphi° - \sqrt{a^2 - b^2}}{r\sin\varphi°}\right)$$

Substituting above in eqn. 2.1, we get

$$\left\{\tan^{-1}\left(\frac{r\cos\varphi° - \sqrt{a^2 - b^2}}{r\sin\varphi°}\right)\right\} + \alpha° = (90 - \varphi°)$$

$$\therefore \alpha° = (90 - \varphi°) - \tan^{-1}\left(\frac{r\cos\varphi° - \sqrt{a^2 - b^2}}{r\sin\varphi°}\right) - - - - - - - - - - - - - - [2.2]$$

$$\therefore \tan\alpha° = \frac{\tan(90 - \varphi°) - \left[\left(r\cos\varphi° - \sqrt{a^2 - b^2}\right)/r\sin\varphi°\right]}{1 + \tan(90 - \varphi°)\left[\left(r\cos\varphi° - \sqrt{a^2 - b^2}\right)/r\sin\varphi°\right]}$$

$$= \left\{\frac{r\cot\varphi°\sin\varphi° - r\cos\varphi° + \sqrt{a^2 - b^2}}{r\sin\varphi°}\right\} \times \left\{\frac{r\sin\varphi°}{r\sin\varphi° + \left[(90 - \varphi°)\left(r\cos\varphi° - \sqrt{a^2 - b^2}\right)\right]}\right\}$$

$$= \frac{r\ \cos\varphi° - r\cos\varphi° + \sqrt{a^2 - b^2}}{r\sin\varphi° + \left(\cot\varphi°\, r\cos\varphi° - \cot\varphi°\sqrt{a^2 - b^2}\right)}$$

$$= \frac{\sqrt{a^2 - b^2}\sin\varphi°}{r\sin^2\varphi° + r\cos^2\varphi° - \cos\varphi°\sqrt{a^2 - b^2}}$$

$$= \frac{\sqrt{a^2 - b^2}\sin\varphi°}{r(\sin^2\varphi° + \cos^2\varphi°) - \cos\varphi°\sqrt{a^2 - b^2}}$$

$$\therefore \tan\alpha° = \frac{\sqrt{a^2 - b^2}\sin\varphi°}{r - \sqrt{a^2 - b^2}\cos\varphi°}$$

$$\cot\alpha° = \frac{r - \sqrt{a^2 - b^2}\cos\varphi°}{\sqrt{a^2 - b^2}\sin\varphi°} - [2.4]$$

Referring figure 2, 'O' is the geometric centre of the ellipse, 'R' is another focus, 'P' is one of the points on ellipse and 'N' is the projection of point 'P' on OA.

Let, $\angle RPN = \delta°;\ \angle OPR = \beta°$

$\delta° - \beta° = (90° - \varphi°)$

$\therefore \beta° = \delta° - (90° - \varphi°) - [2.5]$

$$\text{In Right traingle ENP}, \qquad \tan\delta° = \frac{\sqrt{a^2 - b^2} + r\cos\varphi°}{r\sin\varphi°} - - - - - - - - - - - - - [2.6]$$

$$\therefore \delta° = \tan^{-1}\left(\frac{\sqrt{a^2 - b^2} + r\cos\varphi°}{r\sin\varphi°}\right)$$

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Substituting above in eqn. [2.5], we get

$$\beta^\circ = \tan^{-1}\left(\frac{\sqrt{a^2-b^2}+r\cos\varphi^\circ}{r\sin\varphi^\circ}\right)-(90-\varphi^\circ) ------------------ [2.7]$$

$$\beta^\circ = \tan\left[\tan^{-1}\left(\frac{\sqrt{a^2-b^2}+r\cos\varphi^\circ}{r\sin\varphi^\circ}\right)-(90-\varphi^\circ)\right]$$

$$\therefore\ \tan\beta^0 = \left[\left(\frac{r\cos\varphi^\circ+\sqrt{a^2-b^2}}{r\sin\varphi^\circ}\right)-\tan(90-\varphi^\circ)\right] \div \left[1+\left\{\left(\frac{r\cos\varphi^\circ+\sqrt{a^2-b^2}}{r\sin\varphi^\circ}\right)\times\tan(90-\varphi^\circ)\right\}\right]$$

$$\therefore \tan\beta^\circ = \left[\frac{r\cos\varphi^\circ+\sqrt{a^2-b^2}-(\cot\varphi^\circ\times r\sin\varphi^\circ)}{r\sin\varphi^\circ}\right]\div\left[\frac{r\sin\varphi^\circ+\left(r\cos\varphi^\circ+\sqrt{a^2-b^2}\right)\cot\varphi^\circ}{r\sin\varphi^\circ}\right]$$

$$\therefore \tan\beta^\circ = \left[\frac{r\cos\varphi^\circ+\sqrt{a^2-b^2}-(\cot\varphi^\circ\times r\sin\varphi^\circ)}{r\sin\varphi^\circ}\right]\times\left[\frac{r\sin\varphi^\circ}{r\sin\varphi^\circ+\left(r\cos\varphi^\circ+\sqrt{a^2-b^2}\right)\cot\varphi^\circ}\right]$$

$$\therefore \tan\beta^\circ = \left(\frac{r\cos\varphi^\circ+\sqrt{a^2-b^2}-r\cos\varphi^\circ}{r\sin^2\varphi^\circ+r\cos^2\varphi^\circ+\sqrt{a^2-b^2}\cos\varphi^\circ}\right)$$

$$\therefore \tan\beta^\circ = \frac{\sqrt{a^2-b^2}\sin\varphi^\circ}{r(\sin^2\varphi^\circ+\cos^2\varphi^\circ)+\sqrt{a^2-b^2}\cos\varphi^\circ}$$

$$\tan\beta^0 = \frac{\sqrt{a^2-b^2}\sin\varphi^\circ}{r+\sqrt{a^2-b^2}\cos\varphi^\circ}$$

$$\therefore \cot\beta^\circ = \frac{r+\sqrt{a^2-b^2}\cos\varphi^\circ}{\sqrt{a^2-b^2}\sin\varphi^\circ} ------------------------ [2.8]$$

Subtracting eqn. [2.4] from [2.8]

$$\cot\beta^\circ-\cot\alpha^\circ = \left(\frac{r+\sqrt{a^2-b^2}\cos\varphi^\circ}{\sqrt{a^2-b^2}\sin\varphi^\circ}\right)-\left(\frac{r-\sqrt{a^2-b^2}\cos\varphi^\circ}{\sqrt{a^2-b^2}\sin\varphi^\circ}\right)$$

$$\therefore \cot\beta^\circ-\cot\alpha^\circ = \frac{r+\sqrt{a^2-b^2}\cos\varphi^\circ-r+\sqrt{a^2-b^2}\cos\varphi^\circ}{\sqrt{a^2-b^2}-\sin\varphi^\circ}$$

$$\therefore \cot\beta^\circ-\cot\alpha^\circ = \frac{2\sqrt{a^2-b^2}\cos\varphi^\circ}{\sqrt{a^2-b^2}\sin\varphi^\circ}$$

$$\therefore \cot\beta^\circ-\cot\alpha^\circ = \left(\frac{2\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}}\right)\times\cot\varphi^\circ$$

$$\therefore \cot\beta^\circ-\cot\alpha^\circ = 2\cot\varphi^\circ --------------------- [2.9]$$

The above eqn. 2.9 is the mathematical expression for one of the properties of tangent of ellipse.

Theorem- 3

Referring figure 3, suppose a tangent is drawn to ellipse at point ‘P’ anywhere on ellipse, points ‘T’&‘S’ are the x-intercepts (Eric, 2003) and y-intercept (Eric, 2003) of tangent respectively with ‘X’&‘Y-axis respectively, ‘O’ is centre of the ellipse, OA & OB are semi-major axis & semi-minor axis of the ellipse and $PT = i,\ PS = j,\ OP = r,\ OS = h$ and $OT = d$, then $(d^2+j^2)-(h^2+i^2) = 2(a^2-b^2)$. It is an arbitrary constant.

Derivation of Equations and Proof of the Theorem- 3

We know that

(i) The *canonical equation of ellipse* (Borowski and Borwein, 1991)

$$\Rightarrow \left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1 \quad ------------------------------ [3.1]$$

(ii) The equation of tangent of an ellipse (Bali, 2005) at P (x_1, y_1)

$$\Rightarrow \left(\frac{xx_1}{a^2}\right) + \left(\frac{yy_1}{b^2}\right) = 1 \quad ------------------------------ [3.2]$$

(ii) The equation for slope of tangent of an ellipse (Bali, 2005) at P (x_1, y_1)

$$\Rightarrow \ m = \tan\theta° = -\left(\frac{b^2x_1}{a^2y_1}\right) ------------------------ [3.3]$$

In figure 3, points A, B, C and D are extremities of an ellipse. Point O is the centre of ellipse. $P(x_1,\ y_1)$ is a point anywhere on ellipse. T and S are the intersection point of tangent drawn at point P on x-axis and y-axis respectively. N and M are the projection of point P on x-axis and y-axis respectively. Let the coordinates of 'P' is (u, v). Therefore, $ON = MP = x_1 = u$, $NP = ON = y_1 = v$, OT is the $x -$ intercept of the tangent $= d$, OS is the $y -$ intercept of the tangent $= h$,

$\angle OTP = \alpha°$, $\angle TOP = \beta°$

$\angle XTP = \theta°$ therefore $\angle OTP = \alpha° = 180 - \theta°$

The equation of tangent of ellipse $\Rightarrow \left(\frac{xx_1}{a^2}\right) + \left(\frac{yy_1}{b^2}\right) = 1$

We know that co-ordinates of 'T' is (d, o)

Substituting, $x = d$, $x_1 = u$ and $y = 0$ in above eqn. [3.2], we get

$$\frac{ud}{a^2} = 1$$

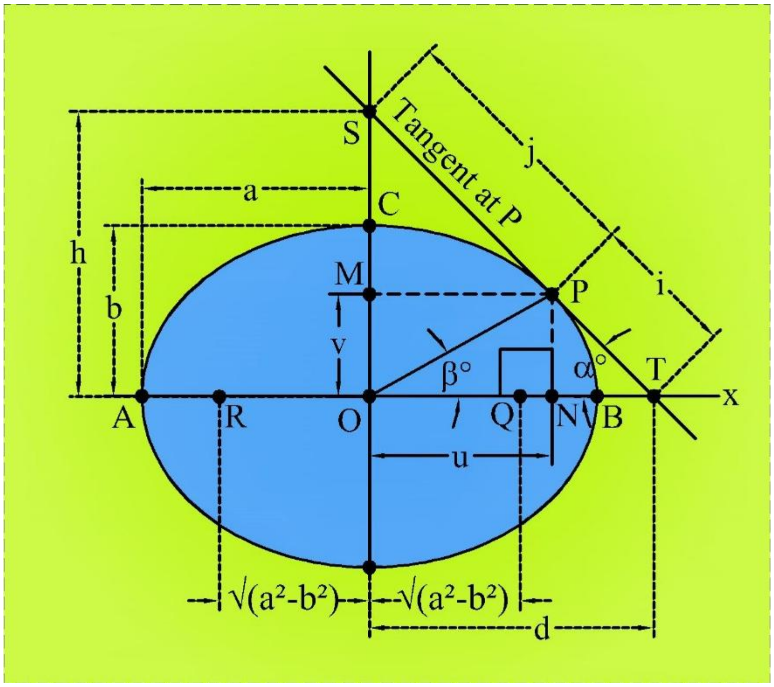

Figure 3: An Ellipse with tangent, semi diameter, x-intercept and y-intercept of the tangent

$$\therefore ON = u = \frac{a^2}{d} ------------------------ [3.4]$$

Substituting, eqn. [3.4] in [3.1], we get

$$PN = \left(\frac{b}{d}\right)\sqrt{d^2 - a^2}$$

$\therefore v = \left(\frac{b}{d}\right)\sqrt{d^2 - a^2}$ ------------------------ [3.5]

In right-angled triangle ONP,

$OP^2 = ON^2 + PN^2$

Substituting eqns. [3.4] & [3.5] in above eqn.

$\therefore OP^2 = r^2 = \frac{1}{d^2}[a^4 + b^2(d^2 - a^2)]$ -----------------[3.6]

In right-angled triangle TNP,

$PT^2 = NT^2 - PN^2$

This can be written as $PT^2 = (OT - ON)^2 + PN^2$

$\therefore i^2 = (d - u)^2 + v^2$

Substituting eqn. [3.4] & [3.5] in above eqn.

$$i^2 = \left(d - \frac{a^2}{d}\right)^2 + \left[\left(\frac{b}{d}\right)\sqrt{d^2 - a^2}\right]^2$$

$\therefore PT^2 = i^2 = \frac{1}{d^2}[a^4 + d^4 + b^2(d^2 - a^2) - 2a^2d^2]$ ---------------- [3.7]

From eqns. [3.6] & [3.7]

$$i^2 - r^2 = \frac{1}{d^2}(d^4 - 2a^2d^2)$$

$$\therefore i^2 - r^2 = \frac{d^2(d^2 - 2a^2)}{d^2}$$

$$\therefore i^2 - r^2 = d^2 - 2a^2$$

This can be written as

$\therefore i^2 - r^2 + 2a^2 = d^2$ --------------------------[3.8]

Similarly, $j^2 - r^2 + 2b^2 = h^2$ --------------------------[3.9]

Subtracting [3.9] from [3.8],

$$\begin{array}{r} i^2 - r^2 + 2a^2 = d^2 \\ (-)\ j^2 - r^2 + 2b^2 = h^2 \\ \hline i^2 - j^2 + 2a^2 - 2b^2 = d^2 - h^2 \end{array}$$

This can be written as

$(i^2 - j^2) + 2(a^2 - b^2) = (d^2 - h^2)$

This can be rewritten as

$(d^2 + j^2) - (h^2 + i^2) = 2(a^2 - b^2)$ ----------------------[3.10]

$2(a^2 - b^2)$is an *arbitrary constant* (Borowski E.J & Borwein J.M)

We already know that $OQ = OR = \sqrt{a^2 - b^2}$

Therefore, $OQ = \sqrt{a^2 - b^2}$

Therefore, $OQ^2 = a^2 - b^2$

Therefore, $2 \times OQ^2 = 2(a^2 - b^2)$

$(d^2 + j^2) - (h^2 + i^2) = 2(a^2 - b^2) = 2 \times OQ^2$ ---------------- [3.10]

This mathematical expression 3.10 is one of the properties of tangent of ellipse.

Theorem- 4

If a tangent drawn at 'P' anywhere on ellipse, 'JQ' & 'KR' are the perpendiculars (Eric, 2003) from the foci upon any tangent and 'LP' is the line segment of normal (Eric, 2003) in between tangent and major axis, then the product of (i) addition of JQ and KR and (ii) LP is always a constant and it is equal to twice the square of minor axis. It can be written in mathematical form as:

$(JQ + KR) \times LP = 2b^2$is an arbitrary constant (Borowski and Borwein, 1991)

Derivation of Equations and Proof of the Theorem-4

Referring figure 4,

Let, 'O' is the centre of ellipse, 'a' & 'b' are the semi major and semi minor axis, 'Q' and 'R' are the foci of the ellipse, TS is the tangent drawn at point 'P' on the ellipse, 'N' is the projection of point 'P' on major axis & 'M' is the projection of point 'P' on minor axis and ∠OTS be the θ°. L is the intersection of normal drawn at P and x-axis.

We know the property, $\mathrm{ON} \times \mathrm{OT} = \mathrm{a}^2$ Refer (Bali, 2005)

$$\therefore \mathrm{ON} = \frac{\mathrm{a}^2}{\mathrm{OT}} \text{------------------------} [4.1]$$

In figure, $\mathrm{NT} = \mathrm{OT} - \mathrm{ON}$

Substituting eqn. 4.1

$$\mathrm{NT} = \mathrm{OT} - \frac{\mathrm{a}^2}{\mathrm{OT}}$$

$$\therefore \mathrm{NT} = \frac{\mathrm{OT}^2 - \mathrm{a}^2}{\mathrm{OT}} \text{------------------------} [4.2]$$

We know the property, $\mathrm{OM} \times \mathrm{OS} = \mathrm{b}^2$ Refer (Bali, 2005)

Substituting, $\mathrm{OM} = \mathrm{NP}$ in above eqn. [Referring figure 4, $\mathrm{OM} = \mathrm{NP}$]

$$\therefore \mathrm{NP} \times \mathrm{OS} = \mathrm{b}^2 \text{------------------------} [4.3]$$

In right angled triangle TOS, $\tan\theta° = \frac{\mathrm{OS}}{\mathrm{OT}}$

$\therefore \mathrm{OS} = \mathrm{OT} \times \tan\theta°$ and substituting in eqn.3

$$\mathrm{NP} \times (\mathrm{OT} \times \tan\theta°) = \mathrm{b}^2$$

$$\therefore \mathrm{NP} = \frac{\mathrm{b}^2}{\mathrm{OT} \times \tan\theta°} \text{------------------------} [4.4]$$

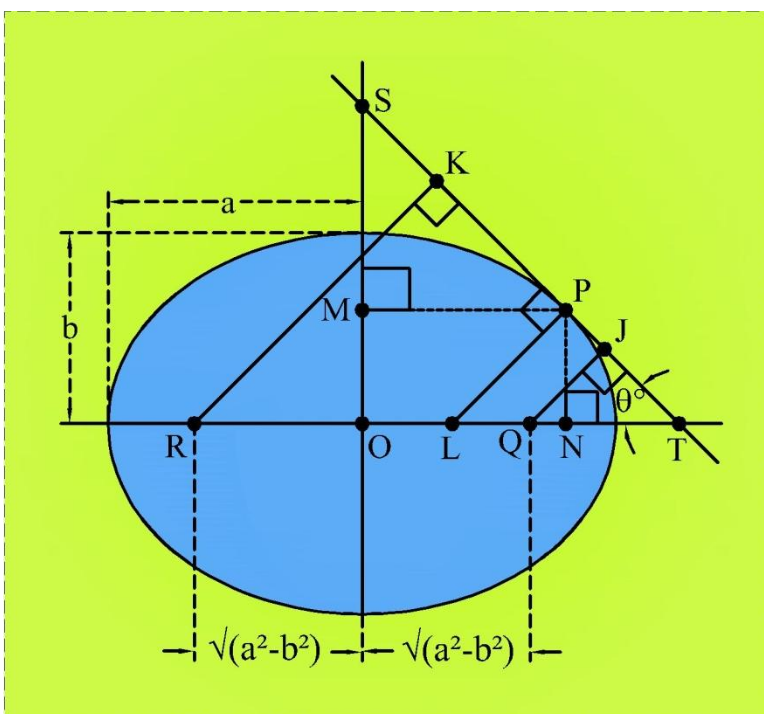

Figure 4: An Ellipse with tangent, normal and perpendiculars of foci to the tangent

In right angled triangle PNT, $\tan\theta° = \frac{\mathrm{NP}}{\mathrm{NT}}$

$$\therefore \mathrm{NP} = \mathrm{NT} \times \tan\theta° \text{ ------------------------ } [4.5]$$

(i) Determination of OT

In right angled triangle TNP,

$$\tan\theta° = \frac{\mathrm{NP}}{\mathrm{NT}}$$

Substituting eqn.4.4 and 4.2 in above eqn. we get

$$= \left(\frac{b^2}{\mathrm{OT} \times \tan\theta°}\right) \times \left(\frac{\mathrm{OT}}{\mathrm{OT}^2 - a^2}\right)$$

$$\tan\theta° = \frac{b^2}{\tan\theta° \, (\mathrm{OT}^2 - a^2)}$$

$$\therefore \tan^2\theta° = \frac{b^2}{\mathrm{OT}^2 - a^2}$$

$$\therefore \mathrm{OT}^2 - a^2 = \frac{b^2}{\tan^2\theta°}$$

$$\therefore \mathrm{OT}^2 = \frac{b^2}{\tan^2\theta°} + a^2$$

$$\therefore \mathrm{OT}^2 = \frac{b^2 + a^2\tan^2\theta°}{\tan^2\theta°}$$

$$\text{Substituting}, \tan\theta° = \frac{\sin\theta°}{\cos\theta°}$$

$$\mathrm{OT}^2 = \frac{a^2\sin^2\theta° + b^2\cos^2\theta°}{\sin^2\theta°}$$

$$\mathrm{OT} = \frac{\sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°}}{\sin\theta°} \text{ ---------------------- } [4.6]$$

(ii) Determination of NP

$$\text{Referring eqn. 4.4,} \qquad \mathrm{NP} = \frac{b^2}{\mathrm{OT} \times \tan\theta°}$$

Substituting eqn. 4.5in above, we get

$$\mathrm{NP} = b^2 \div \left[\left(\frac{\sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°}}{\sin\theta°}\right) \times \tan\theta°\right]$$

$$\text{Put,} \qquad \tan\theta° = \frac{\sin\theta°}{\cos\theta°} \text{ and simplifying, we get}$$

$$\mathrm{NP} = \frac{b^2\cos\theta°}{\sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°}} \text{ ---------------------- } [4.7]$$

(iii) Determination of TQ & TR

$$\text{Referring figure 4,} \qquad \mathrm{TQ} = \mathrm{OT} - \mathrm{OQ}$$

$$\therefore \mathrm{TQ} = \mathrm{OT} - \sqrt{a^2 - b^2}$$

Substituting, eqn. [4.6] in above eqn., we get

$$\mathrm{TQ} = \frac{\sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°}}{\sin\theta°} - \sqrt{a^2 - b^2}$$

$$\therefore \mathrm{TQ} = \frac{\sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°} - \sin\theta°\sqrt{a^2 - b^2}}{\sin\theta°} \text{ ---------------- } [4.8]$$

Similarly,

$$\mathrm{TR} = \frac{\sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°} + \sqrt{a^2\sin^2\theta° - b^2\sin^2\theta°}}{\sin\theta°} \text{ ------------------ } [4.9]$$

(iv) Determination of JQ

International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jpms.htm
2014 Vol. 4 (4) October-December, pp. 19-33/Kalaimaran
Research Article

Let, pointsQ and R are the foci of the ellipse. Points J and K are the intersection of perpendicular drawn from the foci Q and R respectively upon tangent TS.

In right angled triangle TJQ,

$$\mathrm{JQ} = \mathrm{TQ} \times \sin\theta°$$

$$\mathrm{JQ} = \left[\frac{\sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°} - \sin\theta°\sqrt{a^2 - b^2}}{\sin\theta°}\right] \times \sin\theta°$$

$$\therefore \mathrm{JQ} = \sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°} - \sin\theta°\sqrt{a^2 - b^2} \text{ ------------------- } [4.10]$$

(v) Determination of KR

In right angled triangle TKR,

$$\mathrm{KR} = \mathrm{TR} \times \sin\theta°$$

Substituting eqn. [4.9] in above

$$\mathrm{KR} = \left[\frac{\sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°} + \sin\theta°\sqrt{a^2 - b^2}}{\sin\theta°}\right] \times \sin\theta°$$

$$\therefore \mathrm{KR} = \sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°} + \sin\theta°\sqrt{a^2 - b^2} \text{ ----------------- } [4.11]$$

(vi) Determination of LP

In right angled triangle, LNP

$$\cos\theta° = \frac{\mathrm{NP}}{\mathrm{LP}}$$

$$\therefore \mathrm{LP} = \frac{\mathrm{NP}}{\cos\theta°}$$

Substituting eqn. [4.7] in above eqn. we get

$$\mathrm{LP} = \left(\frac{b^2\cos\theta°}{\sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°}}\right) \times \left(\frac{1}{\cos\theta°}\right)$$

$$\therefore \mathrm{LP} = \frac{b^2}{\sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°}} \text{ ------------------------- } [4.12]$$

(vii) Final proof for the theorem

Adding 4.10 and 4.11, we get

$$\mathrm{JQ} + \mathrm{KR} = \left[\left(\sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°} - \sin\theta°\sqrt{a^2 - b^2}\right) + \left(\sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°} + \sin\theta°\sqrt{a^2 - b^2}\right)\right]$$

$$\therefore \mathrm{JQ} + \mathrm{KR} = 2\sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°} \text{ ------------------------- } [4.13]$$

Multiplying eqn. [4.13] by [4.12], we get

$$(\mathrm{JQ} + \mathrm{KR}) \times \mathrm{LP} = \left(2\sqrt{a^2\sin^2\theta° + b^2\cos^2\theta°}\right) \times \left(\frac{b^2}{\sqrt{a^2\sin^2\theta° + b^2\cos^2}}\right)$$

$$\therefore (\mathrm{JQ} + \mathrm{KR}) \times \mathrm{LP} = 2b^2 \text{ ------------------------- } [4.14]$$

The eqn. [4.14] is the mathematical form of the theorem.

Theorem- 5

If a normal (Eric, 2003) drawn at point 'P' anywhere on ellipse, 'GQ', 'RE', are the perpendiculars (Eric, 2003) from the foci (Eric, 2003). 'OK' is the perpendiculars from ellipse centre 'O' upon the normal and 'PT' is the part of tangent (Borowski and Borwein, 1991) in between tangent point and major axis, then the product of 'GQ' & 'RE' is always equal to the product of 'PT' & 'OK'. It can be written in mathematical form as $\mathrm{GQ} \times \mathrm{RE} = \mathrm{PT} \times \mathrm{OK}$

Derivation of Equations and Proof of the Property

Referring figure 5, Let, 'O' is the centre of ellipse, 'a' & 'b' are the semi major and semi minor axis, 'Q' & 'R' are the foci of the ellipse, 'TS' is the tangent & 'PM' is the normal drawn at point 'P' on the ellipse, 'N' and M are the projection of point 'P' on x-axis and y-axis respectively.

'a' is the semi-minor axis (Borowski and Borwein, 1991) and 'b' is the semi- minor axis (Borowski and Borwein, 1991) and let ∠OTS be the θ°.

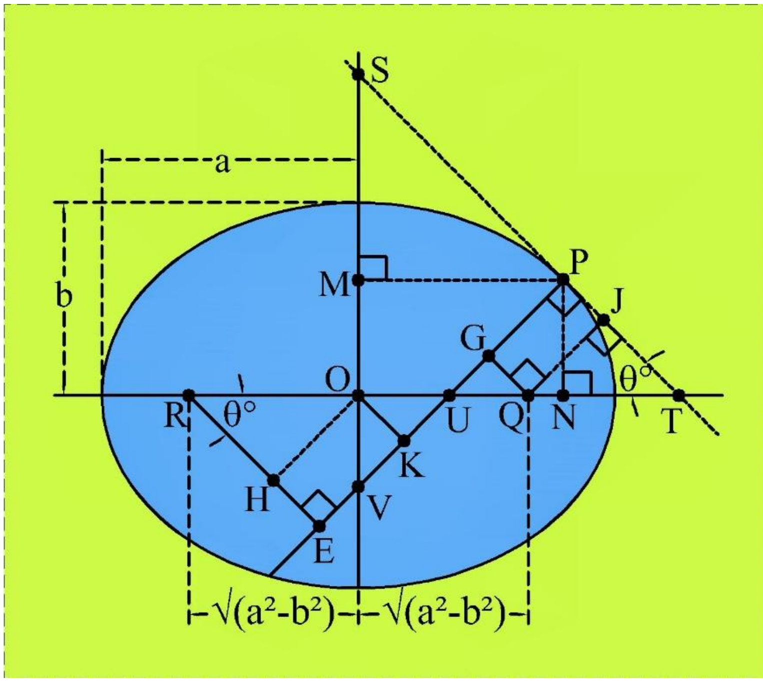

Figure 5: A normal line passing through point P of an ellipse

(i) Determination of TQ

Referring figure 5, $\mathrm{TQ} = \mathrm{OT} - \mathrm{OQ}$

$\therefore \mathrm{TQ} = \mathrm{OT} - \sqrt{\mathrm{a}^2 - \mathrm{b}^2}$

Substituting, eqn. [4.6] in above eqn., we get

$$\mathrm{TQ} = \frac{\sqrt{\mathrm{a}^2 \sin^2\theta + \mathrm{b}^2\cos^2\theta}}{\sin\theta} - \sqrt{\mathrm{a}^2 - \mathrm{b}^2}$$

$$\therefore \mathrm{TQ} = \frac{\sqrt{\mathrm{a}^2\sin^2\theta + \mathrm{b}^2\cos^2\theta} - \sin\theta\sqrt{\mathrm{a}^2 - \mathrm{b}^2}}{\sin\theta} \quad \text{------------ [5.1]}$$

(ii) Determination of JQ

Let, 'Q' & 'R' are the foci of the ellipse. 'JQ' is the parallel line drawn from the foci 'Q' upto the tangent.

$\mathrm{JQ} = \mathrm{TQ} \times \sin\theta$

$$\mathrm{JQ} = \left[\frac{\sqrt{\mathrm{a}^2\sin^2\theta + \mathrm{b}^2\cos^2\theta} - \sin\theta\sqrt{\mathrm{a}^2 - \mathrm{b}^2}}{\sin\theta}\right] \times \sin\theta$$

$$\therefore \mathrm{JQ} = \sqrt{\mathrm{a}^2\sin^2\theta + \mathrm{b}^2\cos^2\theta} - \sin\theta\sqrt{\mathrm{a}^2 - \mathrm{b}^2} \quad \text{------------------ [5.2]}$$

(iii) Determination of PT

PT is the tangent at point 'P'. In right angled triangle TNP,

$$\frac{\mathrm{NT}}{\mathrm{PT}} = \cos\theta$$

$$\therefore \mathrm{PT} = \frac{\mathrm{NT}}{\cos\theta}$$

Substituting eqn. 4.2 in above eqn. we get

$$\mathrm{PT} = \frac{\mathrm{OT}^2 - a^2}{\mathrm{OT}} \times \frac{1}{\cos\theta}$$

Substituting eqn. 4.6 in above eqn. we get

$$\mathrm{PT} = \left(\frac{\left(\frac{a^2 \sin^2 \theta° + b^2 \cos^2 \theta°}{\sin^2 \theta°}\right) - a^2}{\frac{\sqrt{a^2 \sin^2 \theta° + b^2 \cos^2 \theta°}}{\sin \theta°}}\right) \times \frac{1}{\cos\theta}$$

$$\mathrm{PT} = \left(\frac{a^2 \sin^2 \theta° + b^2 \cos^2 \theta° - a^2 \sin^2 \theta°}{\sin^2 \theta°}\right) \times \left(\frac{\sin\theta°}{\sqrt{a^2 \sin^2 \theta° + b^2 \cos^2 \theta°}}\right) \times \frac{1}{\cos\theta}$$

Simplifying

$$\mathrm{PT} = \frac{b^2 \cos^2 \theta}{\sin\theta\cos\theta\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}}$$

$$\therefore \mathrm{PT} = \frac{b^2 \cos\theta}{\sin\theta\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}} \quad ------------------[5.3]$$

(iv) Determination of TU

In right angled triangle, TUP

$$\cos\theta = \frac{\mathrm{PT}}{\mathrm{TU}}$$

$$\therefore \mathrm{TU} = \frac{\mathrm{PT}}{\cos\theta}$$

Substituting eqn. [5.3] in above eqn., we get

$$\mathrm{TU} = \left(\frac{b^2 \cos\theta}{\sin\theta\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}}\right) \times \left(\frac{1}{\cos\theta}\right)$$

$$\therefore \mathrm{TU} = \frac{b^2}{\sin\theta\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}} \quad --------------------[5.4]$$

(v) Determination of UQ

Referring figure 5, $\mathrm{UQ} = \mathrm{TU} - \mathrm{TQ}$

Substituting eqn. [5.4] & [5.7] in above

$$\mathrm{UQ} = \left(\frac{b^2}{\sin\theta\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}}\right) - \left(\frac{\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta} - \sin\theta\sqrt{a^2 - b^2}}{\sin\theta}\right)$$

$$\therefore \mathrm{UQ} = \frac{b^2 - \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\left(\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta} - \sin\theta\sqrt{a^2 - b^2}\right)}{\sin\theta\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}}$$

$$\therefore \mathrm{UQ} = \left[\frac{b^2 - (a^2 \sin^2\theta + b^2 \cos^2\theta) + \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\left(\sin\theta\sqrt{a^2 - b^2}\right)}{\sin\theta\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}}\right]$$

$$\therefore \mathrm{UQ} = \left[\frac{b^2 - a^2 \sin^2\theta - b^2 \cos^2\theta + \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\left(\sin\theta\sqrt{a^2 - b^2}\right)}{\sin\theta\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}}\right]$$

$$\therefore \mathrm{UQ} = \left[\frac{b^2(1 - \cos^2\theta) - a^2 \sin^2\theta + \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\left(\sin\theta\sqrt{a^2 - b^2}\right)}{\sin\theta\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}}\right]$$

$$\therefore \mathrm{UQ} = \left[\frac{b^2 \sin^2\theta - a^2 \sin^2\theta + \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\left(\sin\theta\sqrt{a^2 - b^2}\right)}{\sin\theta\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}}\right]$$

$$\therefore \mathrm{UQ} = \left[\frac{\sin^2\theta(b^2 - a^2) + \sin\theta\sqrt{(a^2 - b^2)a^2 \sin^2\theta + b^2 \cos^2\theta}}{\sin\theta\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}}\right]$$

Simplifying, we get

$$UQ = \left[\frac{\sin\theta(b^2 - a^2) + \sqrt{(a^2 - b^2)(\sin^2\theta + b^2\cos^2\theta)}}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right] \qquad ------------[5.5]$$

(vi) Determination of GQ

Let, 'JQ' is the parallel line drawn from the foci 'Q' upto the tangent. In right-angled triangle UGQ,

$GQ = UQ \times \cos\theta$

Substituting eqn. [5.5] in above

$$GQ = \left[\frac{\sin\theta(b^2 - a^2) + \sqrt{(a^2 - b^2)(\sin^2\theta + b^2\cos^2\theta)}}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right] \times \cos\theta$$

$$\therefore GQ = \left[\frac{\sin\theta\cos\theta(b^2 - a^2) + \cos\theta\sqrt{(a^2 - b^2)(\sin^2\theta + b^2\cos^2\theta)}}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right] \qquad -------[5.6]$$

(vii) Determination of OU

Referring figure 5, $OU = OQ - UQ$

Substituting $OQ = \sqrt{a^2 - b^2}$ in above, we get

$$OU = \sqrt{a^2 - b^2} - \left[\frac{\sin\theta(b^2 - a^2) + \sqrt{(a^2 - b^2)(\sin^2\theta + b^2\cos^2\theta)}}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right]$$

Simplifying, we get

$$OU = \frac{(a^2 - b^2)\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \qquad ----------------------[5.7]$$

(viii) Determination of OK

In right angled triangle, OKU

$$\cos\theta = \frac{OK}{OU}$$

$\therefore OK = OU \times \cos\theta$

Substituting eqn. [5.7] in above eqn., we get

$$\therefore OK = \frac{(a^2 - b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \qquad ----------------------[5.8]$$

(ix) Determination of RE

In right angled triangle, RHO

$$\cos\theta = \frac{RH}{OR}$$

$\therefore RH = OR \times \cos\theta$

$\therefore RH = \sqrt{a^2 - b^2} \times \cos\theta$

Referring figure 5, $RE = RH + HE$

$\therefore RE = \cos\theta\sqrt{a^2 - b^2} + OK$

Substituting eqn. [5.8] in above eqn., we get

$$\therefore RE = \cos\theta\sqrt{a^2 - b^2} + \left[\frac{(a^2 - b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right]$$

$$\therefore RE = \left[\frac{\cos\theta\sqrt{(a^2 - b^2)(\sin^2\theta + b^2\cos^2\theta)} + (a^2 - b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right] \qquad -------[5.9]$$

(x) Final proof for the property

Multiplying. [5.9] & [5.6], we get

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$$\mathrm{RE} \times \mathrm{GQ} = \left[\frac{\cos\theta\sqrt{(a^2-b^2)(\sin^2\theta+b^2\cos^2\theta)}+(a^2-b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}\right] \times \left[\frac{\cos\theta\sqrt{(a^2-b^2)(\sin^2\theta+b^2\cos^2\theta)}-(a^2-b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}\right]$$

Simplifying, we get

$$\mathrm{RE} \times \mathrm{GQ} = \frac{b^2(a^2-b^2)\cos^2\theta}{a^2\sin^2\theta+b^2\cos^2\theta}$$

Multiplying. [5.3] & [5.8], we get

$$\mathrm{PT} \times \mathrm{OK} = \left[\frac{\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}\right] \times \left[\frac{b^2\cos^2\theta}{\sin\theta\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}\right]$$

Simplifying, we get

$$\mathrm{PT} \times \mathrm{OK} = \frac{b^2(a^2-b^2)\cos^2\theta}{a^2\sin^2\theta+b^2\cos^2\theta}$$

$$\therefore \mathrm{RE} \times \mathrm{GQ} = \mathrm{PT} \times \mathrm{OK} \qquad ------------------[5.10]$$

The eqn. [5.10] is the mathematical form of the property.

Theorem- 6

If OP is the focal distance of an ellipse at point 'P', 'AB' is one of the sides of its inscribed parallelogram (Eric, 2003) and θ° is the angle between x–axis and side AB then the square of diameter PS at θ° is always equal to twice of the square of 'AB'. It can be written in mathematical form as $\mathrm{PS}^2 = 2\mathrm{AB}^2$. Where, $\theta° = \tan^{-1}\left(\frac{b}{a}\right)$

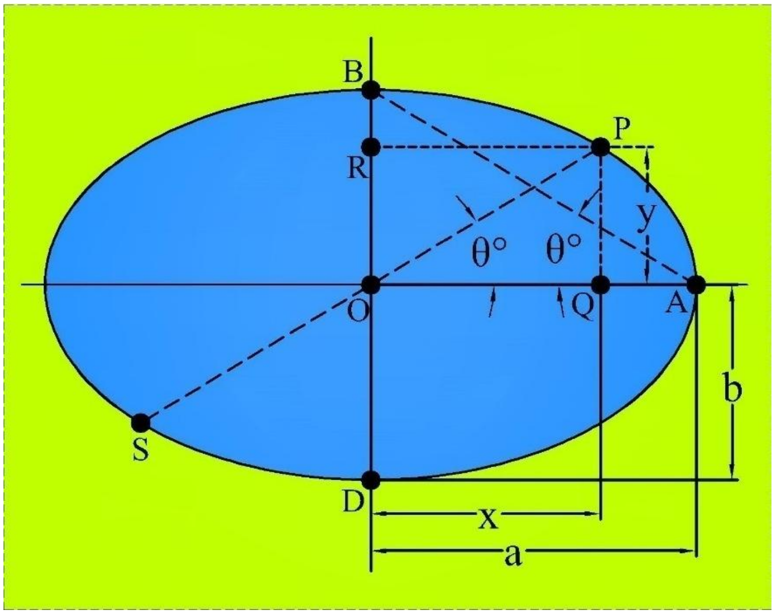

Figure 6: A normal line passing through point P of an ellipse

Derivation of Equations and Proof of the Property

Referring figure 6 Let, 'O' is the centre of ellipse, 'a' & 'b' are the semi major axis and semi minor axis, 'Q' & 'R' are the projection of P on x-axis and y-axis respectively, 'PS' is the diameter of ellipse at point P. 'AB' is one of the sides of the inscribed parallelogram. Let ∠POQbeθ°.

In right – triangle AOB, $\tan(\theta°) = \frac{b}{a}$

Therefore, $\sin(\theta°) = \dfrac{b}{\sqrt{a^2+b^2}}$ ———————————————————— [6.1]

Simillarly, $\cos(\theta°) = \dfrac{a}{\sqrt{a^2+b^2}}$ ———————————————————— [6.2]

The focal distance of point P at θ° from x − axis is

$$OP^2 = \frac{ab}{\sqrt{a^2 sin^2(\theta°) + b^2 cos^2(\theta°)}}$$

Substitute 6.1 & 6.2 in above

$$OP^2 = \frac{a^2b^2}{a^2\left(\frac{b^2}{a^2+b^2}\right) + b^2\left(\frac{a^2}{a^2+b^2}\right)}$$

$$\therefore OP^2 = \frac{a^2b^2}{\frac{a^2b^2}{a^2+b^2} + \frac{a^2b^2}{a^2+b^2}}$$

$$\therefore OP^2 = \frac{a^2b^2}{\left(\frac{2a^2b^2}{a^2+b^2}\right)}$$

$$\therefore OP^2 = \frac{a^2b^2(a^2+b^2)}{2a^2b^2}$$

$$\therefore OP^2 = \frac{a^2+b^2}{2} = \frac{AB^2}{2} \text{ ———————————————— [6.3]}$$

$$\therefore PS^2 = (OP + OS)^2$$

$$\therefore PS^2 = OP^2 + OS^2 + 2(OP)(OS)$$

$$\therefore PS^2 = 2OP^2 + 2OP^2 \quad [\because OP^2 = OS^2]$$

$$\therefore PS^2 = 4OP^2$$

Substituting 6.3 in above eqn.

$$PS^2 = 4\left(\frac{AB^2}{2}\right)$$

$$\therefore PS^2 = 2AB^2 \text{ ———————————————— [6.4]}$$

The eqn. [6.4] is the mathematical form of the property.

CONCLUSION

The properties have been developed with necessary derivation of equations and appropriate drawings. The mathematical expressions of each theorem have also been given at the end of the individual sections. The theorems, which has been defined in this research article is very useful for those doing research or further study in the field of conics & Euclidean geometry, also very useful to the research scholars for reference to the higher level research works, since this is also one of the important properties of an ellipse.

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