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Research Article

ON THE NON-HOMOGENEOUS BIQUADRATIC EQUATION WITH 4 UNKNOWNS $x^3+y^3+2z^3=3xyz+6(k^2+s^2)(x+y)w^3$

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ABSTRACT
The non- homogeneous biquadratic equation with four unknowns given by $x^3+y^3+2z^3=3xyz+6(k^2+s^2)(x+y)w^3$ is considered and analyzed for finding its non zero distinct integral solutions. Introducing the linear transformations $x=u+v,\ \ y=u-v\ \ z=2u$ and employing the method of factorization, different patterns of non zero distinct integer solutions of the equation under the above equation are obtained. A few interesting relations between the integral solutions and the special numbers namely Polygonal numbers, Pyramidal numbers, Centered Polygonal numbers, Centered Pyramidal numbers, Thabit-ibn-Kurrah number, Carol number, Mersenne number are exhibited.

Keywords: *Non-Homogeneous Equation, Integral Solutions, Polygonal Numbers, Pyramidal Numbers and Special Number*
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Notation

$t_{m,n}$ = Polygonal number of rank n with sides m

p_m^n = Pyramidal number of rank n with sides m

$ct_{m,n}$ = Centered Polygonal number of rank n with sides m

cp_m^n = Centered Pyramidal number of rank n with sides m

g_n = Gnomonic number

Tha_n = Thabit-ibn-Kurrah number

$carl_n$ = Carol number

Mer_n = Mersenne number

ky_n = Kynea number

wo_n = Woodhall number

p_n = Pronic number

INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular biquadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity (Carmichael, 1959; Dickson, 1952; Gopalan and Pandichelvi, 2009). In this context one may refer (Gopalan and Sangeetha, 2010; Gopalan and Sangeetha, 2010; Gopalan and Sangeetha, 2011; Gopalan and Sivkami, 2013; Manju *et al.*, 2012; Manju *et al.*, 2011; Sangeetha *et al.*, 2014) for various problems on the biquadratic Diophantine equations. However, often we come across non-homogeneous biquadratic equations and as such one may require its integral solution in its most general form. This paper concern with the non homogeneous biquadratic equation with four unknowns $x^3+y^3+2z^3=3xyz+6(k^2+s^2)(x+y)w^3$ for determining its infinitely many non-zero integral solutions. Also, a few interesting properties among the solutions are presented.

Method of Analysis

The biquadratic equation with four unknowns to be solved for its non-zero distinct integral solution is

$$x^3+y^3+2z^3=3xyz+6(k^2+s^2)(x+y)w^3 \quad (1)$$

Consider the transformations

$$\left.\begin{array}{l} x=u+v \\ y=u-v \\ z=2u \end{array}\right\} \quad (2)$$

On substituting (2) in (1), we get

$$u^2+v^2=(k^2+s^2)w^3 \quad (3)$$

In what follows we illustrate the methods of obtaining patterns of integer solutions to (1)

Pattern 1

Assume $w=p^2+q^2=(p+iq)(p-iq)$ (4)

Using (4) in (3), and employing the method of factorization we get

$$(u+iv)(u-iv)=(k+is)(k-is)(p+iq)^3(p-iq)^3$$

Which is equivalent to the system of equations,

$$u+iv=(k+is)(p+iq)^3$$

$$u-iv=(k-is)(p-iq)^3$$

On equating real and imaginary parts, we obtain

$$u=u(p,q,k,s)=kp^3-3pq^2k+sq^3-3p^2qs$$

$$v=v(p,q,k,s)=sp^3+3pq^2k-kq^3-3p^2qs$$

On substituting u and v in (2) we get the values of x, y and z . The non-zero distinct integrals values of x, y, z and w satisfying (1) are given by

$$x=x(p,q,k,s)=(p^3-3pq^2)(k+s)+(3p^2q-q^3)(k-s)$$

$$y=y(p,q,k,s)=(p^3-3pq^2)(k-s)+(q^3-3p^2q)(k+s)$$

$$z=z(p,q,k,s)=2\left((p^3-3pq^2)(k+s)+(3p^2q-q^3)(k-s)\right)$$

$$w(p,q)=p^2+q^2$$

Properties

1) $x(a,1,k,s)=k\left(2cp_a^6+t_{8,n}-g_a-2\right)+s\left(cp_a^6-ct_{6,a}+2\right)$

2) $\dfrac{y(1,b,2,3)+17b+1-p_b^3}{cp_b^4}=$ Nasty number

3) $\dfrac{z(a,2,s,s)}{2s}=p_a^5+cp_a^3+t_{12,a}-\left(2g_a-ct_{12,a}+7a+3\right)$

4) $w\left(2^n,n2^n\right)=jal_{2n}+wo_n$

5) $x(1,b,1,2)+w(1,b)-cp_b^6+t_{18,b}\equiv 4(\text{mod}10)$

Pattern 2

Rewrite (3) as

$$u^2+v^2=(k^2+s^2)w^3 * 1 \tag{5}$$

Write 1 as, $1=\dfrac{\left(m^2-n^2+2imn\right)\left(m^2-n^2-2imn\right)}{\left(m^2+n^2\right)^2}$ (6)

Using (4) and (6), in (5) it is written in factorizable form as

$$(u+iv)(u-iv)=(k+is)(k-is)(p+iq)^3(p-iq)^3\frac{\left(m^2-n^2+2imn\right)\left(m^2-n^2-2imn\right)}{\left(m^2+n^2\right)^2}$$

Which is equivalent to the system of equations,

$$(u+iv)=(k+is)(p+iq)^3\frac{\left(m^2-n^2+2imn\right)}{\left(m^2+n^2\right)}$$

$$(u-iv)=(k-is)(p-iq)^3\frac{\left(m^2-n^2-2imn\right)}{\left(m^2+n^2\right)}$$

On equating the real and imaginary parts we obtain

$$u=\frac{1}{\left(m^2+n^2\right)}\left(\left(\left(m^2-n^2\right)\left(k(p^3-3pq^2)+s(q^3-3p^2q)\right)\right)+2mn\left(k(3p^2q-q^3)+s(p^3-3pq^2)\right)\right)$$

$$v=\frac{1}{\left(m^2+n^2\right)}\left(\left(\left(m^2-n^2\right)\left(s(p^3-3pq^2)+k(3p^2q-q^3)\right)\right)+2mn\left(s(q^3-3p^2q)+k(p^3-3pq^2)\right)\right)$$

Replacing p by $\left(m^2+n^2\right)P$ and q by $\left(m^2+n^2\right)Q$ in the above equations, we have

$$u=\left(m^2+n^2\right)^2\left(\left(\left(m^2-n^2\right)\left(k(P^3-3PQ^2)+s(Q^3-3P^2Q)\right)\right)+2mn\left(k(3P^2Q-Q^3)+s(P^3-3PQ^2)\right)\right)$$

$$v=\left(m^2+n^2\right)^2\left(\left(\left(m^2-n^2\right)\left(s(P^3-3PQ^2)+k(3P^2Q-Q^3)\right)\right)+2mn\left(s(Q^3-3P^2Q)+k(P^3-3PQ^2)\right)\right)$$

Substituting the values of u and v in (2), the non-zero distinct integral values of x, y, z and w satisfying (1) are given by

$$x=x\left(m,n,k,s,P,Q\right)=\left(m^2+n^2\right)^2\begin{pmatrix}\left(\left(m^2-n^2\right)\left((k+s)(P^3-3PQ^2)+(k-s)(Q^3-3P^2Q)\right)\right)\\+2mn\left((k-s)(3P^2Q-Q^3)+(k+s)(P^3-3PQ^2)\right)\end{pmatrix}$$

$$y=y\left(m,n,k,s,P,Q\right)=\left(m^2+n^2\right)^2\begin{pmatrix}\left(\left(m^2-n^2\right)\left((k-s)(P^3-3PQ^2)+(k+s)(Q^3-3P^2Q)\right)\right)\\+2mn\left((k+s)(3P^2Q-Q^3)+(k-s)(3PQ^2-P^3)\right)\end{pmatrix}$$

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$$z = z(m,n,k,s,P,Q) = 2(m^2+n^2)^2 \begin{pmatrix} \left((m^2-n^2)\left(k(P^3-3PQ^2)+s(Q^3-3P^2Q)\right)\right)+ \\ 2mn\left(k(3P^2Q-Q^3)+s(P^3-3PQ^2)\right) \end{pmatrix}$$

$$w(m,n,P,Q) = (m^2+n^2)^2 (P^2+Q^2)$$

Properties

1) $x(1,1,k,k,P,1) = 8k\left(cp_p^{12} - 6t_{3,p} + t_{8,p}\right)$

2) $y(2,2,s,2s,1,Q) + 504s\left(2p_Q^9 + t_{8,Q}\right) = 504s(6Q+1)$

3) $(3,3,2,s,1,Q) + 58324\left(2p_Q^8 + t_{18,Q} + g_Q - 2\right) = 0$

4) $w(3,4,2^n,1) - 625mer_{2n} = 1250$

Pattern 3

Write 1 as,

$$1 = \frac{(1+i)^{2n}(1-i)^{2n}}{2^{2n}} \tag{7}$$

Using (4) and (7) in (5) and by applying the same procedure in pattern 2, we get the non-zero distinct integral values of x, y, z and w satisfying (1) are given by

$$x = x(n,k,s,p,q) = \cos\frac{n\pi}{2}\left((k+s)(p^3-3pq^2)+(k-s)(3p^2q-q^3)\right)+$$
$$\sin\frac{n\pi}{2}\left((k-s)(p^3-3pq^2)+(k+s)(q^3-3p^2q)\right)$$

$$y = y(n,k,s,p,q) = \cos\frac{n\pi}{2}\left((k-s)(P^3-3pq^2)+(k+s)(q^3-3p^2q)\right)+$$
$$\sin\frac{n\pi}{2}\left((k+s)(3p^2q-q^3))+(k-s)(q^3-3p^2q)\right)$$

$$z = z(n,k,s,p,q) = 2\begin{pmatrix} \cos\frac{n\pi}{2}\left(k(p^3-3pq^2)+s(q^3-3p^2q)\right) \\ -\sin\frac{n\pi}{2}\left(k(3p^2q-q^3)+s(p^3-3pq^2)\right) \end{pmatrix}$$

$$w = w(p,q) = p^2+q^2$$

Properties

1) $x(4,k,k,p,1) - y(4,k,k,p,1) - 2k\left(1+cp_p^6\right) = 0$

2) $w(2^n,2^n) = car\,1_n + ky_n + 1$

3) $z(4,1,1,p,1) - 2w(p,1) + 4t_{9,p} + 20p = 2p_p^3$

4) $y(1,2,1,p,1) + 3p_p^5 \equiv 1 (\bmod 9)$

5) $x(6,1,1,q-1,q) + y(6,1,1,q-1,q) - 6p_q^{10} + 15q + 10g_q = 10$

CONCLUSION

To conclude one may consider biquadratic equation with multivariables(≥5)and search for their non-zero distinct integer solutions along with their corresponding properties.

REFERENCES

Carmichael RD (1959). *The Theory of Numbers and Diophantine Analysis* (Dover publications) New York.

Dickson LE (1952). *History of Theory of Numbers* (Chelsa Publishing company) New York.

Gopalan MA and Pandichelvi V (2009). On the solutions of the Biquadratic equation $\left(x^2-y^2\right)^2=\left(z^2-1\right)^2+w^4$, International conference on Mathematical methods and Computation 24-25.

Gopalan MA and Sangeetha G (2010). Integral solutions of Ternary Quartic equation $x^2-y^2+2xy=z^4$, *Antarctica Journal of Mathematics* **7**(1) 95-101.

Gopalan MA and Sangeetha G (2010). Integral solutions of non-homogeneous Quartic equation $x^4-y^4=\left(2\alpha^2+2\alpha+1\right)\left(z^2-w^2\right)$, *Journal of Food Science and Technology* **4**(3) 15-21.

Gopalan MA and Sangeetha G (2011). Integral solutions of Non-homogeneous biquadratic $x^4+x^2+y^2-y=z^2+z$, *Acta Ciencia Indica* **XXXVII**(4) 799-803.

Gopalan MA and Sivkami B (2013). Integral solutions of quartic equation with four unknowns $x^3+y^3+z^3=3xyz+2(x+y)w^3$, *Antarctica Journal of Mathematics* **10**(2) 151-159.

Manju Somanath, Sangeetha G and Gopalan MA (2012). Integral solutions of biquadratic equation with four unknowns given by $xy+\left(k^2+1\right)z^2=5w^2$, *Pacific-Asian Journal of Mathematics* **6**(2) 185-190.

Manju Somanath, Sangeetha G and Gopalan MA (2011). Integral solutions of non-homogeneous Quartic equation $x^4-y^4=\left(k^2+1\right)\left(z^2-w^2\right)$, *Archimedes Journal of Mathematics* **1**(1) 51-57.

Sangeetha G, Manju Somanath, Gopalan MA and Pushparani (2014). Integral solutions of the homogeneous biquadratic equation with five unknowns $\left(x^3+y^3\right)z=\left(w^2-p^2\right)R^2$, *International conference on Mathematical methods and Computation* 221-226.