International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jpms.htm
2014 Vol. 4 (3) July- September, pp. 45-50/Vidhyalakshmi et al.

Research Article

OBSERVATIONS ON THE SEXTIC EQUATION WITH FOUR UNKNOWNS

$$(x-y)(x^2+y^2)=z(x^2-xy+y^2+7w^5)$$

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ABSTRACT

The Diophantine equation of degree six with four unknowns given by $(x-y)(x^2+y^2)=z(x^2-xy+y^2+7w^5)$ is analyzed for its non-zero integral solutions. A few interesting relations between the solutions and special numbers are given.

Keywords: *Sextic Equation with Four Unknowns, Integral Solutions, Polygonal Numbers*
Mathematics subject classification: 11D41

Notations

$$t_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$$

$$P_n^m=\frac{n(n+1)}{6}\left[(m-2)n+(5-m)\right]$$

$$PT_n=\frac{n(n+1)(n+2)(n+3)}{4!}$$

$$PR_n=n(n+1)$$

$$S_n=6n(n-1)+1$$

$$Ky_n=(2^n+1)^2-2$$

$$CP_n^3=\frac{n}{2}(n^2+1)$$

$$CP_n^6=n^3$$

$$CP_n^{12}=n(2n^2-1)$$

$$CP_n^{17}=\frac{n}{6}(17n^2-11)$$

$$F_{4,n,4}=\frac{n(n+1)^2(n+2)}{12}$$

$$F_{4,n,6}=\frac{n^2(n+1)(n+2)}{6}$$

$$F_{4,n,7}=\frac{n(n+1)(n+2)(5n-1)}{4!}$$

$$F_{5,n,3}=\frac{n(n+1)(n+2)(n+3)(n+4)}{5!}$$

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INTRODUCTION

The theory of diophantine equations offers a rich variety of fascinating problems (Carmichael, 1959; Dickson, 1952; Mordell, 1969; Telang, 1996). Particularly, in (Gopalan *et al.*, 2007; 2010a), sextic equations with 3 unknowns are studied for their integral solutions (Gopalan *et al.*, 2010b; 2012a,b; 2013a,b; 2014) analyse sextic equations with 4 unknowns for their non-zero integer solutions. This communication concerns with yet another sextic equation with 4 unknowns given by $(x-y)(x^2+y^2)=z(x^2-xy+y^2+7w^5)$.Using different methods infinitely many non-zero integer quadruples (x, y, z, w) satisfying the above equation are obtained. Various interesting properties among the values of x, y, z and w are presented.

Method of Analysis

The equation under consideration to be solved is

$$(x-y)(x^2+y^2)=z(x^2-xy+y^2+7w^5) \tag{1}$$

Intoduction of the linear transformations

$$x=u+v,\ y=u-v,\ z=v \tag{2}$$

in (1) leads to

$$v^2+3u^2=7w^5 \tag{3}$$

Equation (3) is solved through different methods and thus, we obtain different patterns of solutions to (1).

Method 1

Assume

$$w=a^2+3b^2 \tag{4}$$

$$7=(2+i\sqrt{3})(2-i\sqrt{3}) \tag{5}$$

Using (4) and (5) in (3) and employing the method of factorization, define

$$v+i\sqrt{3}u=(2+i\sqrt{3})(a+i\sqrt{3}b)^5$$

Equating real and imaginary parts on both sides of the above equation we get

$$v=2a^5-15a^4b-60a^3b^2+90a^2b^3+90ab^4-27b^5$$

$$u=a^5+10a^4b-30a^3b^2-60a^2b^3+45ab^4+18b^5$$

Substituting the values of u,v in (2) the non-zero distinct integral solutions to (1) are given by

$$x=3a^5-5a^4b-90a^3b^2+30a^2b^3+135ab^4-9b^5$$

$$y=-a^5+25a^4b+30a^3b^2-150a^2b^3-45ab^4+45b^5$$

$$z=2a^5-15a^4b-60a^3b^2+90a^2b^3+90ab^4-27b^5$$

$$w=a^2+3b^2$$

Properties

1) $206(x(a,1)+3y(a,1)+370w(a,1)-20t_{9,a^2})$ is a nasty number

2) $6(x(a,b)y(a,b)+z^2(a,b))$ is a nasty number

3) $z(1,b)+9b^3w(1,b)-2160P_{tb}+882p_b^5+555p_{rb}\equiv 2(\mathrm{mod}\, b)$

4) $3(x(a,1)-z(a,1)-a^3w(a,1))-5S_{a^2}+96t_{4,a}+18t_{8,a}+594p_{a-1}^3$ is a perfect square

5) $\dfrac{5x(a,1)+y(a,1)}{14}+15CP_a^{12}-90t_{4,a}+60t_{5,a}$ is fifth power of an integer

6) $3OH_a*t_{4,a}-15p_{ra^2}+61CP_a^6+97(3t_{4,a}-t_{8,a})+t_{212,a}-z(a,1)$ is a cubical integer

Case i
Equation (5) can be written as

$$7=\frac{(1+i3\sqrt{3})(1-i3\sqrt{3})}{4}$$

Proceeding as in method1 and taking a=2A,b=2B we obtain the non-zero integral solutions to (1) as

$$x=2^4(4A^5-40A^4B-120A^3B^2+240A^2B^3+180AB^4-72B^5)$$

$$y=2^4(2A^5+50A^4B-60A^3B^2-300A^2B^3+90AB^4+90B^5)$$

$$z=2^4(A^5-45A^4B-30A^3B^2+270A^2B^3+45AB^4-81B^5$$

$$w=2^2(A^2+3B^2)$$

Case ii
In (5) write '7' as

$$7=\frac{(5+i\sqrt{3})(5-i\sqrt{3})}{4}$$

Proceeding as in case1 the non-zero integral solutions of (1) are given by

$$x=2^4(6A^5+10A^4B-180A^3B^2-60A^2B^3+270AB^4+18B^5)$$

$$y=2^4(-4A^5+40A^4B+120A^3B^2-240A^2B^3-180AB^4+72B^5)$$

$$z=2^4(A^5+25A^4B-30A^3B^2-150A^2B^3+45AB^4+45B^5)$$

$$w=2^2(A^2+3B^2)$$

Method 2
Replace u by αw and v by βw in (3) we get

$$\beta^2+3\alpha^2=7w^3 \tag{6}$$

Using (4) and (5) in (6) and using the method of factorization define

$$(\beta+i\sqrt{3}\alpha)=(2+i\sqrt{3})(a+i\sqrt{3}b)^3 \tag{7}$$

Equating real and imaginary parts of (7) and using (2) we get the non-zero integral solutions to (1) as

$$x=3a^5-3a^4b-18a^3b^2-6a^2b^3-81ab^4+9b^5$$

$$y=-a^5+15a^4b+6a^3b^2+30a^2b^3+27ab^4-45b^5$$

$$z=2a^5-9a^4b-12a^3b^2-18a^2b^3-54ab^4+27b^5$$

$$w=a^2+3b^2$$

Properties

1) $x(2^n,1)+3y(2^n,1)=42(Ky_{2n}-2)$

2) $\dfrac{y(a,1)+y(-a,1)}{30}$ can be written as difference of two squares

3) $z(a,1)-2a^3w(a,1)+9(12F_{4,a,4}-4CP_a^3-2t_{10,a}+5t_{4,a})$ is a cubical integer

4) $120F_{5,a,3}-8t_{3,a^2}-82P_a^5+7t_{4,a}-y(a,1)-z(a,1)\equiv 18(\bmod 51)$

Case iii
In (5) write '7' as

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$$7=\frac{(1+i3\sqrt{3})(1-i3\sqrt{3})}{4}$$

Proceeding as in method2 and taking a=2A,b=2B we obtain the non-zero integral solutions to (1) as

$$x=2^6(A^5-6A^4B-6A^3B^2-12A^2B^3-27AB^4+18B^5)$$

$$y=2^5(A^5+15A^4B-6A^3B^2+30A^2B^3-27AB^4-45B^5)$$

$$z=2^4(A^5-27A^4B-6A^3B^2-54A^2B^3-27AB^4+81B^5)$$

$$w=2^2(A^2+3B^2)$$

Case iv

In (5) write '7' as

$$7=\frac{(5+i\sqrt{3})(5-i\sqrt{3})}{4}$$

Proceeding as in case iii the non-zero integral solutions of (1) are given by

$$x=2^4(6A^5+6A^4B-36A^3B^2+12A^2B^3-162AB^4-18B^5)$$

$$y=2^4(-4A^5+24A^4B+24A^3B^2+48A^2B^3+108AB^4-72B^5)$$

$$z=2^4(5A^5-9A^4B-30A^3B^2-18A^2B^3-135AB^4+27B^5)$$

$$w=2^2(A^2+3B^2)$$

Method 3

Replace u by αw^2 and v by βw^2 in (3) we get

$$\beta^2+3\alpha^2=7w \tag{8}$$

Using (4)and (5) in (8) and proceeding as in method2, we obtain the non-zero integral solutions of (1) as

$$x=3a^5-a^4b+18a^3b^2-6a^2b^3+27ab^4-9b^5$$

$$y=-a^5+5a^4b-6a^3b^2+30a^2b^3-9ab^4+45b^5$$

$$z=2a^5-3a^4b+12a^3b^2-18a^2b^3+18ab^4-27b^5$$

$$w=a^2+3b^2$$

Properties

1) $x(a,1)+y(a,1)-z(a,1)-63=7(6F_{4,a,6}-6P_a^5+7t_{4,a})$

2) $y(a,1)+a^3w(a,1)-24F_{4,a,7}+6CP_a^{17}-2t_{25,a}\equiv 45(\mathrm{mod}\,3)$

3) $x(a,1)-a^3w(1,a)+6F_{4,a,6}-6CP_a^{20}+4t_{4,a}\equiv -9(\mathrm{mod}\,41)$

4)30x(2a,a),6y(a,a),2z(3a,a)and 6w(a,a) are nasty numbers,

Case V

In (5) write '7' as

$$7=\frac{(1+i3\sqrt{3})(1-i3\sqrt{3})}{4}$$

Proceeding as in method2 and taking a=2A, b=2B we obtain the non-zero integral solutions to (1) as

$$x = 2^4(4A^5 - 8A^4B + 24A^3B^2 - 48A^2B^3 + 36AB^4 - 72B^5)$$
$$y = 2^4(2A^5 + 10A^4B + 12A^3B^2 + 60A^2B^3 + 18AB^4 + 90B^5)$$
$$z = 2^4(A^5 - 9A^4B + 6A^3B^2 - 54A^2B^3 + 9AB^4 - 81B^5)$$
$$w = 2^2(A^2 + 3B^2)$$

Case vi

In (5), write '7' as

$$7 = \frac{(5+i\sqrt{3})(5-i\sqrt{3})}{4}$$

Proceeding as in case.v the non-zero integral solutions of (1) are given by

$$x = 2^4(6A^5 + 2A^4B + 36A^3B^2 + 12A^2B^3 + 54AB^4 + 18B^5)$$
$$y = 2^4(-4A^5 + 8A^4B - 24A^3B^2 + 48A^2B^3 - 36AB^4 + 72B^5)$$
$$z = 2^4(5A^5 - 3A^4B + 30A^3B^2 - 18A^2B^3 + 45AB^4 - 27B^5)$$
$$w = 2^2(A^2 + 3B^2)$$

Conclusion

In this paper we have analysed a sextic equation with 4 unknowns for its non-zero distinct integral solutions. As Diophantine equations of sixth degree are rich in variety, one may search for other forms of sextic equation with multi variables for determining their integer solutions

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