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Research Article

ON PARABOLA SOME NEW GEOMETRICAL PROPERTIES

***Ara. Kalaimaran**
Civil Engineering Division, CSIR-Central Leather Research Institute, Chennai-20, Tamil Nadu, India
**Author for correspondence*

ABSTRACT
Parabola is one of the conic sections obtained by the intersections of circular cones by planes.A parabola is defined such that the set of all points in the plane are equidistant from a given line (Directrix) and a given fixed point (Focus). Mr. Menaechmus (c.375-325 BC), a pupil of Eudoxus, tutor to Alexander the Great, and a friend of Plato, is credited with the discovery of the conics. The parabola has many important applications like parabolic antenna, parabolic microphone, automobile headlight reflectors, the design of ballistic missiles and etc. The reflective and refractive properties have many advantages and frequently used in physics. Similarly, itsgeometrical properties are widely used in engineering and many other areas. Some new properties of the parabola have been developed now and defined in this article with necessary analytic geometric equations and illustrated with an example.

Keywords: *Parabola, Focus, Directrix, Axis of Symmetry, Vertex and Co-Ordinate Geometry*

INTRODUCTION

Some Definitions

Circum-circle: A circle that circumscribes a given polygon, passing through all its vertices (Borowski and Borwein, 2005).
Directrix: A fixed line on the convex side of a conic section, interms of which, together with the focus and the eccentricity, the locus points that constitute the conic is defined (Borowski and Borwein, 2005).
Eccentricity: It is the ratio between the distance of any such point from a given fixed point (the Focus) and its distance from a given fixed line (the Directrix) (Borowskiand Borwein, 2005).
Focus: A fixed point on the concave side of the conic section, in terms of which, together with its directrix and its eccentricity, the locus of the points constituting conic is defined(Weisstein Eric W, 2003)
Latus-rectum: A chord that passes through the focus and perpendicular to the major axis of a conic (Borowski and Borwein, 2005).
Normal: A line which is perpendicular to the tangent to the curve (Borowski and Borwein, 2005).
Parabola: A parabola is the projection of an arc of a circle onto an oblique plane through the chord joining the end points of the arc (Borowski and Borwein, 2005).
Paraboloid: A three-dimensional surface or solid that has parabolic sections parallel to Cartesian planes, and either elliptical or hyperbolic sections parallel to the third coordinate plane (Borowski and Borwein, 2005).
Sub-normal: The projection on the x-axisof a line normal to a curve at point$P_0(x_0, y_0)$ and extending from P_0 to the x-axis. (Daintith and Remnie, 2005)
Sub-tangent: A segment of the x-axis lying between the x-coordinate of the point at which a tangent is drawn to a curve and the intercept of the tangent with the axis (Borowski and Borwein, 2005).
Tangent: A line that touches a curve at a point and has the same gradient as that of the curve at the point(Borowski and Borwein, 2005).
Vertex: Either of the points at which the major axis intersects the curve (Borowski and Borwein, 2005).
The area enclosed by a parabola and a line segment, the so-called "parabola segment", was computed by Archimedes via the method of exhaustion in the third century BC, in his the Quadrature(Borowski and Borwein, 2005) of the Parabola. The name "parabola" is due to Apollonius, who discovered many properties of conic sections.

The idea that a parabolic reflector could produce an image was already well known before the invention of the reflecting telescope. When Isaac Newton built the first reflecting telescope in 1668. Parabolic mirrors are used in most modern reflecting telescopes and in satellite dishes and radar receivers.

Important Applications of Parabola

The applications of reflective and refractive properties are used widely in Science and Technology. Some of them are defined briefly as given below:

Parabolic Solar Collector

Parabolic mirrors that look like troughs are used to reflect the sun's rays onto tubes filled with oil. The heated oil is then used to boil water, which sends steam to a turbine. Mechanical drives slowly rotate the mirrors to keep the reflected sunlight focused on the oil-filled tubes.The world's largest solar-thermal complexLuz International is located in California's Mojave Desert. There is a reflector in the Pyrenees Mountains, it can reach 6,000 degrees Fahrenheit just from the Sun.

Ignition of Olympic flame

The Olympic flame is traditionally lit at Olympia, Greece using a parabolic reflector concentrating solar beams.

Satellite Dishes

The shape of a satellite dish is a circular paraboloid. The curvature of the parabolic dish is always greatest near the vertex. Bottom edge of the dish is the vertex of the paraboloid.The axis is positioned towards the satellite. All incoming TV signals that are parallel to the axis of the parabola reflect directly through the focus of the parabola. The incoming waves are concentrated to the receiver module where the focus.

Automobile Headlights and Flash/Torch/Search Lights

In an automobile headlight the smooth inner surface of the headlight is a glass reflector upon which bright aluminium or mercury has been deposited. This part is a powerful parabolic reflector, if a lamp is placed at the focusas the rays parallel to the axis.

Fountains as natural phenomena

A jet of water in fountains forms the shape of a parabola.

Path of a Ball

Galileo was the first to show that the path of an object thrown in space is a parabola. Similarly, the path of the missile launching from one place to another place is the parabola. A falling body has two components of its motion, one horizontal and one vertical. In the horizontal direction, its velocity is constant, but in the vertical direction, it accelerates toward the ground at a rate of 9.8 meters per second. The resulting curve is a downward-pointing parabola.

Suspension Bridge

Since the concrete bridge's deck spans a long distance, it must be very heavy in weight by its own in addition to weight of the heavy load of traffic that it must carry. Due to their elegant structure, suspension bridges are used to transport loads over long distances, whether it be between two distant cities or between two ends of a river. Suspension bridges are able to work efficiently because of their cables, which are interesting from a mathematical perspective.

Parabolic Arches

Parabolic arches were widely adopted in traditional construction with stone masonry especially railway bridges. The mathematical genius of this parabolic profile is that the focus to the curve and the curve to the directrix is the same length. Therefore, when one applies pressure to the curve or arch, the entire arch is able to hold the weight, and it doesn't fail because there isn't a single weak point on the curve.

Whispering gallery

A whispering gallery is most simply constructed in the form of a curved wall, and allows whisper communication from any part of the internal side of the circumference to any other part. The sound is carried by waves, known as whispering gallery waves. Such galleries can also be set up using two parabolic dishes. Sometimes the phenomenon is detected in caves.

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Parabolic Receivers

A person who whispers at the focus of one of the parabolic reflectors can be heard by a person located near the focus of the other parabola.

Design of Highways curves

A parabolic curve that is applied to make a smooth and safe transition between two grades on a roadway or a highway. Parabolic functions have been found suitable for thevertical curves because they provide a constant rate of change of slope and imply equal curve tangents

A parabolic loudspeaker

It is a loudspeaker which seeks to focus its sound in coherent plane waves either by reflecting sound output from a speaker driver to a parabolic reflector aimed at the target audience, or by arraying drivers on a parabolic surface. The parabolic loudspeaker has been used for such diverse purposes as directing sound at faraway targets in performing arts centers and stadia, for industrial testing, for intimate listening at museum exhibits, and as a sonic weapon.

An acoustic mirror

Acoustic mirror is a passive device used to reflect and perhaps to focus (concentrate) sound waves. They were used during World War I to detect troop movements and artillery, and were a major area of study as an anti-aircraft early-warning device prior to the introduction of radar.

Pre-stress concrete

In pre-stress concrete works Parabolic Tendon can be sized and placed such that the upward force exerted by the tendon along the length of the member exactly balances the applied uniformly distributed load. A parabolic tendon, the upward thrust is a uniformly distributed load acting upwards. For a singly harped section, the upward thrust is a single point load and for a doubly harped section, the upward thrust is two point loads acting at the two harping points.

In Bending Moment diagram

The bending moment diagram is parabolic, when the beam structure carrying uniformly distributed load for entire span.

In X-ray Optics

Parabolic Equation and Exact Transparent Boundary Conditions in X-Ray Optics-Application to Waveguides and Whispering Gallery Optics- Research article written by R. M. Feshchenko, and A. V. Popov (Springer publication)

Important Basic formulae for Parabola

Type	1	2	3	4
Quadrant	First	Second	Third	Fourth
Eqn. of Parabola	$y^2 = 4ax$	$x^2 = 4ay$	$y^2 = -4ax$	$x^2 = -4ay$
Eqn. of Directrix	$x = a$	$y = a$	$x = -a$	$y = -a$
Axis of symmetry	$y = 0$	$x = 0$	$y = 0$	$x = 0$
Co-ordinate of Focus	$(a, 0)$	$(0, a)$	$(-a, 0)$	$(0, -a)$
Focal distance of any point (x_1, y_1)	$a + x_1$	$a + y_1$	$a - x_1$	$a - y_1$

Cartesian form of Parabola $y^2 = 4ax$

Length of the Latus rectum $= 4a$

Eqn. of Tangent for a parabola at $(x_1, y_1) \Longrightarrow yy_1 = 2a(x + x_1)$

Eqn. of Normal for a parabola at $(x_1, y_1) \Longrightarrow y - y_1 = 2a(x - x_1)$

If m is the slope of the tangent, its eqn. $\Longrightarrow y = mx + \frac{a}{m}$

If $(\theta°)$ is the slope of the tangent, its eqn. $\Longrightarrow y = x\tan\theta° + a\cot\theta°$

Coordinates of point of contact of tangent and the Parabola is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

Equation of normal with slope (m) is $y = mx - 2am - am^3$

The foot of the normal with slope (m)is $(am^2, -2am)$

Equation of normal with slope of tangent $(\theta°)$ is $y = x\tan\theta° - 2a\tan\theta° - a\tan^3\theta°$

ParametricformofParabola

Parametric eqn. of the Parabola is $(at^2, 2at)$. Where $t = 1/\tan\theta°$
Eqn. of the chord joining two points t_1 and t_2 is $\Longrightarrow (t_1 + t_2)y = 2x + 2at_1t_2$
Eqn. of tangent at t is $ty = x + at^2$
Eqn. of nornal at t is $y + tx = 2at + at^3$
To find the point of intersection of tangent at the point t_1 and t_2 of the parabola $y^2 = 4ax$ is $[at_1t_2, a(t_1 + t_2)]$

Some New Properties

<u>New Property-1(Figure 1)</u>

Figure 2 is the diagram of part of parabola. P is the tangent point on the parabola, T is the point where the tangent meeting the axis of symmetry, A is the vertex, F is the focus and N is the point where the normal meeting the axis of symmetry. PT is the length of the tangent, PN is the length of the normal.
If the tangent and normal at any point 'P' of a parabola meet the primary axes at 'T' and 'G' respectively then the focus bisects the length 'TG' and also focus is the centre of the circum-circle for the right triangle 'PTG'.

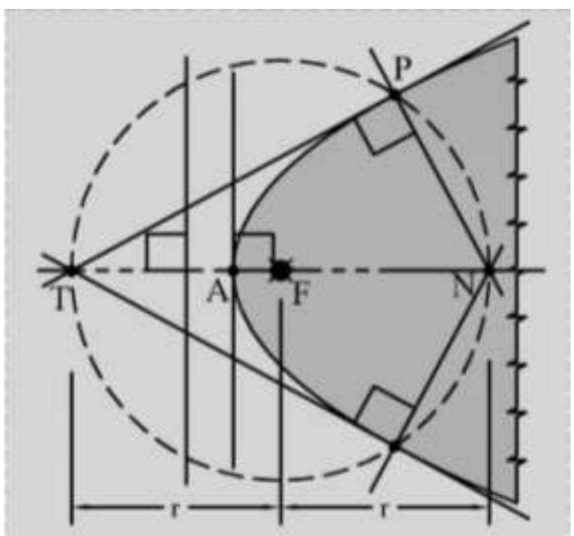

Figure 1: Diagram of Parabola with tangent and Normal

<u>New Property-2(Figure 2)</u>

Figure 2 is the diagram of part of parabola. P is the tangent point on the parabola, T is the point where the tangent meeting the axis of symmetry, A is the vertex, F is the focus, P, Q and R are the three varioustangent points on the parabola, U,VandW are the meeting of the tangents and axis of symmetry respectively and J, M and K are the meeting points of the tangents with the axis parallel to directrix passing through vertex.
The middle point of any tangent of a parabola is lying on axis. The perpendicular bisector of any tangent is passing through its focus.

<u>New Property-3(Figure 3)</u>

Fig. 3 is the diagram of part of parabola. P is the tangent point on the parabola, T is the point where the tangent meeting the axis of symmetry, A is the vertex, B is the junction point of directrix and the axis of symmetry, F is the focus and J is the junction of tangent and directrix. PT is the length of the tangent.
If a tangent drawn on parabola at the point P, A is the vertex and J is the junction of tangent and directrix then A and J are the points lie on periphery of the circle drawn at point P.

It can be written in a mathematical form as
$PA = PJ$

New Property-4(Figure 4)

Fig. 3 is the diagram of part of parabola. Point P is the tangent point on the parabola, A is the vertex and point S is the meeting point of normal and axis of symmetry.

The sum of the squares of Abscissa of the tangent point of the parabola and length of normal is equal to squares of the distance between vertex and meeting point of axis of symmetry and normal.It can be written in a mathematical form as

$AQ^2 + PS^2 = AS^2$

Derivation:

$AQ^2 + PS^2 = (r - a)^2 + \left(2\sqrt{ra}\right)^2$

$AQ^2 + PS^2 = r^2 + a^2 - 2ra + 4ra$

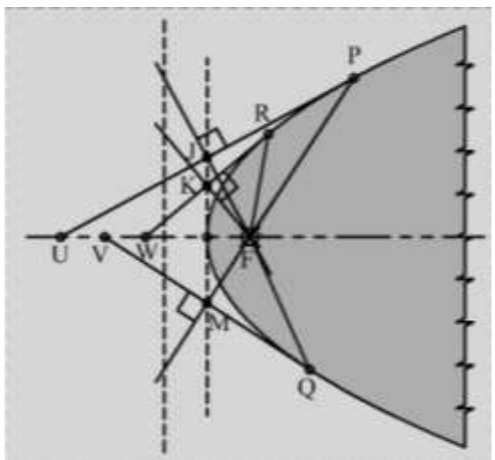

Figure 2: Diagram of Parabola with tangent and Normal

$AQ^2 + PS^2 = r^2 + a^2 + 2ra$

$AQ^2 + PS^2 = (r + a)^2$ ------------------------------ [4.1]

$AS^2 = (r + a)^2$ ------------------------------ [4.2]

Comparing 4.1 and 4.2

$AQ^2 + PS^2 = AS^2$

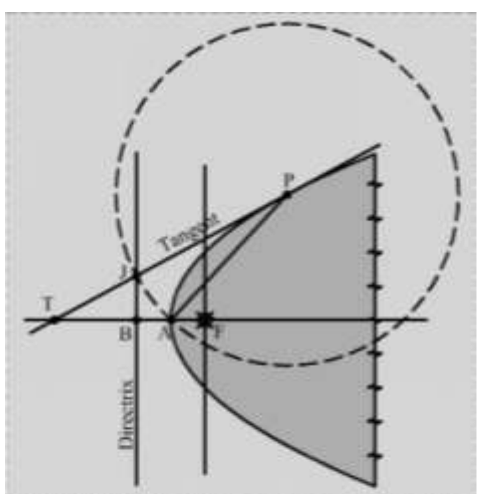

Figure 3: Diagram of Parabola with tangent and Normal

New Property-5 (Figure 4)

The product of sub-tangent and sub-normal is equal to the ordinate of tangent point of the parabola. It can be expressed in a mathematical form as

$TQ \times SQ = PQ^2$

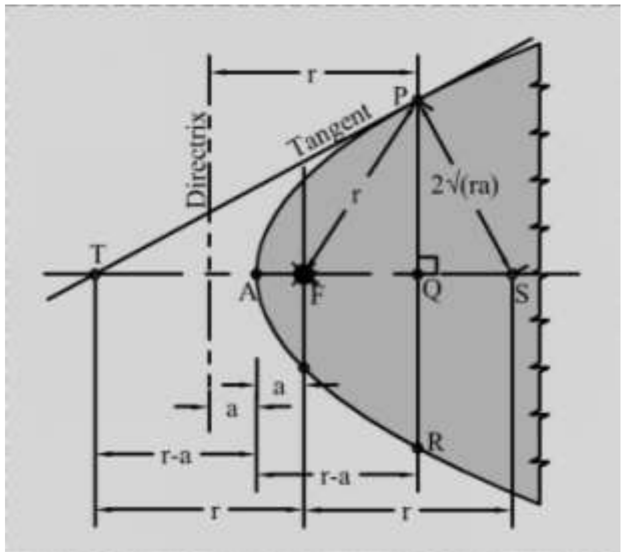

Figure 4: Diagram of Parabola with tangent and Normal

Derivation:

$\mathrm{TQ} \times \mathrm{SQ} = 2(r-a)2a$

$\mathrm{TQ} \times \mathrm{SQ} = 4ar - 4a^2$ −−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−− [5.1]

$\mathrm{PQ}^2 = \left[2\sqrt{a(r-a)}\right]^2$

$\mathrm{PQ}^2 = 4ar - 4a^2$ −−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−− [5.2]

Comparing 5.1 and 5.2

$\mathrm{TQ} \times \mathrm{SQ} = \mathrm{PQ}^2$

New Property-6 (Figure 4)

The sum of the squares of ordinate of the tangent point of the parabola and length of tangent is equal to four times of sum of squares of focal distance of tangent point and distance focus from its vertex. It can also be defined in a mathematical form as

$PQ^2 + PT^2 = 4(PF^2 - AF^2)$

Derivation:

$\mathrm{PQ}^2 + \mathrm{PT}^2 = \left(2\sqrt{a(r-a)}\right)^2 + \left(2\sqrt{r(r-a)}\right)^2$

$\mathrm{PQ}^2 + \mathrm{PT}^2 = 4a(r-a) + 4r(r-a)$

$\mathrm{PQ}^2 + \mathrm{PT}^2 = 4ar - 4a^2 + 4r^2 - 4ar$

$\mathrm{PQ}^2 + \mathrm{PT}^2 = 4r^2 - 4a^2$

$\mathrm{PQ}^2 + \mathrm{PT}^2 = 4(r^2 - a^2)$ −−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−[6.1]

$4(\mathrm{PF}^2 - \mathrm{AF}^2) = 4(r^2 - a^2)$ −−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−−[6.2]

Comparing 6.1 and 6.2

$\mathrm{PQ}^2 + \mathrm{PT}^2 = 4(\mathrm{PF}^2 - \mathrm{AF}^2)$

New Property-7(Ref: Figure 4)

The product of distance of meeting point of tangent to the primary axis from directrix and distance of focus from directrix is equal to subtraction of square of semi latus-rectum from twice of square of half of the normal. It can also be expressed in a mathematical form as

$FQ \times FB = 2FM^2 - FG^2$

Derivation:

$\mathrm{FQ} \times \mathrm{FB} = (r-2a) \times 2a$

$\mathrm{FQ} \times \mathrm{FB} = 2ar - 4a^2$ ------------------------------ [7.1]

$2\mathrm{FM}^2 - \mathrm{FG}^2 = 2\left(\sqrt{ra}\right)^2 - (2a)^2$

$2\mathrm{FM}^2 - \mathrm{FG}^2 = 2ar - 4a^2$ ------------------------------ [7.2]

Comparing 7.1and 7.2,

$\mathrm{FQ} \times \mathrm{FB} = 2\mathrm{FM}^2 - \mathrm{FG}^2$

New Property-8 (Figure 4)

The product of square of the ordinate of a point P and its focal distance is equal to the product of square of the tangent length drawn at the point P and abscissa of focus of the parabola. It can also be defined in a mathematical form as

$$\frac{PQ^2}{PT^2} = \frac{AF}{PF}$$

Derivation

$PT^2 = 4r(r-a)$

$PQ^2 = 4a(r-a)$

$\frac{\mathrm{PQ}^2}{\mathrm{PT}^2} = \frac{4a(r-a)}{4r(r-a)} = \frac{a}{r}$ ------------------------------ [8.1]

$\frac{AF}{PF} = \frac{a}{r}$ ------------------------------ [8.2]

From eqns. 8.1 *and* 8.2, $\frac{\mathrm{PQ}^2}{\mathrm{PT}^2} = \frac{\mathrm{AF}}{\mathrm{PF}}$ ------------------------------[8.3]

New Property-9(Figure 4)

The product of square of the ordinate of a point P and its focal distance is equal to the product of square of the tangent length drawn at the point P and abscissa of focus of the parabola. It can also be defined in a mathematical form as

$$\frac{\mathrm{NF}^2 \times \mathrm{TA}}{\mathrm{TF}} = \frac{\mathrm{NF}^2 \times \mathrm{AQ}}{\mathrm{PF}} = \mathrm{a}(\text{where 'a' is a constant})$$

Derivation:

We know that Right-triangleTQP = TBN

$\therefore \frac{\mathrm{PQ}}{\mathrm{NB}} = \frac{\mathrm{TQ}}{\mathrm{TB}}$

$\therefore \mathrm{NB} = \frac{\mathrm{PQ} \times \mathrm{TB}}{\mathrm{TQ}}$

$\therefore \mathrm{NB} = \frac{2\sqrt{\mathrm{a}(\mathrm{r}-\mathrm{a})} \times (\mathrm{r}-2\mathrm{a})}{\sqrt{\mathrm{r}-\mathrm{a}}}$

$\therefore \mathrm{NB} = (\mathrm{r}-2\mathrm{a})\sqrt{\frac{\mathrm{a}}{\mathrm{r}-\mathrm{a}}}$ ------------------------------[9.1]

In right-triangle FBN, $\mathrm{NB}^2 + \mathrm{BF}^2 = \mathrm{NF}^2$

$\therefore \mathrm{NF}^2 = \mathrm{NB}^2 + \mathrm{BF}^2$

Substituting, 9.1 in above eqn.

$\mathrm{NF}^2 = (\mathrm{r}-2\mathrm{a})^2 \times \left(\frac{\mathrm{a}}{\mathrm{r}-\mathrm{a}}\right) + (2\mathrm{a})^2$

$\mathrm{NF}^2 = \mathrm{r}^2\left(\frac{\mathrm{a}}{\mathrm{r}-\mathrm{a}}\right)$ ------------------------------[9.2]

$\frac{\mathrm{TF}^2 \times \mathrm{AF}}{\mathrm{TA}} = \mathrm{r}^2\left(\frac{\mathrm{a}}{\mathrm{r}-\mathrm{a}}\right)$ ------------------------------ [9.3]

Comparing, eqns. 9.2 and 9.3,

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$$\mathrm{NF}^2 = \frac{\mathrm{TF}^2 \times \mathrm{AF}}{\mathrm{TA}}$$

The above statement can be rewritten as

$$\frac{\mathrm{TF}^2}{\mathrm{NF}^2} = \frac{\mathrm{TA}}{\mathrm{AF}} ------------------------- [9.4]$$

<u>New Property-10(Figure 5)</u>

The ratio conjugate focal distances produced at any point (P) is equal to the ratio of abscissa of the point (P) and abscissa of focus (F). It can also be defined in a mathematical form as

$$\frac{PF}{QF} = \frac{AR}{AF}$$

Derivation:

Let A is the origin $(0,0)$, F is the focus at $(\mathrm{a}, 0)$, P is the point anywhere in the parabola and $\angle\mathrm{AFP} = \theta°$.

$$\text{We know PF} = \frac{2\mathrm{a}}{1+\cos\theta°} ------------------------- [10.1]$$

$$\mathrm{FQ} = \frac{2\mathrm{a}}{1+\cos(180+\theta°)}$$

$$\mathrm{FQ} = \frac{2\mathrm{a}}{1-\cos\theta°} ------------------------- [10.2]$$

$$\frac{\mathrm{PF}}{\mathrm{QF}} = \left(\frac{2\mathrm{a}}{1+\cos\theta°}\right) \div \left(\frac{2\mathrm{a}}{1-\cos\theta°}\right)$$

$$\therefore \frac{\mathrm{PF}}{\mathrm{QF}} = \left(\frac{2\mathrm{a}}{1+\cos\theta°}\right) \times \left(\frac{1-\cos\theta°}{2\mathrm{a}}\right)$$

$$\therefore \frac{\mathrm{PF}}{\mathrm{QF}} = \frac{1-\cos\theta°}{1+\cos\theta°} ------------------------- [10.3]$$

$$\angle\mathrm{PFR} = 180 - \theta°$$

$$\therefore \mathrm{RF} = \mathrm{PF} \times \cos(180 - \theta°)$$

$$\therefore \mathrm{RF} = -\mathrm{PF} \times \cos\theta°$$

Substituting, 10.1 in above eqn.

$$\therefore \mathrm{RF} = \frac{-2\mathrm{a}\cos\theta°}{1+\cos\theta°} ------------------------- [10.4]$$

$$\mathrm{AR} = \mathrm{AF} + \mathrm{RF}$$

Substituting 10.4 and $\mathrm{AF} = \mathrm{a}$ in above eqn.

$$\mathrm{AR} = \mathrm{a} - \left(\frac{2\mathrm{a}\cos\theta°}{1+\cos\theta°}\right)$$

$$\therefore \mathrm{AR} = \frac{\mathrm{a}(1-\cos\theta°)}{1+\cos\theta°} ------------------------- [10.5]$$

$$\frac{\mathrm{AR}}{\mathrm{AF}} = \frac{\mathrm{a}(1-\cos\theta°)}{1+\cos\theta°} \times \frac{1}{a}$$

$$\therefore \frac{\mathrm{AR}}{\mathrm{AF}} = \frac{\mathrm{a}(1-\cos\theta°)}{1+\cos\theta°} \times \frac{1}{a}$$

$$\therefore \frac{\mathrm{AR}}{\mathrm{AF}} = \frac{1-\cos\theta°}{1+\cos\theta°} ------------------------- [10.6]$$

From 10.3 and 10.6,

$$\frac{\mathrm{PF}}{\mathrm{QF}} = \frac{\mathrm{AR}}{\mathrm{AF}} ------------------------- [10.7]$$

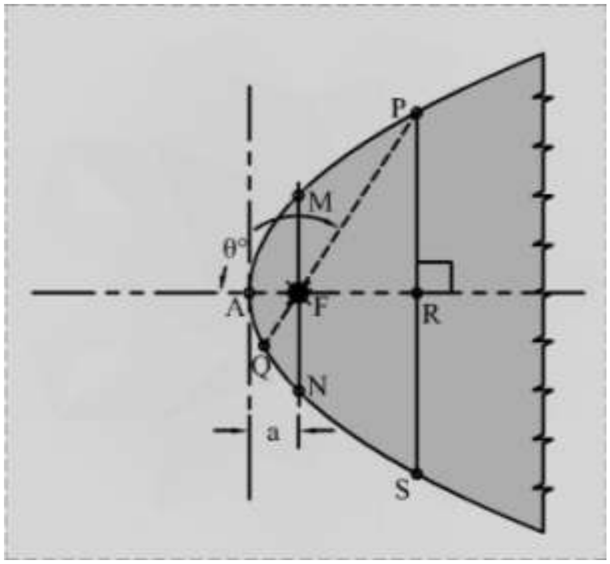

Figure 5: Diagram of Parabola with tangent and Normal

CONCLUSION

The centroid is an important parameter in mechanics and geometric properties especially while evaluating the Moment of Inertia. The author has derived the necessary mathematical formula in order to determine the exact value of centroid of any regular polygon, which is having sides of odd number. The formula has been proved with appropriate examples.

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