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Research Article

COMMON FIXED POINT THEOREM USING CONTROL FUNCTION AND PROPERTY (CLR_G) IN FUZZY METRIC SPACES

***Alok Asati [1], A.D.Singh[1] and Madhuri Asati[2]**
[1]*Department of Mathematics, Govt.M.V.M, Bhopal (India)*
[2]*Department of Applied Science, SKSITS, Indore (India)*
** Author for Correspondence*

ABSTRACT
The aim of this paper is to prove a common fixed point theorem via common limit range property in fuzzy metric space satisfying control function. We generalized the result of Kumar and Fisher (2010).

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Keywords: *Common Fixed Points, Fuzzy Metric Space, Weak Compatible Map, Common Limit Range Property, Fixed Point, Control Function*

INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh in 1965, as a new way to represent the vagueness in everyday life. In mathematical programming problems are expressed as optimizing some goal function given certain constraints, and there are real life problems that consider multiple objectives. Generally it is very difficult to get a feasible solution that brings us to the optimum of all objective functions. A possible method of resolution that is quite useful is the one using fuzzy sets. It was developed extensively by many authors and used in various fields. To use this concept in topology and analysis several researchers have been defined fuzzy metric space in various ways (Aamri and Moutawakil, 2002; Abbas *et al.*, 2009; Balasubramaniam *et al.*, 2002; Chauhan and Kumar, 2013; Chauhan *et al.*, 2012; Cho, 1997; Imdad *et al.*, 2012; Kumar and Chauhan, 2013; Schweizer *et al.*, 1983; Sharma, 2002). George and Veeramani (1994) modified the concept of fuzzy metric space introduced by Kramosil and Michalek (1975) in order to get the Hausdorff topology. Jungck (1996) introduced the notion of compatible maps for a pair of self mapping. Mishra *et al.*, (1994) extended the notion of compatible mappings to fuzzy metric spaces and proved common fixed point theorem in present of continuity of at least one of the mappings, completeness of the underlying space and containment of the range amongst involved mappings. Singh *et al.*, (2005) weakened the notion of compatibility by using the notion of weak compatible mapping in fuzzy metric space and show that every pair of compatible mappings is weak compatible but reverse is not true. The study of common fixed point of non-compatible mappings is also of great interest due to Pant (1998). Sintunavarat *et al.*, (2011) introduced a new concept of property (CLR_g). Recently Chauhan *et al.*, (2013) utilized the notion of common limit range property to prove unified fixed point theorems for weakly compatible mapping in fuzzy metric spaces. We prove common fixed point theorem for weakly compatible mapping with common limit range property.

2. *Preliminaries*

Definition 2.1 [18] A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is called a continuous t norm if $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a*b\le c*d$ whenever $a\le c$ and $b\le d$ $a,b,c,d\in[0,1]$.

Example of t-norms are $a*b=ab$ and $a*b=\min\{a,b\}$.

Definition 2.2 [9] A Triplet $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,\infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

[1] $M(x, y, 0) = 1$,
[2] $M(x, y, t) = M(y, x, t)$,
[3] $M(x, y, t) = 1$ For all $t > 0$ if and only if $x = y$,
[4] $M(x, y, .) : [0,\infty) \to [0,1]$ is continuous,
[5] $M(x, y, t) * (y, z, s) \le M(x, z, t+s)$
[6] $\lim_{t\to\infty} M(x, y, t) = 1$

Note that $M(x, y, t)$ can be considered as the degree of nearness between x and y with respect to t.We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$.

Definition 2.3[9] Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and $M(x, y, t) = \frac{t}{t + d(x, y)}$ for all $x, y \in X$ and all $t > 0$.Then $(X, M, *)$ is a fuzzy metric space. It is called the fuzzy metric space induced b the metric d.

Definition 2.4 [10] Let (X, M, $*$) be a fuzzy metric space. Then

(a) A sequence $\{x_n\}$ in X is said to converges to x in X if for each $\in$ >0 and each t > 0, there exists $n_0 \in$ N such that M $(x_n, x, t) > 1- \in$ for all $n \ge n_0$.
(b) A sequence $\{x_n\}$ in X is said to be Cauchy if for each $\in$ > 0 and each t > 0, there exists $n_0 \in$ N such that M $(x_n, x_m, t) > 1 - \in$ for all $n, m \ge n_0$.
(c) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.5[17] Two maps F and G, from a fuzzy metric space (X, M, $*$) into itself are said to be R-weakly commuting if there exists a positive real number R such that for each $x \in X$
$M(FGx, GFx, Rt) \ge M(Fx, Gx, t)$ for all $t > 0$.

Definition 2.6 [16] Let F and G be maps from a fuzzy metric space $(X, M, *)$ into itself. The maps F and G are said to compatible if
$\lim_{n\to\infty} M(FGx_n, GFx_n, t) = 1$
For all $t > 0$ whenever $\{x_n\}$ is a sequence in X such that
$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = z$ for some $z \in X$.

Definition 2.7[20] Two maps F and G, from a fuzzy metric space (X, M, $*$) into itself are said to be weak-compatible if they commute at their coincidence points,
i.e. $Fx = Gx$

$\Rightarrow FGx = GFx$

Remark 2.8[20] Let (F, G) be a pair of self-maps of a fuzzy metric space (X, M, $*$).Then (F, G) is R-weakly commuting implies that (F, G) is compatible, which implies that (F, G) is weak compatible. But the converse is not true.

Definition 2.9 [6] A pair (A, S) of self-mappings of a fuzzy metric space (X, M, $*$) is said to satisfy the common limit range property with respect to mapping S (briefly, (CLR_S) property),if there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ where $z \in S(X)$

Definition 2.10[6] Two pairs (A, S) and (B, T) of self-mappings of a fuzzy metric space (X, M, $*$) is said to satisfy the common limit range property with respect to mapping S and T (briefly, (CLR_{ST}) property), if there exists a sequence $\{x_n\},\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Sy_n = z$$

Where $z \in S(X) \bigcap T(X)$

Lemma 2.11 [20] Let $(X, \mathrm{M}, *)$ be a fuzzy metric space. Then for all $x, y \in X, M(x, y, \cdot)$ is a non-decreasing function.

Lemma 2.12 [8] Let $(X, \mathrm{M}, *)$ be a fuzzy metric space. If there exists $k \in (0,1)$ such that for all $x, y \in X, M(x, y, kt) \geq \mathrm{M(x, y, t)}, \forall\ t > 0$, then $x = y$.

Lemma 2.13[4] The only t-norm $*$ satisfying $r * r \geq r$ for all $r \in [0,1]$ is the minimum t-norm, that is $a * b = \min\{a, b\}$ for all $a, b \in [0,1]$.

3. Main Result

Theorem 3.1 Suppose that $(X, M, *)$ be a fuzzy metric space with $a * b = \min\{a, b\}$.Let (P, J) and (Q, L) is weak compatible pairs of self maps of X the following conditions satisfies.

[i] $P(X) \subseteq L(X)$, the pair (P, J) satisfies the property (CLR_S),

[ii] $Q(X) \subseteq J(X)$, the pair (Q, L) satisfies the property (CLR_T),

[iii] There exists $k \in (0,1)$ such that for every $x, y \in X$ and $t > 0$

$$M(\mathrm{Px, Qy, kt}) \geq \phi\left[\min\left\{M(Jx, Ly, t), \mathrm{M(Px, Jx, t), M(Qy, Ly, t), M(Px, Ly, t)}\right\}\right]$$

Where $\phi : [0,1] \to [0,1]$ is a continuous function such that $\phi(t) > t$, for each $0 < t < 1$.

[iv]One of the subset $P(X), Q(X), J(X)$ or $L(X)$ is a closed subset of X.

The pair (P, J) and (Q, L) are weak compatible. Then P, Q, J and L have a unique common fixed point in X.

Proof. Let $Q(X) \subseteq J(X)$ and the pair (Q, L) satisfy the property CLR_T then there exists a sequence $\{x_n\}$ in X such that Qx_n and Lx_n converges to Lx for some x in X as $n \to \infty$. Since $Q(X) \subseteq J(X)$ so that there exists a sequence $\{y_n\}$ in X such that $Qx_n = Jy_n$, hence $Jy_n \to Lx$ as $n \to \infty$.

We show that $\lim_{n\to\infty} Py_n = Lx$.Suppose that $\lim_{n\to\infty} Py_n = z$.

Step 1

We put $x = y_n$ and $y = x_n$ in equation (iii)

$$M(\mathrm{Py}_n, \mathrm{Qx}_n, \mathrm{kt}) \geq \phi\left[\min\left\{M(Jy_n, Lx_n, t), \mathrm{M(Py}_n, \mathrm{Jy}_n, \mathrm{t}), \mathrm{M(Qx}_n, \mathrm{Lx}_n, \mathrm{t}), \mathrm{M(Py}_n, \mathrm{Lx}_n, \mathrm{t})\right\}\right]$$

$$M(\mathrm{Py}_n, \mathrm{Qx}_n, \mathrm{kt}) \geq \phi\left[\min\left\{M(Jy_n, Lx_n, t), \mathrm{M(Py}_n, \mathrm{Jy}_n, \mathrm{t}), \mathrm{M(Qx}_n, \mathrm{Lx}_n, \mathrm{t}), \mathrm{M(Py}_n, \mathrm{Lx}_n, \mathrm{t})\right\}\right]$$

Taking limit $n \to \infty$

$$M(z, \mathrm{Lx, kt}) \geq \phi\left[\min\left\{M(Lx, \mathrm{Lx}, t), \mathrm{M}(z, \mathrm{Lx, t}), \mathrm{M}(z, \mathrm{Lx, t}), \mathrm{M(Lx, Lx, t)}\right\}\right]$$

$$M(z,\text{Lx},\text{kt}) \geq \phi\left[\min\{1,\text{M}(z,\text{Lx},\text{t}),\text{M}(z,\text{Lx},\text{t}),1\}\right]$$

$$M(z,\text{Lx},\text{kt}) \geq \phi[\text{M}(z,\text{Lx},\text{t})] > \text{M}(z,\text{Lx},\text{t})$$

Therefore using lemma 2.12

$$z = Lx$$

We have Qx_n, Lx_n, Py_n and Jy_n converges to z.

We show that $Qx = z$

Step2

We put $x = y_n$ and $y = x$ in (iii)

$$M(\text{Py}_n,\text{Qx},\text{kt}) \geq \phi\left[\min\{M(Jy_n,Lx,t),\text{M}(\text{Py}_n,\text{Jy}_n,\text{t}),\text{M}(\text{Qx},\text{Lx},\text{t}),\text{M}(\text{Py}_n,\text{Lx},\text{t})\}\right]$$

Taking limit $n \to \infty$

$$M(z,\text{Qx},\text{kt}) \geq \phi\left[\min\{M(\text{z},\text{z},t),\text{M}(\text{z},\text{z},\text{t}),\text{M}(\text{Qx},\text{z},\text{t}),\text{M}(\text{z},\text{z},\text{t})\}\right]$$

$$M(z,\text{Qx},\text{kt}) \geq \phi\left[\min\{1,1,\text{M}(\text{z},\text{Qx},\text{t}),1\}\right]$$

$$M(z,\text{Qx},\text{kt}) \geq \phi[\text{M}(z,\text{Qx},\text{t})] > \text{M}(\text{z},\text{Qx},\text{t})$$

Therefore using lemma 2.12

$$z = Qx$$

Therefore $z = Qx = Lx$.Since the pair (Q,L) is weak compatible it follows that $Qz = Lz$ also since $Q(X) \subseteq J(X)$ there exists some y in x such that $Qy = Jy(=z)$.

Step 3

We show that $Jy = Py(=z)$.

We put $x = y$ and $y = x_n$ in (iii)

$$M(\text{Py},\text{Qx}_n,\text{kt}) \geq \phi\left[\min\{M(Jy,Lx_n,t),\text{M}(\text{Py},\text{Jy},\text{t}),\text{M}(\text{Qx}_n,\text{Lx}_n,\text{t}),\text{M}(\text{Py},\text{Lx}_n,\text{t})\}\right]$$

Taking limit $n \to \infty$

$$M(\text{Py},z,\text{kt}) \geq \phi\left[\min\{M(\text{z},\text{z},t),\text{M}(\text{Py},z,\text{t}),\text{M}(\text{z},\text{z},\text{t}),\text{M}(\text{Py},\text{z},\text{t})\}\right]$$

$$M(\text{Py},z,\text{kt}) \geq \phi\left[\min\{1,\text{M}(\text{Py},z,\text{t}),1,\text{M}(\text{Py},\text{z},\text{t})\}\right]$$

$$M(\text{Py},z,\text{kt}) \geq \phi[\text{M}(\text{Py},z,\text{t})] > \text{M}(\text{Py},z,\text{t})$$

$$Py = z$$

By lemma 2.12 which implies that $Py = Jy = z$ .But the pair (P,J) is weak compatible.

We have $Pz = Jz$

Step 4

We claim $Pz = z$

We put $x = z$ and $y = x$ in equation (iii)

$$M(\text{Pz},\text{Qx},\text{kt}) \geq \phi\left[\min\{M(Jz,Lx,t),\text{M}(\text{Pz},\text{Jz},\text{t}),\text{M}(\text{Qx},\text{Lx},\text{t}),\text{M}(\text{Pz},\text{Lx},\text{t})\}\right]$$

Taking limit $n \to \infty$

$$M(\text{Pz},z,\text{kt}) \geq \phi\left[\min\{M(Pz,\text{z},t),\text{M}(\text{Pz},Pz,\text{t}),\text{M}(\text{z},\text{z},\text{t}),\text{M}(\text{Pz},\text{z},\text{t})\}\right]$$

$$M(\text{Pz},z,\text{kt}) \geq \phi\left[\min\{M(Pz,z,t),1,1,\text{M}(\text{Pz},\text{z},\text{t})\}\right]$$

$$M(\text{Pz},z,\text{kt}) \geq \phi[M(Pz,z,t)] > M(Pz,z,t)$$

Therefore by using lemma 2.12 we get

$Pz = z$

Which implies that $Pz = Qz = Jz = Lz = z$.

Therefore z is the common fixed point of P, Q, J, and L.

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