International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jpms.htm
2014 Vol. 4 (2) April-June, pp. 118-124/Patel and Shah
Research Article

GENERALIZED FIBONACCI SEQUENCE AND ITS PROPERTIES

***Vandana R. Patel[1] and Devbhadra V. Shah[2]**
[1]*Department of Mathematics, Sir P.T. Science College, Modasa*
[2]*Department of Mathematics, Sarvajanic of P.T. Science College, Surat*
**Author for Correspondence*

ABSTRACT
The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation. In this paper, we study a new generalization $\{G_n\}$, with initial conditions $G_0 = 0$ and $G_1 = 1$ which is generated by the recurrence relation $G_n = aG_{n-1} + bG_{n-2}$, for all $n \geq 2$, where a and b are nonzero real numbers. Some well-known sequences are special cases of this generalization. The Fibonacci sequence is a special case of $\{G_n\}$ with $a = b = 1$. Pell's sequence is $\{G_n\}$ with $a = 2, b = 1$ and the k-Fibonacci sequence is $\{G_n\}$ with $a = k, b = 1$. We shall define Binet's formula and generating function for Generalized Fibonacci sequence $\{G_n\}$. Mainly, Induction method and Binet's formula will be used to establish properties for Generalized Fibonacci sequence$\{G_n\}$.

***Keywords**: Fibonacci Sequence, Lucas Sequence, Generalized Fibonacci Sequence, Generalized Lucas Sequence*

INTRODUCTION
Fibonacci numbers are perhaps most famous for appearing in the rabbit breeding problem, introduced by Leonardo de Pisa in 1202 in his book called Liber Abaci, they remain fascinating and mysterious to people today. However, they also occur in Pascal's triangle (Koshy, 2001), in Pythagorean triples (Koshy, 2001), computer algorithms (Knott, 1996-2014; Stojmenovic, 2000; Fredman and Tarjan, 1987), some areas of algebra (Feingold, 1980; Suck *et al*., 2002; Schork, 2007), graph theory (Chebotarev, 2008; Bogdonowicz, 2008), quasicrystals (Atkins and Geist, 1987; Zubov *et al*., 1994), and many areas of mathematics. They occur in a variety of other fields such as finance, art, architecture, music, etc., (See Knott (1996-2014) for extensive resources on Fibonacci numbers.) The Fibonacci sequence is a source of many identities as appears in the work of Vajda (1989), Harris (1965) and Carlitz (1970).

The Fibonacci sequence $\{F_n\}$ is defined by $F_n = F_{n-1} + F_{n-2}$, for all $n \geq 2$, with initial conditions $F_0 = 0$ and $F_1 = 1$. Also the sequence of Lucas numbers $\{L_n\}$ is defined by $L_n = L_{n-1} + L_{n-2}$, for all $n \geq 2$, with initial conditions $L_0 = 2$ and $L_1 = 1$.

The Binet's formula for Fibonacci sequence and Lucas sequence is given by $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$ and $L_n = \alpha^n + \beta^n = \left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$ respectively.

Where $\alpha = \left(\frac{1+\sqrt{5}}{2}\right)$ =Golden ratio $= 1.618$ and $\beta = \left(\frac{1-\sqrt{5}}{2}\right) = -0.618$

In this paper, we present different properties of the Generalized Fibonacci sequence $\{G_n\}$ which is defined by $G_n = aG_{n-1} + bG_{n-2}$, for all $n \geq 2$ with $G_0 = 0$ and $G_1 = 1$; where a and b are nonzero real numbers.

The few terms of the sequence $\{G_n\}$ are: 0, 1, a, a^2+2ab, $a^4+3a^2b+b^2$, $a^5+4a^3b+3ab^2$, ...and so on. The Generalized Lucas sequence $\{K_n\}$ which is defined by $K_n = aK_{n-1} + bK_{n-2}$, for all $n \geq 2$ with $K_0 = 2$ and $K_1 = 1$; where a and b are nonzero real numbers.

Generating Function for the Generalized Fibonacci Sequence
Generating functions provide a powerful method for solving linear homogeneous recurrence relations. Even though generating functions are typically used in conjunction with linear recurrence relations with constant coefficients, we will systematically make use of them for linear recurrence relations with

nonconstant coefficients. In this section, we consider the generating function for the generalized Fibonacci sequence and derive some of the most interesting identities satisfied by this sequence.

Theorem 2.1. The generating function for the generalized Fibonacci sequence given by $\{G_n\}$ is $(x) = \frac{x}{1-ax-bx^2}$.

Proof: Let $g(x) = G_0 + G_1 x + G_2 x^2 + \cdots + G_n x^n + \cdots$ be the generating function of the generalized Fibonacci sequence $\{G_n\}$, we note that $G_0 = 0, G_1 = 1$.

Now, $g(x) = G_0 + G_1 x + G_2 x^2 + \cdots + G_n x^n + \cdots$

$axg(x) = aG_0 x + aG_1 x^2 + aG_2 x^3 + \cdots + aG_n x^{n+1} + \cdots$

$bx^2 g(x) = bG_0 x^2 + bG_1 x^3 + bG_2 x^4 + \cdots + bG_n x^{n+2} + \cdots$

We will add the power series $g(x)$, $-axg(x)$, $-bx^2 g(x)$ then we get,

$g(x) - axg(x) - bx^2 g(x) = G_0 + (-aG_0 + G_1)x + (-bG_0 - aG_1 + G_2)x^2 + \ldots\ldots\ldots$

Here notice that if we take our rearranged recursion formula $G_n - aG_{n-1} + bG_{n-2} = 0$, with $n = 2$, we get $G_2 - aG_1 + bG_0 = 0$. Thus, the Co efficient of x^2 term in our combined series is zero. In fact using the recursion formula, the co efficient of the terms after the x^2 term we see they are all zero.

Thus We have $(x) - axg(x) - bx^2 g(x) = G_0 + (-aG_0 + G_1)x$, Since $G_0 = 0, G_1 = 1$

$\therefore (1 - ax - bx^2)g(x) = x$

$\therefore\ g(x) = \frac{x}{1-ax-bx^2} = \sum_{n=0}^{\infty} G_n x^n$, which is required generating function.

Binet's Formula for the Generalized Fibonacci Sequence

Koshy refers to the Fibonacci numbers as one of the "two shining stars in the vast array of integer sequences. We may guess that one reason for this reference is the sheer quantity of interesting properties this sequence possesses. Further still, almost all of these properties can be derived from Binet's formula. A main objective of this paper is to demonstrate that many of the properties of the Fibonacci sequence can be stated and proven for a much larger class of sequences, namely the generalized Fibonacci sequence. Therefore, we will state and prove Binet's formula for the generalized Fibonacci sequence.

Theorem 3.1. (Binet's Formula) The terms of the generalized Fibonacci sequence $\{G_n\}$ are given by $G_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$; where $\alpha = \frac{a+\sqrt{a^2+4b}}{2}$ and $\beta = \frac{a-\sqrt{a^2+4b}}{2}$

Proof: We first express function $g(x)$ for G_n as a sum of partial fractions.

Let $(1 - ax - bx^2) = (1 - \alpha x)(1 - \beta x)$

Now Consider $g(x) = \frac{x}{1-ax-bx^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$

$\therefore x = A(1 - \beta x) + B(1 - \alpha x)$

If $x = \frac{1}{\beta}$ then $\frac{1}{\beta} = B\left(1 - \frac{\alpha}{\beta}\right) \Rightarrow \frac{1}{\beta} = B\left(\frac{\beta-\alpha}{\beta}\right)$

$\Rightarrow B = \frac{1}{\beta - \alpha} \Rightarrow B = \frac{-1}{\alpha - \beta}$

Similarly, If We take $x = \frac{1}{\alpha}$ then we have $\frac{1}{\alpha} = A(1 - \frac{\beta}{\alpha}) \Rightarrow\ A = \frac{1}{\alpha-\beta}$

$$\therefore g(x) = \frac{x}{1 - ax - bx^2} = \frac{\frac{1}{\alpha - \beta}}{1 - \alpha x} + \frac{\frac{-1}{\alpha - \beta}}{1 - \beta x}$$

$$\therefore g(x) = \frac{1}{\alpha-\beta} \sum_{n=0}^{\infty} \alpha^n x^n - \frac{1}{\alpha-\beta} \sum_{n=0}^{\infty} \beta^n x^n$$

$$\therefore g(x) = \sum_{n=0}^{\infty} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) x^n$$

But, $g(x) = \sum_{n=0}^{\infty} G_n \ x^n$

$\therefore G_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ Which is Binet's formula for given generalized Fibonacci sequence.

Properties of Generalized Fibonacci Sequence

Theorem 4.1. Sum of first n terms of the Generalized Fibonacci sequence $\{G_n\}$ is $G_1 + G_2 + \cdots + G_n = \sum_{i=1}^{n} G_i = \frac{T_{n+1} + bT_n - 1}{a+b-1}$

Theorem 4.2. Sum of the first n terms with odd indices is

$$G_1 + G_3 + \cdots + G_{2n-1} = \sum_{i=1}^{n} G_{2i-1} = \frac{1}{1-b}(1 + a[G_2 + G_4 + \cdots + G_{2n}] - G_{2n+1})$$

This identity becomes

$G_{2n+1} - 1 = a[G_0 + G_2 + G_4 + \cdots + G_{2n-2}] + (b-1)\ [G_1 + G_3 + \cdots + G_{2n-1}]$

Theorem 4.3. Sum of the first n terms with even indices is

$$G_2 + G_4 + \cdots + G_{2n} = \sum_{i=1}^{n} G_{2i} = \frac{1}{1-b}(a[G_1 + G_3 + \cdots + G_{2n+1}] - G_{2n+2})$$

This identity becomes

$G_{2n+2} = (b-1)[G_0 + G_2 + G_4 + \cdots + G_{2n}] + a[G_1 + G_3 + \cdots + G_{2n+1}]$

Theorem 4.4. Multiplication of two consecutive generalized Fibonacci numbers is given by

$G_n G_{n+1} = a[G_n^2 + bG_{n-1}^2 + b^2 G_{n-2}^2 + \cdots + b^{n-1} G_1^2 + b^n G_0^2]$

Proof: $G_n = aG_{n-1} + bG_{n-2}$ and $G_{n+1} = aG_n + bG_{n-1}$

$\therefore G_n G_{n+1} = aG_n^2 + \ bG_n G_{n-1}$

$= aG_n^2 + \ b[aG_{n-1}^2 + bG_{n-1}G_{n-2}]$

$= aG_n^2 + \ abG_{n-1}^2 + b^2 G_{n-1}G_{n-2}$

$= aG_n^2 + \ abG_{n-1}^2 + b^2[aG_{n-2}^2 + bG_{n-2}G_{n-3}]$

$= aG_n^2 + \ abG_{n-1}^2 + ab^2 G_{n-2}^2 + b^3 G_{n-2}G_{n-3}$ ……………………………………………..

$= aG_n^2 + \ abG_{n-1}^2 + ab^2 G_{n-2}^2 + \cdots + ab^{n-1}G_1^2 + \ b^n[aG_0^2 + bG_0 G_{-1}]$

$= aG_n^2 + \ abG_{n-1}^2 + ab^2 G_{n-2}^2 + \cdots + ab^{n-1}G_1^2 + ab^n G_0^2$

$\therefore G_n G_{n+1} = a[G_n^2 + bG_{n-1}^2 + b^2 G_{n-2}^2 + \cdots + b^{n-1} G_1^2 + b^n G_0^2]$

$$\therefore G_n G_{n+1} = a \sum_{k=0}^{n} G_n^2 b^{n-k}$$

Theorem 4.5. If $U = \begin{bmatrix} 0 & b \\ 1 & a \end{bmatrix}$ then $U^n = \begin{bmatrix} bG_{n-1} & G_n \\ bG_n & G_{n+1} \end{bmatrix}$

Proof: we prove this result by using principal mathematical induction,

For *n=1* , we have $U = \begin{bmatrix} bG_0 & G_1 \\ bG_1 & G_2 \end{bmatrix}$

$= \begin{bmatrix} 0 & b \\ 1 & a \end{bmatrix}$

Thus result to be proved is true for *n=1*.

Suppose it is true for *n=k*.

i.e. Let $U^k = \begin{bmatrix} bG_{k-1} & G_k \\ bG_k & G_{k+1} \end{bmatrix}$ be true.

Now $U^{k+1} = U^k U$

$= \begin{bmatrix} bG_{k-1} & G_k \\ bG_k & G_{k+1} \end{bmatrix} \begin{bmatrix} 0 & b \\ 1 & a \end{bmatrix}$

$= \begin{bmatrix} bG_k & aG_k + bG_{k-1} \\ bG_{k+1} & aG_{k+1} + bG_k \end{bmatrix}$
$= \begin{bmatrix} bG_k & G_{k+1} \\ bG_{k+1} & G_{k+2} \end{bmatrix}$
Thus result is true for *n=k+1*.
This proves the result by induction.

Theorem 4.6. Prove that : $G_{m+n} = G_m G_{n+1} + bG_{m-1}G_n$; $For\ 0 \leq m \leq n$
Proof: We prove this result by mathematical induction,
For *n=1*, $G_{m+1} = G_m G_2 + bG_{m-1}G_1$
$\therefore aG_m + bG_{m-1} = aG_m + bG_{m-1}$
$\because G_1 = 1, G_2 = a$
$\therefore$ result is true for *n=1*.
Suppose result is true for *n=k*.
$\therefore\ G_{m+k} = G_m G_{k+1} + bG_{m-1}G_k$
Now we have to show that result is true for *n=k+1*.
$\therefore\ G_{m+k+1} = aG_{m+k} + bG_{m+k-1}$
$= a[G_m G_{k+1} + bG_{m-1}G_k] + b[G_m G_k + bG_{m-1}G_{k-1}]$
$= G_m[aG_{k+1} + bG_k] + bG_{m-1}[aG_k + bG_{k-1}]$
$= G_m G_{k+2} + bG_{m-1}G_{k+1}$
$\therefore\ G_{m+k+1} = G_m G_{k+1+1} + bG_{m-1}G_{k+1}$
$\therefore$ result is true for *n=k+1*.
$\therefore$ by induction, we can say that, $G_{m+n} = G_m G_{n+1} + bG_{m-1}G_n$
Note : if we take $n = n - m \geq 0$ then above result can be written as
$G_{m+n-m} = G_m G_{n-m+1} + bG_{m-1}G_{n-m}$
i.e. $G_n = G_m G_{n-m+1} + bG_{m-1}G_{n-m}$
above identity also prove using matrix method,
We know that $U^m \times U^n = U^{m+n}$
We have, If $U = \begin{bmatrix} 0 & b \\ 1 & a \end{bmatrix}$ then $U^n = \begin{bmatrix} bG_{n-1} & G_n \\ bG_n & G_{n+1} \end{bmatrix}$
$\therefore \begin{bmatrix} bG_{m-1} & G_m \\ bG_m & G_{m+1} \end{bmatrix} \times \begin{bmatrix} bG_{n-1} & G_n \\ bG_n & G_{n+1} \end{bmatrix} = \begin{bmatrix} bG_{m+n-1} & G_{m+n} \\ bG_{m+n} & G_{m+n+1} \end{bmatrix}$
$\therefore \begin{bmatrix} b^2 G_{m-1}G_{n-1} + bG_m G_n & bG_{m-1}G_n + G_m G_{n+1} \\ b^2 G_m G_{n-1} + bG_{m+1}G_n & bG_m G_n + G_{m+1}G_{n+1} \end{bmatrix} = \begin{bmatrix} bG_{m+n-1} & G_{m+n} \\ bG_{m+n} & G_{m+n+1} \end{bmatrix}$
Now equating the corresponding entries, then
$\therefore b^2 G_{m-1}G_{n-1} + bG_m G_n = bG_{m+n-1} \Longrightarrow G_{m+n-1} = G_m G_n + bG_{m-1}G_{n-1}$
$\therefore bG_{m-1}G_n + G_m G_{n+1} = G_{m+n} \Longrightarrow G_{m+n} = G_m G_{n+1} + bG_{m-1}G_n$
$\therefore b^2 G_m G_{n-1} + bG_{m+1}G_n = bG_{m+n} \Longrightarrow G_{m+n} = G_{m+1}G_n + bG_m G_{n-1}$
$\therefore bG_m G_n + G_{m+1}G_{n+1} = G_{m+n+1} \Longrightarrow G_{m+n+1} = G_{m+1}G_{n+1} + bG_m G_n$

Theorem 4.7. For any nonnegative integer n we have, $G_{n+1}G_{n-1} - G_n^2 = (-1)^n b^{n-1}$
Proof: $G_n = aG_{n-1} + bG_{n-2}$
$G_{n-1} = aG_{n-2} + bG_{n-3}$ and
$G_{n+1} = aG_n + bG_{n-1}$
Now, $G_{n+1}G_{n-1} - G_n^2 = (aG_n + bG_{n-1})G_{n-1} - G_n^2$
$= aG_n G_{n-1} + bG_{n-1}^2 - G_n^2$
$= bG_{n-1}^2 + G_n(aG_{n-1} - G_n)$
$= bG_{n-1}^2 + G_n(-bG_{n-2}) = bG_{n-1}^2 - bG_n G_{n-2}$
$= -b(G_n G_{n-2} - G_{n-1}^2)$

We can now repeat the above process on the last line to obtain,
$= (-b)^2[G_{n-1}G_{n-3} - G_{n-2}^2]$
$= (-b)^3[G_{n-2}G_{n-4} - G_{n-3}^2]$ ……………………………..
$= (-1)^n b^n [G_1 G_{-1} - G_0^2]$
$= (-1)^n b^{n-1}$ ($\because G_1 = 1$ & $G_{-1} = \frac{1}{b}$ $\therefore G_{n+1}G_{n-1} - G_n^2 = (-1)^n b^{n-1}$
which is known as **cassini's** identity
This identity also is prove using matrix ,
We have, $U = \begin{bmatrix} 0 & b \\ 1 & a \end{bmatrix}$ then $|U| = -b$ and
$U^n = \begin{bmatrix} bG_{n-1} & G_n \\ bG_n & G_{n+1} \end{bmatrix}$ then $|U^n| = bG_{n-1}G_{n+1} - bG_n^2$
$\therefore |U^n| = b[G_{n+1}G_{n-1} - G_n^2]$
$\therefore (-b)^n = b[G_{n+1}G_{n-1} - G_n^2]$
$\therefore (-1)^n b^{n-1} = [G_{n+1}G_{n-1} - G_n^2]$
$\therefore G_{n+1}G_{n-1} - G_n^2 = (-1)^n b^{n-1}$

Theorem 4.8. prove that : $\alpha^n = G_n\alpha + bG_{n-1}$ and $\beta^n = G_n\beta + bG_{n-1}$
Proof: we prove this result by induction, if *n=1* then $\alpha = G_1\alpha + bG_0$
So, $\alpha = \alpha$ as $G_1 = 1$ *and* $G_0 = 0$
There for result is true for *n=1*.
Also, *n=2* then $\alpha^2 = \frac{1}{2}\left(a^2 + 2b + a\sqrt{a^2 + 4b}\right)$ and
$G_2\alpha + bG_1 = \frac{1}{2}\left(a^2 + 2b + a\sqrt{a^2 + 4b}\right)$
Thus result is true for *n=2*.
Suppose result is true for *n=k*.
$\therefore \alpha^k = G_k\alpha + bG_{k-1}$ is true
Now we have to show that result is true for *n=k+1*.
$\therefore \alpha^{k+1} = \alpha\alpha^k$
$= \alpha^2 G_k + \alpha bG_{k-1}$
$= (G_2\alpha + bG_1)G_k + \alpha bG_{k-1}$ ($\because \alpha^2 = G_2\alpha + bG_1$
$= \alpha G_2 G_k + bG_1 G_k + \alpha bG_{k-1}$
$= \alpha(aG_k + bG_{k-1}) + bG_k$ ($\because G_1 = 1$ & $G_2 = a$
$= \alpha G_{k+1} + bG_k$
Thus result is true for *n= k+1*.
So, by mathematical induction we can say that,
$\alpha^n = G_n\alpha + bG_{n-1}$, for all $n \in N$
Similarly, we can prove $\beta^n = G_n\beta + bG_{n-1}$

Theorem 4.9. Prove that : $G_n - c^{n-1} = (a-c)G_{n-1} + [(a-c)c + b][G_0c^{n-2} + G_1c^{n-3} + \ldots +$ *Gn−2 , where 1≤c≤a*
Proof: We prove this result by mathematical induction,
For *n=2,*
$\therefore G_2 - c^{2-1} = (a-c)G_{2-1} + [(a-c)c + b]G_0$
$\therefore aG_1 + bG_0 - c = (a-c)G_1$
$\therefore (a-c)G_1 = (a-c)G_1$
Thus result is true for *n=2*.
Suppose result is true for *n=k*.
$\therefore G_k - c^{k-1} = (a-c)G_{k-1} + [(a-c)c + b][G_0c^{k-2} + G_1c^{k-3} + \cdots + G_{k-2}]$
So, we have to only show that result is true for *n=k+1*.

$\therefore (a-c)G_{k+1-1} + [(a-c)c+b][G_0c^{k+1-2} + G_1c^{k+1-3} + \cdots + G_{k+1-2}] =$
$(a-c)G_k + [(a-c)c+b][G_{k-1} + cG_{k-2} + c^2G_{k-3} + \cdots + c^{k-2}G_1 + c^{k-1}G_0]$
$= (aG_k + bG_{k-1}) - cG_k + b\left(cG_{k-2} + c^2G_{k-3} + \cdots + c^{k-2}G_1 + c^{k-1}G_0\right)$
$+\ a\left(cG_{k-1} + c^2G_{k-2} + \cdots + c^{k-1}G_1 + c^kG_0\right)$
$- c\left(cG_{k-1} + c^2G_{k-2} + \cdots + c^{k-1}G_1 + c^kG_0\right)$
$= G_{k+1} - cG_k + c(aG_{k-1} + bG_{k-2}) + c^2(aG_{k-2} + bG_{k-3}) + \cdots + c^{k-1}(aG_1 + bG_0) - c^2G_{k-1}$
$- c^3G_{k-2} - \cdots - c^{k-1}G_2 - c^kG_1 - c^{k+1}G_0$
$= G_{k+1} - cG_k + cG_k + c^2G_{k-1} + \cdots + c^{k-1}G_2 - c^2G_{k-1} - \cdots - c^{k-1}G_2 - c^k$
$= G_{k+1} - c^k$
$\therefore G_{k+1} - c^k = (a-c)G_k + [(a-c)c+b][G_0c^{k-1} + G_1c^{k-2} + \cdots + G_{k-1}]$
Thus result is true for *n=k+1*.
Hence the result.
Note that if *c=a* in above result then we have,
$G_n - a^{n-1} = b[G_0a^{n-2} + G_1a^{n-3} + \cdots + G_{n-2}]$
i.e. $G_n = a^{n-1} + b\sum_{i=1}^{n-2} a^{i-1}G_{n-1-i}$

Connection Formulae

Theorem 5.1. Prove that : $K_n = G_n + 2bG_{n-1}$,
Proof: $K_n = aK_{n-1} + bK_{n-2}$ with $K_0 = 2$ & $K_1 = 1$
Also, $G_n = aG_{n-1} + bG_{n-2}$ with $G_0 = 0$ & $G_1 = 1$
We prove this result by mathematical induction;
For *n=2*, $K_2 = aK_1 + bK_0 = a + 2b$
and $G_2 + 2bG_1 = a + 2b$
$\therefore K_2 = G_2 + 2bG_1$ is true.
Thus result is true for *n=2*.
Suppose result is true for *n=k*.
$\therefore K_k = G_k + 2bG_{k-1}$
Now we have to show that given result is true for *n=k+1*.
$\therefore K_{k+1} = K_k + bK_{k-1}$
$= a[G_k + 2bG_{k-1}] + b[G_{k-1} + 2bG_{k-2}]$
$= [aG_k + bG_{k-1}] + 2b[aG_{k-1} + bG_{k-2}]$
$= G_{k+1} + 2bG_k$
$\therefore K_{k+1} = G_{k+1} + 2bG_{k+1-1}$
Thus result is true for *n=k+1*.
There for by induction, $K_n = G_n + 2bG_{n-1}$

Theorem 5.2. Prove that : $G_nK_n = G_{2n} + (1-a)G_n^2$
Proof: We know that : $G_{m+n} = G_mG_{n+1} + bG_{m-1}G_n$
$\therefore G_{n+n} = G_nG_{n+1} + bG_{n-1}G_n$
$\therefore G_{2n} = G_n[G_{n+1} + bG_{n-1}]$
$\therefore G_{2n} = G_n[aG_n + bG_{n-1} + bG_{n-1}]$
$\therefore G_{2n} = G_n[aG_n - G_n + G_n + 2bG_{n-1}]$
$\therefore G_{2n} = G_n[(a-1)G_n + K_n]$ ($\because K_n = G_n + 2bG_{n-1}$
$\therefore G_{2n} = (a-1)G_n^2 + G_nK_n$
$\therefore G_nK_n = G_{2n} + (1-a)G_n^2$

Theorem 5.3. For generalized Fibonacci sequence defined by $G_n = aG_{n-1} + bG_{n-2}$ with relatively prime integers a and b then for all $m \geq 1$, G_m and bG_{m-1} are relatively prime.
Proof: First we claim that G_m is relatively prime to *b*.

We know that : $G_m = a^{m-1} + b\sum_{i=1}^{m-2} a^{i-1}G_{m-1-i}$
Suppose $d > 1$ is divisor of G_m and b then d must be divide a^{m-1}
i.e. d/b & d / a^{m-1}
which is impossible as a and b are relatively prime.
So, $gcd(G_m, b) = 1$
Now we prove that G_m and G_{m-1} are relatively prime.
Suppose $d > 1$ is divisor of G_m and G_{m-1}.
But we know that , $G_{m-1}G_{m+1} - G_m^2 = (-1)^m b^{m-1}$
$\therefore d > 1$ is also a divisor of b^{m-1}.
Thus, $d > 1$ is divisor of G_m and b, but $gcd(G_m, b) = 1$
$\therefore d > 1$ is not a divisor of G_m and G_{m-1}.
$\therefore gcd(G_m, G_{m-1}) = 1$
Thus, $gcd(G_m, b) = 1$ and $gcd(G_m, G_{m-1}) = 1$
So, $gcd(G_m, bG_{m-1}) = 1$
$\therefore G_m$ and bG_{m-1} are relatively prime.

CONCLUSION

In this paper, we describe comparable identities of Generalized Fibonacci sequence. we have also developed connection formulas for Generalized Fibonacci sequence and Lucas sequence. It is easy to find out new identities simply by varying the pattern of known identities and using inductive logic to guess new results.

REFERENCES

Atkins J and Geist R (1987). Fibonacci numbers and computer algorithms. *The College Mathematics Journal* **18** 328-336.

Bogdonowicz ZR (2008). Formulas for the number of spanning trees in a fan. *Applied Mathematical Sciences* **2** 781-786.

Carlitz (1970). Some Fibonacci and Lucas identities. *The Fibonacci Quarterly* **8** 61-73.

Chebotarev P (2008). Spanning forests and the golden ratio. *Discrete Applied Mathematics* **156** 813-821.

Feingold AJ (1980). A hyperbolic GCM Lie algebra and the Fibonacci numbers. *Proceedings of the American Mathematical Society* **80** 379-385.

Fredman ML and Tarjan RE (1987). Fibonacci heaps and their uses in improved network optimization algorithms, *Journal of the ACM* **34** 596-615.

Harris VC (1965). On Identities involving Fibonacci Numbers. *The Fibonacci Quarterly* **3** 214-218.

J de Souza, Curado EMF and Rego-Monteiro MA (No Date). Generalized Heisenberg algebras and Fibonacci series. *Journal of Physics A: Mathematical and General* **39** 10415-10425.

Koshy T (2001). *Fibonacci and Lucas Numbers with Applications* (Wiley, New York).

Ron Knott (1996-2014). Fibonacci numbers and the Golden section. Available: http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/

Schork M (2007). Generalized Heisenberg algebras and k-generalized Fibonacci numbers. *Journal of Physics A: Mathematical and Theoretical* **40** 4207-4214.

Stojmenovic I (2000). Recursive algorithms in computer science courses: Fibonacci numbers and binomial coefficients. *IEEE Transactions on Education* **43** 273-276.

Suck JB, Schreiber M and Häussler P (2002). *Quasicrystals: An Introduction to Structure, Physical Properties, and Applications* (Springer-Verlag, Berlin Heidelberg).

Vajda S (1989). *Fibonacci and Lucas Numbers and the Golden Section* (Ellis Horwood Limited, Chichester, England).

Zubov VI and Teixeira JN Rabelo (1994). Anharmonic effects in Fibonacci-chain quasicrystals. *Physical Review B* **49**(3) 8671-8678.