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Research Article

ON SOME INTEGRAL PROPERTIES OF ALEPH FUNCTION, MULTIVARIABLE'S GENERAL CLASS OF POLYNOMIALS ASSOCIATED WITH FEYNMAN INTEGRALS

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ABSTRACT
The idea of the present paper is to discuss certain integral properties of Aleph function and multivariable's general class of polynomials. We establish certain new double integral relations pertaining to a product to a multivariable's general class of polynomials and Aleph function.

Keywords: *Feynman Integrals, Multivariable's General Class of Polynomials, Aleph Function*

INTRODUCTION

The Aleph (χ) - function, introduced by Sudland *et al.,* (1998), however the notation and complete definition is presented here in the following manner in terms on the Mellin- Barnes type integrals

$$\chi[z]=\chi^{m,n}_{x_i,y_i,\tau_i;r}\left[z\,\Big|^{(a_j,A_j)_{1,n}\,[\tau_i(a_{ji},A_{ji})]_{n+1,x_i,r}}_{(b_j,B_j)_{1,m}\,[\tau_i(b_{ji},B_{ji})]_{m+1,y_i,r}}\right]=\frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{x_i,y_i,\tau_i;r}(-s)\,z^{(s)}ds \tag{1}$$

For all $z\neq 0$ where $\omega=\sqrt{(-1)}$ and

$$\Omega^{m,n}_{x_i,y_i,\tau_i;r}(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}\tau_i\prod_{j=n+1}^{x_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=1}^{m}\Gamma(1-b_{ji}-B_{ji}s)} \tag{2}$$

The integration path $L=L_{i\gamma\infty},\gamma\in R$ extends from $\gamma-i\infty\, to\, \gamma+i\infty$, and is such that the poles, assumed to be simple of $\Gamma(1-a_j-A_js)$,j=1,...,n do not coincide with the pole of $\Gamma(b_j+B_js)$,j=1,...,m the parameter x_i, y_i are non-negative integers satisfying: $0\le n\le x_i, 0\le m\le y_i, \tau_i>0$ for i= 1,...,r.

The A_j ,B_j ,A_{ji} ,B_{ji} >0 and a_j ,b_j ,a_{ji} ,$b_{ji}\in$ C. The empty product in (2) is interpreted as unity. The existence conditions for the defining integral (1) are giving below

$$\phi_l>0, |\arg(z)|<\frac{\pi}{2}\phi_l\ , l=1,...,r \tag{3}$$

$$\phi_l\ge 0, |\arg(z)|<\frac{\pi}{2}\phi_l\ \ and\ R\{\xi_l\}<0 \tag{4}$$

Where

$$\phi_l=\sum_{j=1}^{n}A_j+\sum_{j=1}^{m}B_j-\tau_l\left(\sum_{j=n+1}^{x_l}A_{jl}+\sum_{j=m+1}^{y_l}B_{jl}\right) \tag{5}$$

$$\xi_l=\sum_{j=1}^{n}b_j+\sum_{j=1}^{m}a_j+\tau_l\left(\sum_{j=n+1}^{x_l}b_{jl}+\sum_{j=m+1}^{y_l}a_{jl}\right)+\frac{1}{2}(x_l-y_l), l=1,...,r \tag{6}$$

For detailed account of Aleph (χ)-function see Sudland *et al.,* (2001 and 1998). Feynman integrals (Edwards, 1922; Grosche and Steiner, 1998).

The general Class of polynomials $S^{m_1,...,m_r}_{n_1,...,n_r}\left[x_1,...,x_r\right]$ of r variables defined and represented as follows (Srivastava, 1985):

$$S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[x_1,\dots,x_r]=\sum_{k_1}^{\frac{n_1}{m_1}}\dots\sum_{k_r}^{\frac{n_r}{m_r}}\prod_{i=1}^{r}\left\{\frac{(-n)_{m_ik_i}}{\lfloor k_i}A_{n_i,k_i}\;x_i^{k_i}\right\} \tag{7}$$

Where $n_i, m_i = 1,\dots; m_i \neq 0, \forall i \in 1,2,\dots,r$; the coefficients $A(k_1,k_2,\dots,k_r),(k_i \geq 0)$ are arbitrary constant, real or complex. The general class of polynomials (Srivastava and Garg, 1987) is capable of reducing to a number of familiar multivariable polynomials by suitable specializing the arbitrary coefficients $A(k_1,k_2,\dots,k_r),(k_i \geq 0)$

The general class of multivariable polynomials is defined by Srivastava and Garg (1987)

$$S_L^{h_1,\dots,h_r}[X_1,\dots,X_r]=\sum_{k_1,\dots,k_r=0}^{h_1k_1+\dots+h_rk_r\leq L}(-L)_{h_1k_1+\dots+h_rk_r}A(L,k_1,\dots,k_r)\frac{X_1^{k_1}}{\lfloor k_1}\dots\dots\frac{X_r^{k_r}}{\lfloor k_r} \tag{8}$$

Where $h_1,h_2,\dots,h_r$ are arbitrary positive integers and $A(L;k_1,k_2,\dots,k_r),(L;h_i \geq N; i=1,2,\dots,r)$ Coefficients are arbitrary constant, real or complex.

Evidently the case r =1 of the polynomials (8) would correspond to the polynomials given by Srivastava (1972)

$$S_h^L[X]=\sum_{k=0}^{\frac{L}{h}}\frac{(-L)_{L,k}}{\lfloor k}A_{L,k}X^k\{L\in N=(0,1,2,\dots)\} \tag{9}$$

MAIN RESULTS: 1

We shall establish the following results

(A)

$$\int_0^1\int_0^1\left[\left(\frac{1-p}{1-pq}q\right)^{\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1-pq}{(1-p)(1-q)}\right)\right]S_L^{m_1,\dots,m_r}\left[\frac{1-p}{1-pq}wq\right]\chi_{x_i,y_i,\tau_i;r}^{m,n}\left[\frac{1-q}{1-pq}w\right]dpdq$$

$$=\sum_{k_1,\dots,k_r=0}^{m_1k_1,\dots,m_rk_r\leq L}(-L)_{m_1k_1,\dots,m_rk_r}A(L;k_1,\dots,k_r)\left(\prod_{i=1}^{r}\frac{w^{k_i}}{\lfloor(k_i)}\right)\Gamma\left(\sigma+\prod_{i=1}^{r}k_i\right)$$

$$\chi_{x_{i+1},y_{i+1},\tau_i;r}^{m,n+1}\left[w\left|\begin{matrix}(1-\rho;1),\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,x_i;r}\\ \left(b_j,B_j\right)_{1,m},\left[\tau_i\,(b_{ji},B_{ji})\right]_{m+1,y_i;r},\left[\tau_i\left(1-\prod_{i=1}^{r}k_i-\sigma-\rho,1\right)\right]\end{matrix}\right.\right] \tag{10}$$

The above result is valid under the conditions (3), (4); $\mathrm{Re}[\sigma+\rho+b_j/\rho_j]>0, |\arg\omega|<\frac{T\pi}{2},(1\leq j\leq m)$ and a,b are positive. Also $0<p<1$ and $0<q<1$.

Proof:-We have

$$S_L^{m_1,\dots,m_r}\left[\frac{(1-p)}{1-pq}wq\right]\chi_{x_i,y_i,\tau_i;r}^{m,n}\left[\frac{(1-q)}{1-pq}w\right]=$$

$$\sum_{k_1,\dots,k_r=0}^{h_1k_1+\dots+h_rk_r\leq L}(-L)_{h_1k_1+\dots+h_rk_r}A(L,k_1,\dots,k_r)\prod_{i=1}^{r}\left\{\frac{1}{\lfloor k_i}\left(\frac{1-p}{1-pq}wq\right)^{k_i}\right\}$$

$$\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}\tau_i\prod_{j=n+1}^{x_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=1}^{m}\Gamma(1-b_{ji}-B_{ji}s)}\left(\frac{1-q}{1-pq}w\right)^{-s}ds \tag{11}$$

Multiplying both sides of (11) by

$\left[\left(\frac{1-p}{1-pq}q\right)^{\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1-pq}{(1-p)(1-q)}\right)\right]$ and integrating with respect to p and q between 0 and 1 for both the variable and get the result.

(B)

$$\int_0^{\infty}\int_0^{\infty} f(\eta+w)\,\eta^{\sigma-1}w^{\rho-1}S_L^{m_1,\dots,m_r}[\eta]\,\chi^{m,n}_{x_i,y_i,\tau_i;r}[w]\,d\eta\,dw$$

$$=\sum_{k_1,\dots,k_r=0}^{m_1k_1,\dots,m_rk_r\le L}(-L)_{m_1k_1,\dots,m_rk_r}A(L;k_1,\dots,k_r)\left(\prod_{i=1}^{r}\frac{w^{k_i}}{\lfloor k_i}\right)\Gamma\left(\sigma+\prod_{i=1}^{r}k_i\right)\int_0^{\infty}f(\xi)\,\xi^{\sigma+\rho+\prod_{i=1}^{r}k_i-1}$$

$$\chi^{m,n+1}_{x_{i+1},y_{i+1},\tau_i;r}\left[\left|\xi\right.\begin{matrix}(1-\rho;1),(a_j,A_j)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,x_i;r}\\ \left(b_j,,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,y_i;r},\left[\tau_i\left(1-\prod_{i=1}^{r}k_i-\sigma-\rho,1\right)\right]\end{matrix}\right]d\xi \tag{12}$$

The above result is valid under the conditions (3), (4) ; $\mathrm{Re}[\sigma+\rho+b_j/\rho_j]>0, |\arg\omega|<\frac{T\pi}{2}, (1\le j\le m)$ and a,b are positive. Also $0<\eta<\infty$ and $0<w<\infty$.

Proof:-We have

$$S_L^{m_1,\dots,m_r}[\eta]\,\chi^{m,n}_{x_i,y_i,\tau_i;r}[w]=\sum_{k_1,\dots,k_r=0}^{h_1k_1+\dots+h_rk_r\le L}(-L)_{h_1k_1+\dots+h_rk_r}A(L,k_1,\dots,k_r)\prod_{i=1}^{r}\left\{\frac{1}{\lfloor k_i}(\eta)^{k_i}\right\}$$

$$\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}\tau_i\prod_{j=n+1}^{x_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=1}^{m}\Gamma(1-b_{ji}-B_{ji}s)}(w)^{-s}\,ds \tag{13}$$

Multiplying both sides of (13) by

$\left[f(\eta+w)(\eta)^{\sigma-1}(w)^{\rho-1}\right]$ and integrating with respect to η and w between 0 and ∞ for both the variable and get the result.

(C)

$$\int_0^1\int_0^1\psi(\eta w)\,(1-w)^{\rho-1}(1-\eta)^{\sigma-1}w^{\sigma}S_L^{m_1,\dots,m_r}[w(1-\eta)]\,\chi^{m,n}_{x_i,y_i,\tau_i;r}[(1-w)]\,d\eta\,dw=$$

$$=\sum_{k_1,\dots,k_r=0}^{m_1k_1,\dots,m_rk_r\le L}(-L)_{m_1k_1,\dots,m_rk_r}A(L;k_1,\dots,k_r)\left(\prod_{i=1}^{r}\frac{w^{k_i}}{\lfloor k_i}\right)\Gamma\left(\sigma+\prod_{i=1}^{r}k_i\right)\int_0^{1}f(\xi)\,\xi^{\sigma+\rho+\prod_{i=1}^{r}k_i-1}$$

$$\chi^{m,n+1}_{x_{i+1},y_{i+1},\tau_i;r}\left[\frac{(1-\rho,1),(a_j,A_j)_{1,n},[\tau_i(a_{ji},A_{ji})]_{n+1,x_i;r}}{(b_j,\beta_j)_{1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,y_i;r},\tau_i[1-\prod_{i=1}^{r}k_i-\sigma-\rho;1]}\Big|(1-\xi)\right] \tag{14}$$

The above result is valid under the conditions (3), (4); $\mathrm{Re}[\sigma+\rho+b_j/\rho_j]>0, |\arg\omega|<\frac{T\pi}{2}, (1\le j\le m)$ and a,b are positive. Also $0<\eta<1$ and $0<w<1$.

Proof:-We have

$$S_L^{m_1,\ldots,m_r}[w(1-\eta)]\,\chi^{m,n}_{x_i,y_i,\tau_i;r}[1-w]=\sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\le L}(-L)_{h_1k_1+\ldots+h_rk_r}A(L,k_1,\ldots,k_r)\prod_{i=1}^{r}\left\{\frac{1}{\lfloor k_i}\left[w(1-\eta)\right]^{k_i}\right\}$$

$$\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}\tau_i\prod_{j=n+1}^{x_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=1}^{m}\Gamma(1-b_{ji}-B_{ji}s)}(1-w)^{-s}\,ds \tag{15}$$

Multiplying both sides of (15) by $\left[\psi(\eta w)(1-\eta)^{\sigma-1}(1-w)^{\rho-1}w^{\sigma}\right]$ and integrating with respect to η and w between 0 and 1for both the variable and get the result.

(D)

$$\int_0^1\!\!\int_0^1\left[\left\{\frac{q(1-p)}{(1-pq)}\right\}^{a+\sigma}\left\{\frac{1-q}{1-pq}\right\}^{\sigma}\left(\frac{1}{(1-p)}\right)\right]S_L^{m_1,\ldots,m_r}\left[\frac{q(1-p)}{(1-pq)}\right]\chi^{m,n}_{x_i,y_i,\tau_i;r}\left[\frac{wq(1-p)}{(1-pq)}\right]dpdq$$

$$=\sum_{k_1,\ldots,k_r=0}^{m_1k_1,\ldots,m_rk_r\le L}(-L)_{m_1k_1,\ldots,m_rk_r}A(L;k_1,\ldots,k_r)\frac{1}{\lfloor k_i}\Gamma(\rho+1)$$

$$\chi^{m,n+1}_{x_{i+1},y_{i+1},\tau_i;r}\left[w\left|\begin{matrix}\left(1-a-\sigma-\prod_{i=1}^{r}k_i,1\right),(a_j,A_j)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,x_i;r}\\(b_j,B_j)_{1,m},\left[\tau_i(b_{ji},B_{ji})\right]_{m+1,y_i;r},\left[\tau_i\left(-a-\sigma-\rho-\prod_{i=1}^{r}k_i,1\right)\right]\end{matrix}\right.\right] \tag{16}$$

The above result is valid under the conditions (3), (4); $\mathrm{Re}[\sigma+\rho+b_j/\rho_j]>0, |\arg\omega|<\frac{T\pi}{2}, (1\le j\le m)$ and a,b are positive. Also $0<p<1$ and $0<q<1$.

Proof:-We have

$$S_L^{m_1,\ldots,m_r}\left[\frac{q(1-p)}{1-pq}\right]\chi^{m,n}_{x_i,y_i,\tau_i;r}\left[\frac{wq(1-q)}{1-pq}\right]=\sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\le L}(-L)_{h_1k_1+\ldots+h_rk_r}A(L,k_1,\ldots,k_r)\prod_{i=1}^{r}\left\{\frac{1}{\lfloor k_i}\left[\frac{q(1-p)}{1-pq}\right]^{k_i}\right\}$$

$$\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}\tau_i\prod_{j=n+1}^{x_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=1}^{m}\Gamma(1-b_{ji}-B_{ji}s)}\left[\frac{wq(1-q)}{1-pq}\right]^{-s}ds \tag{17}$$

Multiplying both sides of (17) by $\left[\left\{\frac{q(1-p)}{1-pq}q\right\}^{\sigma+a}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1}{(1-p)}\right)\right]$ and integrating with respect to p and q between 0 and 1 for both the variable and get the result.

MAIN RESULTS: 2

If we use the multivariable's general Class of polynomials $S_{n_1,...,n_r}^{m_1,...,m_r}\left[x_1,...,x_r\right]$ defined and represented as follows (Sudland *et al.*, 2001) then we also get new results.

(A1)

$$\int_0^1\int_0^1\left[\left(\frac{(1-p)}{1-pq}q\right)^{\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{(1-pq)}{(1-p)(1-q)}\right)\right]S_{n_1,...,n_r}^{m_1,...,m_r}\left[\frac{(1-p)}{1-pq}wq\right]\chi_{x_i,y_i,\tau_i;r}^{m,n}\left[\frac{(1-q)}{1-pq}w\right]dpdq$$

$$=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\left\{\frac{(-n_i)_{m_ik_i}}{\lfloor k_i}A_{n_i,k_i}w^{k_i}\right\}\Gamma(\sigma+\prod_{i=1}^{r}k_i)$$

$$\chi_{x_{i+1},y_{i+1},\tau_i;r}^{m,n+1}\left[w\,\middle|\,\begin{matrix}(1-\rho;1),\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,x_i;r}\\ \left(b_j,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,y_i;r},\left[\tau_i\left(1-\prod_{i=1}^{r}k_i-\sigma-\rho,1\right)\right]\end{matrix}\right] \tag{18}$$

The above result is valid under the conditions (3), (4); $\mathrm{Re}[\sigma+\rho+b_j/\rho_j]>0, |\arg\omega|<\frac{T\pi}{2}, (1\le j\le m)$ and a,b are positive. Also $0<p<1$ and $0<q<1$.

Proof:-We have

$$S_{n_1,...,n_r}^{m_1,...,m_r}\left[\frac{(1-p)}{1-pq}wq\right]\chi_{x_i,y_i,\tau_i;r}^{m,n}\left[\frac{(1-q)}{1-pq}w\right]=\sum_{k_1=0}^{\frac{n_1}{m_1}}\cdots\sum_{k_r=0}^{\frac{n_r}{m_r}}\prod_{i=1}^{r}\left\{\frac{(-n)_{m_ik_i}}{\lfloor k_i}A_{n_i,k_i}\left(\frac{1-p}{1-pq}wq\right)^{k_i}\right\}$$

$$=\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}\tau_i\prod_{j=n+1}^{x_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=1}^{m}\Gamma(1-b_{ji}-B_{ji}s)}\left(\frac{1-q}{1-pq}w\right)^{-s}ds \tag{19}$$

Multiplying both sides of (19) by

$\left[\left(\frac{1-p}{1-pq}q\right)^{\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1-pq}{(1-p)(1-q)}\right)\right]$ and integrating with respect to p and q between 0 and 1 for both the variable and get the result.

(B1)

$$\int_0^\infty\int_0^\infty f(\eta+w)\,\eta^{\sigma-1}w^{\rho-1}S_{n_1,...,n_r}^{m_1,...,m_r}[\eta]\,\chi_{x_i,y_i,\tau_i;r}^{m,n}[w]\,d\eta dw$$

$$=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\left\{\frac{(-n_i)_{m_ik_i}}{\lfloor k_i}A_{n_i,k_i}w^{k_i}\right\}\Gamma(\sigma+\prod_{i=1}^{r}k_i)\int_0^\infty f(\xi)\,\xi^{\sigma+\rho+\prod_{i=1}^{r}k_i-1}$$

$$\chi_{x_{i+1},y_{i+1},\tau_i;r}^{m,n+1}\left[\xi\,\middle|\,\begin{matrix}(1-\rho;1),\left(a_j,A_j\right)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,x_i;r}\\ \left(b_j,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,y_i;r},\left[\tau_i\left(1-\prod_{i=1}^{r}k_i-\sigma-\rho,1\right)\right]\end{matrix}\right]d\xi \tag{20}$$

The above result is valid under the conditions (3), (4) ; $\mathrm{Re}[\sigma+\rho+b_j/\rho_j]>0, |\arg\omega|<\frac{T\pi}{2}, (1\le j\le m)$ and a,b are positive. Also $0<\eta<\infty$ and $0<w<\infty$.

Proof:-We have

$$S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[\eta]\,\chi_{x_i,y_i,\tau_i;r}^{m,n}[w] = \sum_{k_1=0}^{n_1/m_1}\dots\sum_{k_r=0}^{n_r/m_r}\prod_{i=1}^{r}\left\{\frac{(-n)_{m_ik_i}}{\lfloor k_i}A_{n_i,k_i}(\eta)^{k_i}\right\}$$

$$\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}\tau_i\prod_{j=n+1}^{x_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=1}^{m}\Gamma(1-b_{ji}-B_{ji}s)}(w)^{-s}\,ds \tag{21}$$

Multiplying both sides of (21) by

$\left[f(\eta+w)(\eta)^{\sigma-1}(w)^{\rho-1}\right]$ and integrating with respect to η and w between 0 and ∞ for both the variable and get the result.

(C1)

$$\int_0^1\int_0^1 \psi(\eta w)\,(1-w)^{\rho-1}(1-\eta)^{\sigma-1}\,w^{\sigma}\,S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[w(1-\eta)]\,\chi_{x_i,y_i,\tau_i;r}^{m,n}[(1-w)]\,d\eta\,dw =$$

$$\sum_{k_1=0}^{[n_1/m_1]}\dots\sum_{k_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\left\{\frac{(-n_i)m_ik_i}{\lfloor k_i}A_{n_i,k_i}\,w^{k_i}\right\}\Gamma(\sigma+\prod_{i=1}^{r}k_i)\int_0^1 f(\xi)\,\xi^{\sigma+\rho+\prod_{i=1}^{r}k_i-1}\,d\xi$$

$$\chi_{x_{i+1},y_{i+1},\tau_i;r}^{m,n+1}\left[\frac{(1-\rho,1),(a_j,A_j)_{1,n},[\tau_i(a_{ji},A_{ji})]_{n+1,x_i;r}}{(b_j,\beta_j)_{1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,y_i;r},\tau_i[1-\prod_{i=1}^{r}k_i-\sigma-\rho;1]}\Big|(1-\xi)\right] \tag{22}$$

The above result is valid under the conditions (3), (4) ; $\mathrm{Re}[\sigma+\rho+b_j/\rho_j]>0, |\arg\omega|<\frac{T\pi}{2}, (1\le j\le m)$ and a,b are positive. Also $0<\eta<1$ and $0<w<1$.

Proof:-We have

$$S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[w(1-\eta)]\,\chi_{x_i,y_i,\tau_i;r}^{m,n}[1-w] = \sum_{k_1=0}^{n_1/m_1}\dots\sum_{k_r=0}^{n_r/m_r}\prod_{i=1}^{r}\left\{\frac{(-n)_{m_ik_i}}{\lfloor k_i}A_{n_i,k_i}[w(1-\eta)]^{k_i}\right\}$$

$$\frac{1}{2\pi\omega}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}\tau_i\prod_{j=n+1}^{x_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=1}^{m}\Gamma(1-b_{ji}-B_{ji}s)}(1-w)^{-s}\,ds \tag{23}$$

Multiplying both sides of (23) by

$\left[\psi(\eta w)(1-\eta)^{\sigma-1}(1-w)^{\rho-1}w^{\sigma}\right]$ and integrating with respect to η and w between 0 and 1for both the variable and get the result.

(D1)

$$\int_0^1\int_0^1\left[\left\{\frac{q(1-p)}{(1-pq)}\right\}^{a+\sigma}\left\{\frac{1-q}{1-pq}\right\}^{\rho}\left(\frac{1}{(1-p)}\right)\right]S_{n_1,\dots,n_r}^{m_1,\dots,m_r}\left[\frac{q(1-p)}{(1-pq)}\right]\chi_{x_i,y_i,\tau_i;r}^{m,n}\left[\frac{wq(1-p)}{(1-pq)}\right]dpdq$$

$$=\sum_{k_1=0}^{[n_1/m_1]}\dots\sum_{k_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\left\{\frac{(-n_i)_{m_ik_i}}{\lfloor k_i}A_{n_i,k_i}\right\}\Gamma(\rho+1)$$

$$\chi_{x_{i+1},y_{i+1},\tau_i;r}^{m,n+1}\left[\begin{matrix}\left(1-a-\sigma-\prod_{i=1}^{r}k_i,1\right),(a_j,A_j)_{1,n}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,x_i;r}\\ (b_j,B_j)_{1,m},\left[\tau_i(b_{ji},B_{ji})\right]_{m+1,y_i;r},\left[\tau_i\left(-a-\sigma-\rho-\prod_{i=1}^{r}k_i,1\right)\right]\end{matrix}\Bigg|w\right] \quad (24)$$

The above result is valid under the conditions (3), (4) ; $\mathrm{Re}[\sigma+\rho+b_j/\rho_j]>0, |\arg\omega|<\frac{T\pi}{2}, (1\le j\le m)$ and a,b are positive. Also $0<p<1$ and $0<q<1$.

Proof:-We have

$$S_{n_1,\dots,n_r}^{m_1,\dots,m_r}\left[\frac{q(1-p)}{1-pq}\right]\chi_{x_i,y_i,\tau_i;r}^{m,n}\left[\frac{wq(1-q)}{1-pq}\right]=\sum_{k_1=0}^{\frac{n_1}{m_1}}\dots\sum_{k_r=0}^{\frac{n_r}{m_r}}\prod_{i=1}^{r}\left\{\frac{(-n)_{m_ik_i}}{\lfloor k_i}A_{n_i,k_i}\left[\frac{q(1-p)}{1-pq}\right]^{k_i}\right\}$$

$$\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}\tau_i\prod_{j=n+1}^{x_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=1}^{m}\Gamma(1-b_{ji}-B_{ji}s)}\left[\frac{wq(1-q)}{1-pq}\right]^{-s}ds \quad (25)$$

Multiplying both sides of (25) by

$\left[\left\{\frac{q(1-p)}{1-pq}q\right\}^{\sigma+a}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1}{(1-p)}\right)\right]$ and integrating with respect to p and q between 0 and 1 for both the variable and get the result.

SPECIAL CASES

1 If we choosing

$\tau_1=\tau_2=\dots=\tau_r=1$ in equation (18), (20), (22) and (24) then Aleph –function reduce to I-function (Agrawal, 2011)

(A2)

$$\int_0^1\int_0^1\left[\left(\frac{1-p}{1-pq}q\right)^{\sigma}\left(\frac{1-q}{1-pq}\right)^{\rho}\left(\frac{1-pq}{(1-p)(1-q)}\right)\right]S_{n_1,\dots,n_r}^{m_1,\dots,m_r}\left[\frac{1-p}{1-pq}wq\right]I_{x_i,y_i;r}^{m,n}\left[\frac{1-q}{1-pq}w\right]dpdq$$

$$=\sum_{k_1=0}^{[n_1/m_1]}\dots\sum_{k_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\left\{\frac{(-n_i)_{m_ik_i}}{\lfloor k_i}A_{n_i,k_i}w^{k_i}\right\}\Gamma(\sigma+\prod_{i=1}^{r}k_i)$$

$$I_{x_{i+1},y_{i+1};r}^{m,n+1}\left[\begin{matrix}(1-\rho;1),(a_j,A_j)_{1,n}\left[\left(a_{ji},A_{ji}\right)\right]_{n+1,x_i;r}\\ (b_j,B_j)_{1,m},\left[(b_{ji},B_{ji})\right]_{m+1,y_i;r},\left[\left(1-\prod_{i=1}^{r}k_i-\sigma-\rho,1\right)\right]\end{matrix}\Bigg|w\right] \quad (26)$$

The above result is valid under the conditions (3), (4); $\mathrm{Re}[\sigma+\rho+b_j/\rho_j]>0, |\arg\omega|<\frac{T\pi}{2}, (1\le j\le m)$ and a,b are positive. Also $0<p<1$ and $0<q<1$

(B2)

$$\int_0^\infty\int_0^\infty f(\eta+w)\,\eta^{\sigma-1}w^{\rho-1}S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[\eta]\,I_{x_i,y_i;r}^{m,n}[w]\,d\eta dw$$

$$=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\left\{\frac{(-n_i)_{m_ik_i}}{\lfloor k_i}A_{n_i,k_i}w^{k_i}\right\}\Gamma(\sigma+\prod_{i=1}^{r}k_i)\int_0^\infty f(\xi)\xi^{\sigma+\rho+\prod_{i=1}^{r}k_i-1}$$

$$\mathrm{I}_{x_{i+1},y_{i+1};r}^{m,n+1}\left[\begin{matrix}(1-\rho;1),(a_j,A_j)_{1,n}\left[(a_{ji},A_{ji})\right]_{n+1,x_i;r}\\(b_j,B_j)_{1,m},\left[(b_{ji},B_{ji})\right]_{m+1,y_i;r},\left[\left(1-\prod_{i=1}^{r}k_i-\sigma-\rho,1\right)\right]\end{matrix}\middle|\xi\right]d\xi \qquad (27)$$

The above result is valid under the conditions (3), (4) ; $\mathrm{Re}[\sigma+\rho+b_j/\rho_j]>0, |\arg\omega|<\frac{T\pi}{2}, (1\le j\le m)$ and a,b are positive. Also $0<\eta<\infty$ and $0<w<\infty$.

(C2)

$$\int_0^1\int_0^1\psi(\eta w)\,(1-w)^{\rho-1}(1-\eta)^{\sigma-1}w^{\sigma}S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[w(1-\eta)]\,I_{x_i,y_i;r}^{m,n}[(1-w)]\,d\eta dw=$$

$$\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\left\{\frac{(-n_i)_{m_ik_i}}{\lfloor k_i}A_{n_i,k_i}w^{k_i}\right\}\Gamma(\sigma+\prod_{i=1}^{r}k_i)\int_0^1 f(\xi)\xi^{\sigma+\rho+\prod_{i=1}^{r}k_i-1}d\xi$$

$$\mathrm{I}_{x_{i+1},y_{i+1};r}^{m,n+1}\left[\frac{(1-\rho,1),(a_j,A_j)_{1,n},[(a_{ji},A_{ji})]_{n+1,x_i;r}}{(b_j,\beta_j)_{1,m},[(b_{ji},\beta_{ji})]_{m+1,y_i;r},[1-\prod_{i=1}^{r}k_i-\sigma-\rho;1]}\middle|(1-\xi)\right] \qquad (28)$$

The above result is valid under the conditions (3), (4) ; $\mathrm{Re}[\sigma+\rho+b_j/\rho_j]>0, |\arg\omega|<\frac{T\pi}{2}, (1\le j\le m)$ and a,b are positive. Also $0<\eta<1$ and $0<w<1$.

(D2)

$$\int_0^1\int_0^1\left[\left\{\frac{q(1-p)}{(1-pq)}\right\}^{a+\sigma}\left\{\frac{1-q}{1-pq}\right\}^{\rho}\left(\frac{1}{(1-p)}\right)\right]S_{n_1,\dots,n_r}^{m_1,\dots,m_r}\left[\frac{q(1-p)}{(1-pq)}\right]I_{x_i,y_i;r}^{m,n}\left[\frac{wq(1-p)}{(1-pq)}\right]dpdq$$

$$=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\prod_{i=1}^{r}\left\{\frac{(-n_i)_{m_ik_i}}{\lfloor k_i}A_{n_i,k_i}\right\}\Gamma(\rho+1)$$

$$\mathrm{I}^{m,n+1}_{x_{i+1},y_{i+1};r}\left[\begin{matrix}\left(1-a-\sigma-\prod_{i=1}^{r}k_i,1\right),\left(a_j,A_j\right)_{1,n}\left[\left(a_{ji},A_{ji}\right)\right]_{n+1,x_i;r}\\ \left(b_j,B_j\right)_{1,m},\left[\left(b_{ji},B_{ji}\right)\right]_{m+1,y_i;r},\left[\left(-a-\sigma-\rho-\prod_{i=1}^{r}k_i,1\right)\right]\end{matrix}\middle|w\right] \quad (29)$$

The above result is valid under the conditions (3), (4) ; $\mathrm{Re}[\sigma+\rho+b_j/\rho_j]>0, |\arg\omega|<\frac{T\pi}{2}, (1\le j\le m)$ and a,b are positive. Also $0<x<1$ and $0<y<1$.

2. If we have putting

$\tau_1=\tau_2=...=\tau_r=1$ and r=1 in equation (18), (20) (22) and (24) then Aleph –function reduce to H-function [1, equation (20), (21), (22), (23)]

3. If we Putting $n_i=1$, $m_i=1; i=1,2,...,r$ in equation (18), (20), (22)and (24) then we get [4,equation (2.1), (2.3), (2.5), (2.7)]

Conclusion

The result obtained here are basic in nature and are likely to find useful applications in the study of simple and multiple variable hyper geometric series which in turn are useful in electrical networks, statistical mechanics and probability theory.

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