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Research Article

C_λ – RATE SEQUENCE SPACES OF DIFFERENCE SEQUENCE DEFINED BY A MODULUS FUNCTION

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ABSTARCT
Recall that a C_λ method is obtained by deleting a set of rows from the Cesáro matrix C_1. The purpose of this article is to introduce the sequence spaces $C_\lambda^0(p,f,s,\pi)$, $C_\lambda^c(p,f,s,\pi)$ and $C_\lambda^\infty(p,f,s,\pi)$ using a modulus function f. Several properties of this spaces, and some inclusion relations have been examined.

Keywords: *Modulus Function, Rate Sequence Spaces, Difference Sequence, Paranorm,* C_λ*-Summability Method*
AMS Clasification: *40C05, 40D25, 40G05, 42A05, 42A10*

INTRODUCTION

The notion of modulus function was introduced by Nakano [12] and further investigated by Ruckle [14], Maddox [9], Tripathy and Chandra [15] and many others. A function $f:[0,\infty)\to[0,\infty)$ is called a modulus if

(i) $f(x)=0$ if and only if $x=0,$

(ii) $f(x+y)\le f(x)+f(y),$

(iii) f is increasing,

(iv) f is continuous from the right at 0.

It is immediate from (ii) and (iv) that f is continuous everywhere on $[0,\infty)$. The idea of difference sequence was first introduced by Kizmaz [6] write $\Delta x_k = x_k - x_{k+1}$ for $k=1,2,3,....$ Let ω denote the space of all comlex-valued sequences, $\Delta:\omega\to\omega$ be the difference defined by $\Delta x=(\Delta x_k)_{k=1}^{\infty}$. Let ω denote the space of all real or complex-valued sequence. It can be topologized with the seminorms $p_n(x)=|x_n|$, $(n=1,2,...)$, any vector subspace X of ω is a sequence space. A sequence space X with a vector space topology τ, is a K-space provided that the inclusion map $i:(X,\tau)\to\omega$, $i(x)=x$, is continuous. If, in addition, τ is complete, metrizable and locally convex then (X,τ) is an FK-space. So an FK-space is a complete, metrizable locally convex topological vector space of sequences for which the coordinate functionals $P_n(x)=x_n$, $(n=1,2,\ldots)$, are continuous. The basic properties of FK-spaces may be found in [3], [16], [17] and [19].

Ruckle [14] used the idea of a modulus function f to construct a class of FK spaces

$$L(f)=\left\{x=(x_k):\sum_{k=1}^{\infty} f(|x_k|)<\infty\right\}.$$

Let $\pi=(\pi_n)$ be a sequence of positive numbers i.e, $\pi_n>0,\forall n\in\mathbb{N}$ and X an $FK-$space. We shall consider the sets of sequences $x=(x_n)$

$$X_\pi=\{x\in w:\left(\frac{x_n}{\pi_n}\right)\in X\}.$$

The set X_π may be considered as FK-space. We shall call them as rate spaces (see, [4] and [5]). Let F be an infinite subset of $\mathbb{N}$ and F as the range of a strictly increasing sequence of positive integers, say

$F=\{\lambda(n)\}_{n=1}^{\infty}$. The Cesáro submethod C_λ is defined as

$$(C_\lambda x)_n=\frac{1}{\lambda(n)}\sum_{k=1}^{\lambda(n)}x_k,\ (n=1,2,...),$$

where $\{x_k\}$ is a sequence of a real or complex numbers. Therefore, the C_λ-method yields a subsequence of the Cesáro method C_1, and hence it is regular for any λ. C_λ is obtained by deleting a set of rows from Cesáro matrix. The basic properties of C_λ – method can be found in [1] and [13]. We need the following inequality throughout the paper. Let $p=(p_k)$ be a sequence of positive real numbers with $G=\sup_k p_k$ and $D=\max(1,2^{G-1})$. Then, it is well known that for all $a_k,\ b_k\in\mathbb{C}$, the field of complex numbers, for all $k\in\mathbb{N}$,

$$|a_k+b_k|^{p_k}\le D\left(|a_k|^{p_k}+|b_k|^{p_k}\right) \quad (1)$$

Also for any complex μ,

$$|\mu|^{p_k}\le\max(1,|\mu|^G) \quad (2)$$

see in [11]. Motivating by Maddox [9], Jürimäe [4] and Başarır [2] we make the following definitions: The following sequence spaces

$$C_\lambda^\infty(\pi)=\left\{x\in w:\ \sup_k\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)}\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right]<\infty\right\},$$

$$C_\lambda^0(\pi)=\left\{x\in w:\ \left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)}\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right]\to 0\ (k\to\infty)\right\},$$

$$C_\lambda^c(\pi)=\left\{x\in w:\ \left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)}\left|\Delta\left(\frac{x}{\pi}\right)_i-L\right|\right]\to 0\ (k\to\infty),\ \text{for some } L>0\right\},$$

$$C_\lambda^\infty(p,\pi)=\left\{x\in w:\ \sup_k\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)}\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right]^{p_k}<\infty\right\},$$

$$C_\lambda^0(p,\pi)=\left\{x\in w:\ \left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)}\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right]^{p_k}\to 0\ (k\to\infty)\right\},$$

$$C_\lambda^c(p,\pi)=\left\{x\in w:\ \left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)}\left|\Delta\left(\frac{x}{\pi}\right)_i-L\right|\right]^{p_k}\to 0\ (k\to\infty),\ for\,some\,L>0\right\},$$

$$C_\lambda^\infty(p,s,\pi)=\left\{x\in w:\ \sup_k k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)}\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right]^{p_k}<\infty,\ s\ge 0\right\},$$

$$C_\lambda^0(p,s,\pi)=\left\{x\in w:\ k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)}\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right]^{p_k}\to 0\ (k\to\infty),\ s\ge 0\right\}$$

and

$$C_\lambda^c(p,s,\pi)=\left\{x\in w:\ k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)}\left|\Delta\left(\frac{x}{\pi}\right)_i-L\right|\right]^{p_k}\to 0\ (k\to\infty),\ s\ge 0,\ for\,some\,L>0\right\}.$$

RESULTS

In this section, C_λ – rate sequence spaces of difference sequence is defined by a modulus function, and several theorems on this subject are given.

Definition 1. *Let* f *be a modulus function. Then we define the following sets of sequences*

$$C_{\lambda}^{\infty}(p,f,s,\pi)=\left\{x\in w:\ \sup_k k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k}<\infty,\ s\geq 0\right\},$$

$$C_{\lambda}^{0}(p,f,s,\pi)=\left\{x\in w:\ k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k}\to 0\ (k\to\infty),\ s\geq 0\right\}$$

and

$$C_{\lambda}^{c}(p,f,s,\pi)=\left\{x\in w:\ k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i-L\right|\right)\right]^{p_k}\to 0\ (k\to\infty),\ \ s\geq 0,\ for\,some\,L>0\right\},$$

where $\Delta\left(\frac{x}{\pi}\right)_k=\frac{x_k}{\pi_k}-\frac{x_{k+1}}{\pi_{k+1}}$.

Theorem 1.

(i) *For any modulus function* f, $C_{\lambda}^{0}(p,f,s,\pi)$, $C_{\lambda}^{c}(p,f,s,\pi)$ *and* $C_{\lambda}^{\infty}(p,f,s,\pi)$ *are linear spaces over the complex field* $\mathbb{C}$.

(ii) *Let* f *be any modulus. Then* $C_{\lambda}^{0}(p,f,s,\pi)\subset C_{\lambda}^{c}(p,f,s,\pi)\subset C_{\lambda}^{\infty}(p,f,s,\pi)$.

Proof. (i) Let $x,y\in C_{\lambda}^{0}(p,f,s,\pi)$. For any $\alpha,\ \beta\in\mathbb{C}$, there exist integers M and N such that $|\alpha|\leq M_{\alpha}$ and $|\beta|\leq N_{\beta}$. By definition of modulus function and inequality (1) we have

$$k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{\alpha x+\beta y}{\pi}\right)_i\right|\right)\right]^{p_k}$$

$$\leq k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{\alpha\, x}{\pi}\right)_i\right|\right)+\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{\beta\, y}{\pi}\right)_i\right|\right)\right]^{p_k}$$

$$\leq D\,M_{\alpha}^{G}\ k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k}+D\,N_{\beta}^{G}\ k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{y}{\pi}\right)_i\right|\right)\right]^{p_k}.$$

This implies that $\alpha x+\beta y\in C_{\lambda}^{0}(p,f,s,\pi)$, and completes the proof. The others cases are routine works in view of the above theorem.

(ii) The proof of the inclusion $C_{\lambda}^{0}(p,f,s,\pi)\subset C_{\lambda}^{c}(p,f,s,\pi)$ is routine verification. So, we leave it to the reader. Let $x\in C_{\lambda}^{c}(p,f,s,\pi)$. Then there is some L such that

$$k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i-L\right|\right)\right]^{p_k}\to 0\ (k\to\infty),\ s\geq 0.$$

From inequality (1), we get

$$k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k}=k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i-L+L\right|\right)\right]^{p_k}$$

$$\leq D\,k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i-L\right|\right)\right]^{p_k}+D\,k^{-s}\left[f(|L|)\right]^{p_k}.$$

There exissts an integer M such that $|L|\leq M$. Therefore we have

$$k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k}$$

$$\leq D\,k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\right]^{p_k} + D\,k^{-s}\left[Mf(|1|)\right]^{G}.$$

This shows that $x \in C_\lambda^\infty(p, f, s, \pi)$, and completes the proof.

Theorem 2. $C_\lambda^0(p, f, s, \pi)$, $C_\lambda^c(p, f, s, \pi)$ *and* $C_\lambda^\infty(p, f, s, \pi)$ are *linear topological spaces paranormed by*

$$g(x) = \sup_k k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k/M},$$

where $M = \max(1,\ G = \sup_k p_k)$.

The proof follows by using standart techniques and the fact that every paranormed space is a topological linear space $[18,\ p.37]$. So we omit the details.

Theorem 3. $C_\lambda^0(p, f, s, \pi)$, $C_\lambda^c(p, f, s, \pi)$ *and* $C_\lambda^\infty(p, f, s, \pi)$ *are complete in their paranorm topologies.*

Proof. Let $\left(x^{(j)}\right)$ be a Cauchy sequence in $C_\lambda^0(p, f, s, \pi)$, where
$\left(x^{(j)}\right) = \left(x_1^j, x_2^j, \ldots\right)$, $\forall j \in \mathbb{N}$.

Hence for a given $\varepsilon > 0$ there exists N that $g\left(x^{(j)} - x^{(t)}\right) < \varepsilon$ for all $j, t > N$, that is

$$\sup_k k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x^{(j)}}{\pi}\right)_i - \Delta\left(\frac{x^{(t)}}{\pi}\right)_i\right|\right)\right]^{p_k/M} < \varepsilon, \text{ for all } j, t > N. \qquad (3)$$

This implies that for each fixed i, $\left|\Delta\left(\frac{x^{(j)}}{\pi}\right)_i - \Delta\left(\frac{x^{(t)}}{\pi}\right)_i\right| < \varepsilon$. So $\left(\Delta\left(\frac{x^{(j)}}{\pi}\right)_i\right)$ is a Cauchy sequence in $\mathbb{C}$, but $\mathbb{C}$ is complete so for $\forall i \in \mathbb{N}$ we have $\Delta\left(\frac{x^{(j)}}{\pi}\right)_i \to \Delta\left(\frac{x}{\pi}\right)_i$ $(j \to \infty)$. From (3) we get

$$\sup_k k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x^{(j)}}{\pi}\right)_i - \Delta\left(\frac{x^{(t)}}{\pi}\right)_i\right|\right)\right]^{p_k} < \varepsilon^M, \text{ for all } j, t > N.$$

So for each fixed k

$$k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x^{(j)}}{\pi}\right)_i - \Delta\left(\frac{x^{(t)}}{\pi}\right)_i\right|\right)\right]^{p_k} < \varepsilon^M, \text{ for all } j, t > N.$$

By taking $t \to \infty$ in the above inequality we obtain

$$k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x^{(j)}}{\pi}\right)_i - \Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k} < \varepsilon^M, \text{ for all } j > N.$$

Since k is arbitrary we have

$$\sup_k k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x^{(j)}}{\pi}\right)_i - \Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k} < \varepsilon^M, \text{ for all } j > N,$$

this implies $g\left(x^{(j)} - x\right) < \varepsilon$, for $j > N$.

Now we have to show that $x \in C_\lambda^0(p, f, s, \pi)$. Since $p_k / M \leq 1$ and $M \geq 1$, using Minkowski's inequality and definition of modulus function, we can get

$$k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k/M}$$
$$= k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i + \Delta\left(\frac{x^{(j)}}{\pi}\right)_i - \Delta\left(\frac{x^{(j)}}{\pi}\right)_i\right|\right)\right]^{p_k/M}$$
$$\leq k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - \Delta\left(\frac{x^{(j)}}{\pi}\right)_i\right|\right) + \frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x^{(j)}}{\pi}\right)_i\right|\right)\right]^{p_k/M}$$
$$\leq k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - \Delta\left(\frac{x^{(j)}}{\pi}\right)_i\right|\right)\right]^{p_k/M}$$
$$+k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x^{(j)}}{\pi}\right)_i\right|\right)\right]^{p_k/M}.$$

Taking supremum of such k' s we obtain

$$\sup_k k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k/M}$$
$$\leq \sup_k k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - \Delta\left(\frac{x^{(j)}}{\pi}\right)_i\right|\right)\right]^{p_k/M}$$
$$+\sup_k k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x^{(j)}}{\pi}\right)_i\right|\right)\right]^{p_k/M}$$
$$\leq \varepsilon + g\left(x^{(j)}\right).$$

Hence $x \in C_\lambda^0(p, f, s, \pi)$, and the proof is completed. Similarly it can be shown that the spaces $C_\lambda^c(p, f, s, \pi)$ and $C_\lambda^\infty(p, f, s, \pi)$ are also complete.

Theorem 4. *Let* $\inf p_k = r > 0$. *Then*

(i) *If* $x \to L\left[C_\lambda^c(\pi)\right]$ *then* $x \to L\left[C_\lambda^c(p, f, s, \pi)\right]$,

(ii) *If* $x \to L\left[C_\lambda^c(p, s, \pi))\right]$ *then* $x \to L\left[C_\lambda^c(p, f, s, \pi)\right]$,

(iii) *If* $\gamma = \lim_{t\to\infty} \frac{f(t)}{t} > 0$ *then* $C_\lambda^c(p, s, \pi) = C_\lambda^c(p, f, s, \pi)$.

Proof. (i) Let $x \to L\left[C_\lambda^c(\pi)\right](k \to \infty)$. Since f modulus function, we have

$$\lim_{k\to\infty}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\right]$$
$$= \left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\lim_{k\to\infty}\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\right]$$
$$= 0.$$

Since $\inf p_k = r > 0$ then

$$\lim_{k\to\infty}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\right]^r = 0,$$

so, for $0 < \varepsilon < 1, \exists k_0 \ni$ for all $k > k_0$,

$$\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\right]^r < \varepsilon < 1$$

and since $p_k \geq r$ for all k,

$$\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\frac{x_i}{\pi_i} - L\right|\right)\right]^{p_k}$$

$$\leq \left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\right]^r < \varepsilon$$

then we obtain

$$\lim_{k\to\infty}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\right]^{p_k} = 0.$$

Since (k^{-s}) is bounded, we can get

$$\lim_{k\to\infty} k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\right]^{p_k} = 0.$$

Hence $x \in C_\lambda^c(p, f, s, \pi)$.

(ii) Suppose that $x \in C_\lambda^c(p, s, \pi)$, so that

$$S_k = k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)}\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\right]^{p_k} \to 0\ (k\to\infty).$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$. Now we can write

$$R_1 = \left\{k \in \mathrm{N} : \left|\Delta\left(\frac{x}{\pi}\right)_i - L\right| > \delta\right\}.$$

$$R_2 = \left\{k \in \mathrm{N} : \left|\Delta\left(\frac{x}{\pi}\right)_i - L\right| \leq \delta\right\},$$

For $\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right| > \delta$,

$$\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right| < \left|\Delta\left(\frac{x}{\pi}\right)_i - L\right| \delta^{-1} < 1 + \left[\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right| \delta^{-1}\right],$$

where $k \in R_1$ and $[t]$ denotes the integer part of t. By definition of modulus function we can get for $\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right| > \delta$,

$$\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)$$

$$\leq \left(\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\delta^{-1}\right) f(1)$$

$$\leq 2\, f(1)\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\delta^{-1}.$$

For $\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right| \leq \delta$,

$$\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right) < \varepsilon,$$

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where $k \in R_2$. Therefore,

$$k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\right]^{p_k}$$

$$= k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\right]^{p_k} + k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\right]^{p_k}$$

where the first term over $k \in R_2$ and the second over $k \in R_1$. Then we obtain

$$k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\right]^{p_k}$$

$$\leq k^{-s}\varepsilon^{G} + \left[2 f(1)\delta^{-1}\right]^{G} S_k \to 0\ (k \to \infty).$$

This implies that $x \in C_\lambda^c(p, f, s, \pi)$.

(iii) For proof, we have to show that $C_\lambda^c(p, f, s, \pi) \subset C_\lambda^c(p, s, \pi)$. Let $x \in C_\lambda^c(p, f, s, \pi)$. For any modulus function we have $\gamma = \lim_{t\to\infty}\frac{f(t)}{t} = \inf\left\{\frac{f(t)}{t} : t > 0\right\}$ in [4]. Let $\gamma > 0$. By definition of γ we have $\gamma t \leq f(t)$ for all $t > 0$. Hence we can get

$$k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)}\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\right]^{p_k}$$

$$\leq k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)}\gamma^{-1} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\right]^{p_k}$$

$$\leq \gamma^{-G} k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i - L\right|\right)\right]^{p_k}$$

so $x \in C_\lambda^c(p, s, \pi)$.

Theorem 5. *Let* f_1 *and* f_2 *be two modulus and* $s, s_1, s_2 \geq 0$. *Then,*

(i) $C_\lambda^0(p, f_1, s, \pi) \cap C_\lambda^0(p, f_2, s, \pi) \subset C_\lambda^0(p, f_1 + f_2, s, \pi)$,

(ii) $s_1 \leq s_2$ *implies* $C_\lambda^0(p, f, s_1, \pi) \subset C_\lambda^0(p, f, s_2, \pi)$.

Proof. (i) Let $x \in C_\lambda^0(p, f_1, s, \pi) \cap C_\lambda^0(p, f_2, s, \pi)$. From inequality (1), we have

$$\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)}(f_1 + f_2)\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k}$$

$$= \left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f_1\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right) + \frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f_2\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k}$$

$$\leq D\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f_1\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k} + D\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f_2\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k}.$$

Since (k^{-s}) is bounded, we can get

$$k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)}(f_1 + f_2)\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k}$$

$$\leq D k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f_1\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k} + D k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f_2\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k}.$$

So this completes the proof (i).

(ii) Let $s_1 \le s_2$. Then $k^{-s_2} \le k^{-s_1}$ for all $k \in \mathbb{N}$. Hence we have

$$k^{-s_2}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k} \le k^{-s_1}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k},$$

this inequality implies that $C_\lambda^0(p,f,s_1,\pi) \subset C_\lambda^0(p,f,s_2,\pi)$.

The proof of following results is routine work in view of Theorem 5.

Corollary 1. *Let* f_1 *and* f_2 *be two modulus and* $s,s_1,s_2 \ge 0$. *Then*

(i) $C_\lambda^c(p,f_1,s,\pi) \cap C_\lambda^c(p,f_2,s,\pi) \subset C_\lambda^c(p,f_1+f_2,s,\pi)$,

(ii) $C_\lambda^\infty(p,f_1,s,\pi) \cap C_\lambda^\infty(p,f_2,s,\pi) \subset C_\lambda^\infty(p,f_1+f_2,s,\pi)$,

(iii) $s_1 \le s_2$ *implies* $C_\lambda^c(p,s_1,\pi) \subset C_\lambda^c(p,s_2,\pi)$,

(iv) $s_1 \le s_2$ *implies* $C_\lambda^\infty(p,f,s_1,\pi) \subset C_\lambda^\infty(p,f,s_2,\pi)$.

Theorem 6. *Let* f *be a modulus function, then*

(i) $(\ell_\infty)_\pi = \{x \in w : \left(\frac{x_k}{\pi_k}\right) \in \ell_\infty\} \subset C_\lambda^\infty(p,f,s,\pi)$,

(ii) *If* f *is bounded then* $C_\lambda^\infty(p,f,s,\pi) = w$.

Proof. (i) Let $x \in (\ell_\infty)_\pi$. Since $\left(\frac{x_k}{\pi_k}\right)$ is bounded $\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)$ is also bounded, hence we can get

$$k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k} \le k^{-s}[H\ f(1)]^{p_k}$$
$$\le k^{-s}[H\ f(1)]^{G} < \infty.$$

So $x \in C_\lambda^\infty(p,f,s,\pi)$.

(ii) Let f is bounded. Then for any $x \in w$

$$k^{-s}\left[\frac{1}{\lambda(k)}\sum_{i=1}^{\lambda(k)} f\left(\left|\Delta\left(\frac{x}{\pi}\right)_i\right|\right)\right]^{p_k} \le k^{-s}M^{p_k}$$
$$\le k^{-s}M^{G} < \infty,$$

therefore we obtain $C_\lambda^\infty(p,f,s,\pi) = w$ and this completes the proof.

REFERENCES

Armitage DH and Maddox IJ (1989). A new type of Cesáro mean. *Analysis* **9**(1-2) 195-204.

Başarır M (1990). Some New Sequence Spaces and Related Matrix Transformations, *100. Year University Press* **6** 32-38.

Johann Boos (2006). *Classical and Modern Methods in Summability* (Oxford University Press,Oxford).

Jürimäe E (1994). Matrix mappings between rate-spaces and spaces with speed. *Acta et Commentationes Universitatis Tartuensis* **970** 29-52.

Jürimäe E (1994). Properties of domains of mappings between rate-spaces and spaces with speed. *Acta et Commentationes Universitatis Tartuensis* **970** 53-64.

Kizmaz H (1981). On certain sequence spaces. *CMB: Canadian Mathematical Bulletin* **24**(2) 169-176.

Maddox IJ (1968). Paranormed Sequence Spaces Generated by Infinite Matrices. *Mathematical Proceedings of the Cambridge Philosophical Society* **64** 335-340.

Maddox IJ (1969). Some Properties of Paranormed Sequence Spaces. *Journal of the London Mathematical Society* **1**(2) 316-322..

Maddox IJ (1970). *Elements of Functional Analysis* first edition (Chambridge Univ. Press).

Maddox IJ (1986). Sequence spaces defined by a modulus. *Mathematical Proceedings of the Cambridge Philosophical Society* **100** 161-166.

Research Article

Maddox IJ (1987). Inclusion between FK spaces and Kuttner's theorem. *Mathematical Proceedings of the Cambridge Philosophical Society* **101** 523-527.
Nakano H (1953). Concave modulars. *Journal of the Mathematical Society of Japan* **5** 29-49.
Osikiewicz JA (2000). Equivalance Results for Cesáro Submethods. *Analysis* **20** 35-43.
Ruckle WH (1973). FK spaces in which the sequence of coordinate vectors is bounded. *CJM: Canadian Journal of Mathematics* **25** 973-978.
Tripathy BC and Chandra P (2011). On Some Generalized Difference Paranormed Sequence Spaces Associated with Multiplier Sequences Defined by Modulus Function. *Analysis in Theory and Applications* **27**(1) 21-27
Wilansky A (1964). *Functional Analysis* (Blaisdell Press).
Wilansky A (1978). *Modern Methods in Topolgical Vector Spaces* (McGraw-Hill).
Wilansky A (1984). *Summability Through Functional Analysis* (North Holland).
Zeller K (1951). *Allgemeine Eigenschaften von Limitierungsverfahren. Mathematische Zeitschrift* 463-487.