International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jpms.htm
2014 Vol. 4 (2) April-June, pp. 19-28/Prabakaran and Vinoba
Research Article

EXISTENCE OF SOLUTIONS FOR QUASILINEAR IMPULSIVE NEUTRAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS IN BANACH SPACES

***R.Prabakaran[1] and V.Vinoba[2]**
[1]*Department of Science and Humanities, Sri Krishna College of Technology, Coimbatore-42, India*
[2]*Department of Mathematics, K.N. Government Arts College for Women, Thanjavur, India*
**Author for Correspondence*

ABSTRACT
In this article, we prove the existence of mild solutions for quasilinear impulsive integro-differential equations with nonlocal initial conditions. The results are obtained by using semigroup theory and the Banach fixed point theorem. An example is provided to illustrate the theory.

Keywords: *Existence, Neutral Differential Equation, Impulsive Differential Equation, Fixed Point Theorem*
2000 AMS Subject Classification: *34A37, 47D06, 47H10, 34K40*

INTRODUCTION
An Abstract differential equations widely used in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and their rates of change in space and/or time are known or postulated. This type of equations arise in various applications such as in the flow of fluid through fissured rocks, thermodynamics, and shear in second-order fluids. For more details, we refer the reader to (Brill, 1977; Lagnese, 1972; Lightboure and Rankin, 1983). This is illustrated in classical mechanics where the motion of a body is described by its position and velocity as the time varies. It is well known that the systems described by partial differential equations can be expressed as abstract differential equations (Pazy, 1983). These equations occur in various fields of study and each system can be represented by different forms of differential or integro-differential equations in Banach spaces. Using the method of semigroups, solutions of nonlinear and semilinear evolution equations have been discussed by Pazy (1983). The study of abstract quasilinear initial value problems with nonlocal conditions was initiated by Bahuguna (1995) (Dong et al., 2008; Kanagaraj and Balachandran, 2002). Such problems with nonlocal conditions have been extensively studied in the literature (Balachandran and Chandrasekaran, 1996, 1997; Balachandran *et al.,* 1997; Balachandran *et al.*, 2002; Balachandran *et al.*, 1999; Balachandran and Uchiyama, 2000; Xue, 2009). Park *et al.*, (2011) established the existence of solutions of quasilinear evolution equations in Banach spaces.
A neutral functional differential equation is one in which the derivatives of the past history or derivatives of functionals of the past history are involved as well as the present state of the system. Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention during the last few decades. There are also a number of applications in which the delayed argument occurs in the derivative of the state variable as well as in the independent variable, as in the so called neutral differential difference equations. The literature for neutral functional differential equations is the book by Hale and Lunel (1993) and the references therein. Hernandez (2001) established the existence results for partial neutral functional differential equations with nonlocal conditions. He made use of fixed point theorems and the results mentioned in Pazy (1983). The theory of impulsive differential equation (Lakshmikantham *et al.*, 1989; Samoilenko and Perestyuk, 1995) is much richer than the corresponding theory of differential equations without impulsive effects. The impulsive condition

$$\Delta u(t_i) = u(t_i^+) - u(t_i^-) = I_i(u(t_i^-)), \quad i = 1, 2, \ldots, m,$$

is a combination of traditional initial value problems and short-term perturbations whose duration is negligible in comparison with the duration of the process. Lin and Liu (2003) discussed the iterative methods for the solution of impulsive functional differential systems.

Research Article

Motivated by the above approach, the goal of this paper is to use the fixed point theorem to obtain the mild solution of the quasilinear impulsive neutral integro-differential equation with nonlocal conditions in Banach space.

Preliminaries

In this section, consider the quasilinear impulsive neutral integrodifferential equation with nonlocal conditions of the form

$$\frac{d}{dt}[u(t)+g(t,u(t))] = A(t,u(t))u(t) + f(t,u(t)) + \int_0^t h(t,s,u(s))ds, \quad t\in I, t\neq t_k, \tag{1}$$

$$u(0)+\sum_{i=1}^{n} c_i u(t_i) = u_0 \tag{2}$$

$$\Delta u(t_k) = I_k(u_{t_k}), \quad k=1,2,\ldots,m, \tag{3}$$

where $0 \le t_1 < t_2 < \ldots < t_p \le b$, $A(t,u)$ is the infinitesimal generator of a C_0-semigroup in the Banach space X. And $u_0 \in X$, $f: I\times X \to X$, $h: I^2\times X\to X$, $g: I\times X\to X$, $q: X\to X$ and $I_k : X\to X$ are appropriate functions and the symbol $\Delta u(t_k)$ represent the jump of function u at t, which is defined by $\Delta u(t_k) = u(t^+) - u(t^-)$. Here $I=[0,b]$. In this paper, we establish the existence of a quasilinear impulsive neutral integro-differential equation with nonlocal conditions using Banach fixed point theorem.

Let X be a Banach space with norm $\|\cdot\|$. Let $\mathrm{PC}([0,b],X)$ consist of functions u from $[0,b]$ into X, such that $x(t)$ is continuous at $t\neq t_i$ and left continuous at $t=t_i$ and the right limit $x(t_i^+)$ exists, for $i=1,2,3,\ldots,n$. Evidently $\mathrm{PC}([0,b],X)$ is a Banach space with the norm

$$\|x\|_{\mathrm{PC}} = \sup_{t\in[0,b]} \|x(t)\|.$$

Definition 2.1 *If the evolution family* $\{U_u(t,s)\}_{0\le s\le t\le b}$ *is equicontinuous then the following properties hold:*

(i) $U_u(t,s)U_u(s,r) = U_u(t,r),\ (t,s,r)\in[0,b]\times[0,b]\times[0,b]$ *and all* $x\in X$;

(ii) For each $x\in X$, the functions for $(t,s)\to U_u(t,s)x$ is continuous and $U_u(t,s)\in \mathrm{L}(X)$ for every $t\ge s$ and

(iii) The function $t\to U_u(t,s),$ for $(s,t]\in \mathrm{L}(X),$ is differentiable with $\dfrac{\partial U_u(t,s)}{\partial t} = A_u(t,u)U_u(t,s),$ *for almost all* $t,s\in[0,b].$

In this article, we assume that there exists an operator E on $\mathrm{D}(E) = X$ given by the formula

$$E = [I + \sum_{i=1}^{n} c_i U(t_i,0)]^{-1}$$

with The existence of E can be observed from the following fact (Byszewski, 1991). Suppose that $\|U(t_i,0)\| \le \mathrm{C}e^{-\delta t_i} (i=1,2,\ldots,n)$ where δ is a positive constant and $\mathrm{C}\le 1$. If $\sum_{i=1}^{p} |c_i| e^{-\delta t_i} < 1/\mathrm{C}$ then $\left\|\sum_{i=1}^{p} c_i U(t_i,0)\right\| < 1$. So such an operator E exists on X.

First we study the following system

$$(u(t))' = A(t,u(t))u(t) + f(t,u(t)) \qquad t\in(0,b], \tag{4}$$

$$u(0)+\sum_{i=1}^{n} c_i u(t_i)=u_0 \tag{5}$$

Definition 2.2 A continuous solution u of the integral equation

$$\begin{aligned} u(t) \quad &= U(t,0)Eu_0-\sum_{i=1}^{n} c_i U(t,0)E\{\int_0^{t_i} U(t_i,s)\,f(s,u(s))ds\} \\ &+\int_0^t U(t,s)\,f(s,u(s))ds \end{aligned} \tag{6}$$

is said to be a mild solution of problem (4)-(5) on I .

Remark: 2.1 A mild solution of the quasilinear integro-differential (4)-(5) satisfies the condition (5), for (6)

$$u(0)=Eu_0-\sum_{i=1}^{n} c_i E\int_0^{t_i} U(t_i,s)\,f(s,u(s))ds$$

and

$$u(t_j)=U(t_j,0)Eu_0-\sum_{i=1}^{n} c_i U(t_j,0)E\{\int_0^{t_i} U(t_i,s)\,f(s,u(s))ds\}+\int_0^{t_j} U(t_j,s)\,f(s,u(s))ds$$

Therefore,

$$\begin{aligned} u(0)+\sum_{j=1}^{n} c_j u(t_j) &= [I+\sum_{j=1}^{n} c_j U(t_j,0)]Eu_0 \\ &-[I+\sum_{j=1}^{n} c_j U(t_j,0)]\sum_{i=1}^{n} c_i E\{\int_0^{t_i} U(t_i,s)\,f(s,u(s))ds\} \\ &+\sum_{j=1}^{n} c_j \int_0^{t_j} U(t_j,s)\,f(s,u(s))ds \\ &= u_0 \end{aligned}$$

RESULTS

To prove the existence result, we use the following hypotheses:

(M1) $A(t,u(t))$ is the infinitesimal generator of C_0-semigroup $t\in \mathsf{R}$ on Banach space X. There exist constants $M_1 \geq 1$ and $M_2 \geq 1$ such that $\|U(t,s)\| \leq M_1$ and $\|A(t,u)U(t,s)\| \leq M_2$ for every $t\in[0,b]$. Furthermore let $c=\sum_{i=1}^{p} |c_i|$.

(M2) The function $f: I\times X \to X$ satisfies the following condition:

(i) For each $t\in I$, the function $f(t,\cdot): X\to X$ is continuous and for each $x\in X$, the function $f(t,\cdot): X\to X$ is strongly measurable.

(ii) There exists a constant $L_f > 0$ such that

$\|f(t,u)\| \leq L_f$, for $t\in I$ and $u\in X$.

and

$\|f(t,u_1)-f(t,u_2)\| \leq L_f \|u_1-u_2\|$, for $t\in I$ and $u_i \in X$, $i=1,2$.

Theorem: 3.1 *Assume that the conditions* (M1)-(M2) *are satisfied. Then for every* $u_0\in D(X)$ *the problem (4)-(5) has at least one mild solution on* I *provided that there exist a constant* $r>1$ *with* $M_1\|Eu_o\|+c\|E\|M_1^2 L_f b+M_1 b L_f \leq r.$

Proof: Let S be the subset of $C(I,X)$ defined by, $S=\{u:u(t)\in C(I,X),\|u(t)\|\leq r \text{ for } t\in I\}$. We define a mapping $P:S\to S$ by

$$(Pu)(t)=U(t,0)Eu_0-\sum_{i=1}^{n}c_iU(t,0)E\{\int_0^{t_i}U(t_i,s)f(s,u(s))ds\}+\int_0^t U(t,s)f(s,u(s))ds$$

First we show that the operator P maps S into itself. Now

$$\|(Pu)(t)\|\leq\|U(t,0)Eu_0\|+\left\|\sum_{i=1}^{n}c_iU(t,0)E\{\int_0^{t_i}U(t_i,s)f(s,u(s))ds\}\right\|$$
$$+\left\|\int_0^t U(t,s)f(s,u(s))ds\right\|$$
$$\leq M_1\|Eu_o\|+c\|E\|M_1^2L_fb+M_1bL_f$$
$$\leq r$$

Let $u_1,u_2\in S$ then

$$\|(Pu_1)(t)-(Pu_2)(t)\|\leq\left\|\sum_{i=1}^{n}c_iU(t,0)E\{\int_0^{t_i}U(t_i,s)[f(s,u_1(s))-f(s,u_2(s))]ds\}\right\|$$
$$+\left\|\int_0^t U(t,s)[f(s,u_1(s))-f(s,u_2(s))]ds\right\|$$
$$\leq bcM^2\|E\|L_f\|u_1-u_2\|+bML_f\|u_1-u_2\|$$
$$\leq\left[bcM^2\|E\|L_f+bML_f\right]\|u_1-u_2\|$$

Since the mapping P is contraction and hence by Banach fixed point theorem there exists a unique fixed point $u\in S$ such that $(Pu)(t)=u(t)$. This fixed point is then the solution of the problem (4)-(5).

Quasilinear Neutral Integro-Differential System

Now consider the following first order quasilinear neutral integro-differential system

$$\frac{d}{dt}[u(t)+g(t,u(t))]=A(t,u(t))u(t)+f(t,u(t))+\int_0^t h(t,s,u(s))ds \quad t\in(0,b], \tag{7}$$

$$u(0)+\sum_{i=1}^{n}c_iu(t_i)=u_0 \tag{8}$$

To prove the existence result of the system (7)-(8). For this we impose following conditions:

(M3) The function $g:I\times X\to X$ satisfies the following condition:

(i) For each $t\in I$, the function $g(t,\cdot):X\to X$ is continuous and for each $x\in X$, the function $g(t,\cdot):X\to X$ is strongly measurable.

(ii) There exist constants $L_g>0$ and $L_0>0$ such that

$$\|g(t,u(t))\|\leq L_g, \quad \text{for } t\in I \text{ and } u\in X$$

$$\|g(0,u(0))\|\leq \tilde{L}_0, \quad \text{for } t\in I \text{ and } u\in X$$

and

$\|g(t,u_1)-g(t,u_2)\|\leq L_g\|u_1-u_2\|$, for $t\in I$ and $u_i\in X$, $i=1,2$.

(M4) The function $h:I^2\times X\to X$ satisfies the following condition:

(i) For each $t,s\in I$, the function $h(t,s,\cdot):X\to X$ is continuous and for each $x\in X$, the function

$h(t,s,\cdot): X \to X$ is strongly measurable.

(ii) There exists a constant $L_h > 0$ such that

$\|h(t,s,u)\| \le L_h$, for $t \in I$ and $u \in X$.

and

$\|h(t,s,u_1) - h(t,s,u_2)\| \le L_h\|u_1 - u_2\|$, for $t \in I$ and $u_i \in X$, $i = 1,2$.

Definition 4.1 A continuous solution u of the integral equation

$$\begin{aligned} u(t) &= U(t,0)Eu_0 + U(t,0)g(0,u(0)) - \int_0^t A(t,u(t))U(t,s)g(s,u(s))ds \\ &\quad - \sum_{i=1}^n c_i U(t,0)E\Big\{U(t_i,0)g(0,u(0)) - \int_0^{t_i} A(t_i,u(t_i))U(t_i,s)g(s,u(s))ds \\ &\quad + \int_0^{t_i} U(t_i,s)[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau]ds\Big\} \\ &\quad + \int_0^t U(t,s)[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau]ds \end{aligned} \tag{9}$$

is said to be a mild solution of problem (7)-(8) on I .

Remark: 4.1 A mild solution of the neutral integro-differential (7)-(8) satisfies the condition (8), for (9)

$$u(0) = Eu_0 + g(0,u(0)) - \sum_{i=1}^n c_i E\{U(t_i,0)g(0,u(0)) - \int_0^{t_i} A(t_i,u(t_i))U(t_i,s)g(s,u(s))ds$$
$$+ \int_0^{t_i} U(t_i,s)[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau]ds\}$$

and

$$u(t_j) = U(t_j,0)Eu_0 + U(t_j,0)g(0,u(0)) - \int_0^t A(t_j,u(t_j))U(t_j,s)g(s,u(s))ds$$
$$- \sum_{i=1}^n c_i U(t_j,0)E\{U(t_i,0)g(0,u(0)) - \int_0^{t_i} A(t_i,u(t_i))U(t_i,s)g(s,u(s))ds$$
$$+ \int_0^{t_i} U(t_i,s)[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau]ds\}$$
$$+ \int_0^{t_j} U(t_j,s)[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau]ds$$

Therefore,

$$u(0) + \sum_{j=1}^n c_j u(t_j) = u_0$$

Theorem: 4.2 *If assumptions* (M1)-(M4) *hold, then for every* $u_0 \in D(X)$ *the problem (7)-(8) has at least one mild solution on* I *provided that there exist a constant* $r_1 > 1$ *with*

$$M_1\|Eu_o\| + M_1 \tilde{L}_0 + bM_2L_g$$
$$+ c\|E\|M_1[bM_2L_g + M_1 \tilde{L}_0 + M_1 b[L_f + bL_h] + M_1] + bM_1[L_f + bL_h] \le r_1.$$

Proof: Let S be the subset of $\mathrm{C}(I,X)$ defined by, $S = \{u : u(t) \in \mathrm{C}(I,X), \|u(t)\| \le r_1$ for $t \in I\}$. We define a mapping $P:S\to S$ by

$$(Pu)(t)=U(t,0)Eu_0+U(t,0)g(0,u(0))-\int_0^t A(t,u(t))U(t,s)g(s,u(s))ds$$
$$-\sum_{i=1}^{n}c_iU(t,0)E\{U(t_i,0)g(0,u(0))-\int_0^{t_i}A(t_i,u(t_i))U(t_i,s)g(s,u(s))ds$$
$$+\int_0^{t_i}U(t_i,s)[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau]ds\}$$
$$+\int_0^t U(t,s)[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau]ds$$

Now we show that the operator P maps S into itself.

$$\|(Pu)(t)\|\le\|U(t,0)Eu_0\|+\|U(t,0)g(0,u(0))\|+\left\|\int_0^t A(t,u(t))U(t,s)g(s,u(s))ds\right\|$$
$$+\Big\|\sum_{i=1}^{n}c_iU(t,0)E\{U(t_i,0)g(0,u(0))-\int_0^{t_i}A(t_i,u(t_i))U(t_i,s)g(s,u(s))ds$$
$$+\int_0^{t_i}U(t_i,s)[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau]ds\}\Big\|$$
$$+\left\|\int_0^t U(t,s)[f(s,u(s))+\int_0^s h(s,\tau,u(\tau))d\tau]ds\right\|$$

$$\le M_1\|Eu_o\|+M_1\tilde{L}_0+bM_2L_g$$
$$+c\|E\|M_1[bM_2L_g+M_1\tilde{L}_0+M_1b[L_f+bL_h]+M_1]+bM_1[L_f+bL_h]$$
$$\le r_1.$$

Let $u_1,u_2\in S$ then

$$\|(Pu_1)(t)-(Pu_2)(t)\|\le\|U(t,0)[g(0,u_1(0))-g(0,u_2(0))]\|$$
$$+\left\|\int_0^t A(t,u(t))U(t,s)[g(s,u_1(s))-g(s,u_2(s))]ds\right\|$$
$$+\Big\|\sum_{i=1}^{n}c_iU(t,0)E\{U(t_i,0)[g(0,u_1(0))-g(0,u_2(0))]$$
$$-\int_0^{t_i}A(t_i,u(t_i))U(t_i,s)[g(s,u_1(s))-g(s,u_2(s))]ds$$
$$+\int_0^{t_i}U(t_i,s)[(f(s,u_1(s))-f(s,u_2(s)))+\int_0^s(h(s,\tau,u_1(\tau))-h(s,\tau,u_2(\tau)))d\tau]ds\}\Big\|$$
$$+\left\|\int_0^t U(t,s)[(f(s,u_1(s))-f(s,u_2(s)))+\int_0^s(h(s,\tau,u_1(\tau))-h(s,\tau,u_2(\tau)))d\tau]ds\right\|$$

$$\le[M_1\tilde{L}_g+bM_2L_g+bM_1\{L_f+bL_h\}$$
$$+c\|E\|M_1[M_1\tilde{L}_g+bM_2L_g+bM_1\{L_f+bL_h\}]]\|u_1-u_2\|$$

Since the mapping P is contraction and hence by Banach fixed point theorem there exists a unique fixed point $u \in S$ such that $(Pu)(t) = u(t)$. This fixed point is then the solution of the problem (7)-(8).

Quasilinear Neutral Impulsive Integrodifferential System

Now we prove the existence result of the system (1)-(3). For this we impose following conditions:

(M5) $I_k : X \to X, k = 1,2,...,m,$ are continuous and there exist constants d_i such that

$$\|I_k\| = d_i, \qquad k = 1,2,...,m$$

and

$$\|I_k(u_1) - I_k(u_2)\| \le d_i \|u_1 - u_2\|$$

for all $u_1, u_2 \in X$ and $k = 1,2,...,m$

Definition 5.1 A continuous solution u of the integral equation

$$\begin{aligned} u(t) &= U(t,0)Eu_0 + U(t,0)g(0,u(0)) - \int_0^t A(t,u(t))U(t,s)g(s,u(s))ds \\ &\quad - \sum_{i=1}^n c_i U(t,0)E\{U(t_i,0)g(0,u(0)) - \int_0^{t_i} A(t_i,u(t_i))U(t_i,s)g(s,u(s))ds \\ &\quad + \int_0^{t_i} U(t_i,s)[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau]ds + \sum_{0<t_k<t_i} U(t_i,t_k)I_k u(t_k)\} \\ &\quad + \int_0^t U(t,s)[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau]ds + \sum_{0<t_k<t} U(t,t_k)I_k u(t_k) \end{aligned} \tag{10}$$

is said to be a mild solution of problem (1)-(3) on I .

Remark: 5.1 A mild solution of the neutral integro-differential (1)-(3) satisfies the condition (2), for (10)

$$u(0) = Eu_0 + g(0,u(0)) - \sum_{i=1}^n c_i E\{U(t_i,0)g(0,u(0)) - \int_0^{t_i} A(t_i,u(t_i))U(t_i,s)g(s,u(s))ds$$
$$+ \int_0^{t_i} U(t_i,s)[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau]ds + \sum_{0<t_k<t_i} U(t_i,t_k)I_k u(t_k)\}$$

and

$$\begin{aligned} u(t_j) &= U(t_j,0)Eu_0 + U(t_j,0)g(0,u(0)) - \int_0^{t_j} A(t_j,u(t_j))U(t_j,s)g(s,u(s))ds \\ &\quad - \sum_{i=1}^n c_i U(t_j,0)E\{U(t_i,0)g(0,u(0)) - \int_0^{t_i} A(t_i,u(t_i))U(t_i,s)g(s,u(s))ds \\ &\quad + \int_0^{t_i} U(t_i,s)[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau]ds + \sum_{0<t_k<t_i} U(t_i,t_k)I_k u(t_k)\} \\ &\quad + \int_0^{t_j} U(t_j,s)[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau]ds + \sum_{0<t_k<t_j} U(t_j,t_k)I_k u(t_k) \end{aligned}$$

Therefore,

$$u(0) + \sum_{j=1}^n c_j u(t_j) = u_0$$

Theorem: 5.2 *Assume that the conditions* (M1)-(M5) *are satisfied. Then for every* $u_0 \in D(X)$ *the problem (1)-(2) has at least one mild solution on* I *provided that there exist a constant* $r_2 > 1$ *with*

$$\|Eu_o\| M_1 + M_1 \tilde{L}_0 + bM_2 L_g + bM_1[L_f + bL_h] + M_1 \sum_{k=1}^m d_i$$

$$+c\|E\|M_1[bM_2L_g + M_1\tilde{L}_0 + M_1b[L_f + bL_h] + M_1\sum_{k=1}^{m} d_i] \le r_2.$$

Proof: Let S be the subset of $C(I,X)$ defined by, $S=\{u : u(t)\in C(I,X), \|u(t)\|\le r_2 \text{ for } t\in I\}$.We define a mapping $P:S\to S$ by

$$(Pu)(t) = U(t,0)Eu_0 + U(t,0)g(0,u(0)) - \int_0^t A(t,u(t))U(t,s)g(s,u(s))ds$$
$$-\sum_{i=1}^{n} c_i U(t,0)E\{U(t_i,0)g(0,u(0)) - \int_0^{t_i} A(t_i,u(t_i))U(t_i,s)g(s,u(s))ds$$
$$+\int_0^{t_i} U(t_i,s)[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau]ds + \sum_{0<t_k<t_i} U(t_i,t_k)I_k u(t_k)\}$$
$$+\int_0^t U(t,s)[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau]ds + \sum_{0<t_k<t} U(t,t_k)I_k u(t_k)$$

Now we show that the operator P maps S into itself.

$$\|(Pu)(t)\| = \|U(t,0)Eu_0\| + \|U(t,0)g(0,u(0))\| + \left\|\int_0^t A(t,u(t))U(t,s)g(s,u(s))ds\right\|$$
$$+\left\|\sum_{i=1}^{n} c_i U(t,0)E\{U(t_i,0)g(0,u(0)) - \int_0^{t_i} A(t_i,u(t_i))U(t_i,s)g(s,u(s))ds\right.$$
$$\left.+\int_0^{t_i} U(t_i,s)[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau]ds + \sum_{0<t_k<t_i} U(t_i,t_k)I_k u(t_k)\}\right\|$$
$$+\left\|\int_0^t U(t,s)[f(s,u(s)) + \int_0^s h(s,\tau,u(\tau))d\tau]ds\right\| + \left\|\sum_{0<t_k<t} U(t,t_k)I_k u(t_k)\right\|$$
$$\le \|Eu_o\|M_1 + M_1\tilde{L}_0 + bM_2L_g + bM_1[L_f + bL_h] + M_1\sum_{k=1}^{m} d_i$$
$$+c\|E\|M_1[bM_2L_g + M_1\tilde{L}_0 + M_1b[L_f + bL_h] + M_1\sum_{k=1}^{m} d_i]$$
$$\le r_2.$$

Let $u_1,u_2\in S$ then

$$\|(Pu_1)(t)-(Pu_2)(t)\| \le \|U(t,0)[g(0,u_1(0)) - g(0,u_2(0))]\|$$
$$+\left\|\int_0^t A(t,u(t))U(t,s)[g(s,u_1(s)) - g(s,u_2(s))]ds\right\|$$
$$+\left\|\sum_{i=1}^{n} c_i U(t,0)E\{U(t_i,0)[g(0,u_1(0)) - g(0,u_2(0))]\right.$$
$$-\int_0^{t_i} A(t_i,u(t_i))U(t_i,s)[g(s,u_1(s)) - g(s,u_2(s))]ds$$
$$+\int_0^{t_i} U(t_i,s)[(f(s,u_1(s)) - f(s,u_2(s))) + \int_0^s (h(s,\tau,u_1(\tau)) - h(s,\tau,u_2(\tau)))d\tau]ds$$

$$+\sum_{0<t_k<t_i} U(t_i,t_k)[I_k u_1(t_k)-I_k u_2(t_k)]\}\Big\|$$

$$+\left\|\int_0^t U(t,s)[(f(s,u_1(s))-f(s,u_2(s)))+\int_0^s (h(s,\tau,u_1(\tau))-h(s,\tau,u_2(\tau)))d\tau]ds\right\|$$

$$+\left\|\sum_{0<t_k<t} U(t,t_k)[I_k u_1(t_k)-I_k u_2(t_k)]\right\|$$

$$\leq [M_1 \tilde{L}_g + bM_2L_g + bM_1\{L_f + bL_h\} + M_1\sum_{k=1}^{m} d_i$$

$$+c\|E\|M_1[M_1\tilde{L}_g + bM_2L_g + bM_1\{L_f + bL_h\} + M_1\sum_{k=1}^{m} d_i]]\|u_1-u_2\|$$

Since the mapping P is contraction and hence by Banach fixed point theorem there exists a unique fixed point $u \in S$ such that $(Pu)(t) = u(t)$. This fixed point is then the solution of the problem (1)-(3).

Example

Consider the partial integrodifferential equation of the form

$$\frac{\partial}{\partial t}[z_t(t,y) - \int_0^{\pi} b_1(s,y)z_t(s,y)ds] = a(t,y)z_{yy}(t,y)$$

$$+b_2(s,y)(t,\int_0^t sinz_s(s,y)e^{-z_s(sins,y)}ds)+\int_0^a l(t,\tau)z_\tau(t,y)d\tau,\ y\in[0,\pi], t\in \mathrm{I}, \quad (11)$$

$$z(t,0) = z(t,\pi) = 0, \quad t \in \mathrm{I}, \quad (12)$$

$$z(0,y) + \sum_{i=1}^{m} e_i \Phi_{t_i}(s,y) = z_0(y) \quad 0 < y < 1, t \in \mathrm{I}; \quad (13)$$

$$\Delta z|_{t=t_i} = I_i(z(y)) = (\gamma_i(z(y)) + t_i)^{-1}, z \in X, 1 \leq i \leq m, \quad (14)$$

where $a(t,y)$ is continuous on $0 \leq y \leq \pi, t \in \mathrm{I}$ and the constant e_i, γ_i are small.

Let us take $X = \mathrm{L}^2[0,\pi]$ to be endowed with the usual norm $\|\cdot\|_{\mathrm{L}_2}$. and let

$$g(t,u(t)) = \int_0^{\pi} b_1(s,y)z_t(s,y)ds$$

$$f(t,u(t)) = b_2(s,y)(t,\int_0^t sinz(s,y)e^{-z(sins,y)}ds)$$

$$\int_0^t h(t,s,u(s))ds = \int_0^a l(t,\tau)z(t,y)d\tau$$

$$\sum_{i=1}^{m} c_i u(t_i) = \sum_{i=1}^{m} e_i \Phi_{t_i}(s,y)$$

$$I_k(u(t_k)) = (\gamma_i(z(y)) + t_i)^{-1}.$$

Then the above problem can be formulated abstractly as

$$\frac{d}{dt}[u(t)+g(t,u(t))] = A(t,u(t))u(t) + f(t,u(t)) + \int_0^t h(t,s,u(s))ds, \quad t \in I, t \neq t_k,$$

$$u(0) + \sum_{i=1}^{n} c_i u(t_i) = u_0$$

$$\Delta u(t_k) = I_k(u_{t_k}), \quad k = 1,2,\ldots,m,$$

for each $t > 0$. So all the conditions of the Theorem 5.2 are satisfied. Hence the equation (11)-(14) has a mild solution.

REFERENCES

Balachandran K and Chandrasekaran M (1996). Existence of solutions of a delay differential equation with nonlocal condition. *Indian Journal of Pure and Applied Mathematics* **27** 443-449.

Balachandran K and Chandrasekaran M (1997). Existence of solutions of nonlinear integrodifferential equations with nonlocal condition. *Journal of Applied Mathematics and Stochastic Analysis* **10** 279-288.

Balachandran K and Ilamaran S (1997). Existence and uniqueness of mild and strong solutions of a Volterra integrodifferential equation with nonlocal conditions. *Tamkang Journal of Mathematics* **28** 93-100.

Balachandran K, Park JY and Chandrasekaran M (2002). Nonlocal Cauchy problem for delay integrodifferential equations of Sobolev type in Banach spaces. *Applied Mathematics Letters* **15** 845-854.

Balachandran K, Park DG and Kwun YC (1999). Nonlinear integrodifferential equations of Sobolev type with nonlocal conditions in Banach spaces. *Communications of the Korean Mathematical Society* **14** 223-231.

Balachandran K and Uchiyama K (2000). Existence of solutions of nonlinear integrodifferential equations of Sobolev type with nonlocal condition in Banach spaces. *Proceedings of the Indian Academy of Sciences (Mathematical Sciences)* **110** 225-232.

Bahuguna D (1995). Quasilinear integrodifferential equations in Banach spaces. *Nonlinear Analysis.* **24** (1995), 175-183.

Brill H (1977). A semilinear Sobolev evolution equation in Banach space. *Journal of Differential Equations* **24** 412-425.

Byszewski L (1991). Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. *Journal of Mathematical Analysis and Applications* **162** 494-505.

Dong Q, Li G and Zhang J (2008). Quasilinear nonlocal integrodifferential equations in Banach spaces. *Electronic Journal of Differential Equations* **19** 1-8.

Hale JK and Verduyn Lunel SM (1993). *Introduction to Functional-Differential Equations* (Springer-Verlag, New York).

Hernandez ME (2001). Existence results for partial neutral functional differential equations with nonlocal conditions. *Cadernos de Matemática* **2** 239-250.

Kanagaraj M and Balachandran K (2002). Existence of solutions of quasilinear integrodifferential equations in Banach spaces. *Far East Journal of Mathematical Sciences* **4** 337-350.

Lagnese JE (1972). Exponential stability of solutions of differential equation of Sobolev type. *SIAM Journal on Mathematical Analysis* **3** 625-636.

Lightboure JH III and Rankin SM III (1983). A partial functional differential equation of Sobolev type. *Journal of Mathematical Analysis and Applications* **93** 328-337.

Lakshmikantham V, Bainov DD and Simeonov PS (1989). *Theory of Impulsive Differential Equations*, (World Scientific, Singapore).

Lin Y and Liu JH (2003). *S*emilinear integrodifferential equations with nonlocal Cauchy problem. *Journal of Integral Equations and Applications* **15** 79-93.

Park DG, Balachandran K and Paul Samuel F (2011). Regularity solutions of abstract quasilinear delay differential equations. *Journal of the Korean Mathematical Society* **48** 585-597.

Pazy A (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations* (Springer-Verlag, New York).

Samoilenko AM and Perestyuk NA (1995). *Impulsive Differential Equations* (World Scientific, Singapore).

Xue X (2009). Nonlocal nonlinear differential equations with measure of noncompactness in Banach spaces. *Nonlinear Analysis: Theory, Methods & Applications* **70** 2593-2601.