International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jpms.htm
2014 Vol. 4 (2) April-June, pp. 1-7/Ara

Research Article

A PROPERTY DEVELOPED ON SECANT FROM A FIXED POINT ANYWHERE ON THE PRINCIPAL AXES OF AN ELLIPSE

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ABSTRACT

Ellipse is one of the conic sections. It is sort of elongated circle. It is the locus of a point that moves in such a way that the ratio of its distance from a fixed point to its distance from a fixed line equals to constant 'e'. An ellipse is defined by two points, each called a focus. In any ellipse, the sum of the focal distances is constant. The size of the ellipse is determined by the sum of these two distances. The sum of these distances is equal to the length of the major axis which is the longest diameter of the ellipse. A line that intersects an ellipse at two points is called secant of ellipse. The author has derived necessary equations with analytic geometry and developed a new property for secant generated at a fixed point anywhere on the principal axes of an ellipse and defined very clearly along with relevant drawings and proved with appropriate examples

Keywords: *Ellipse, Foci, Secant, Major Axis and Minor Axis*

INTRODUCTION

An *ellipse* (Eric, 2003) is the set of all points in a plane such that the sum of the distances from two fixed points called *foci* (Eric, 2003) is a given constant. Things that are in the shape of an ellipse are said to be elliptical. In the 17th century, a mathematician Mr. Johannes Kepler discovered that the orbits along which the planets travel around the Sun are ellipses with the Sun at one of the foci, in his First law of planetary motion. Later, Isaac Newton explained that this as a corollary of his law of universal gravitation. One of the physical properties of ellipse is that sound or light rays emanating from one focus will reflect back to the other focus. This property can be used, for instance, in medicine. A point inside the ellipse which is the midpoint of the line *segment* (Eric, 2003) linking the two foci is called centre. The longest and shortest diameters of an ellipse is called *Major axis* (Eric, 2003) and *Minor axis* (Eric, 2003) respectively. The two points that define the ellipse is called foci. The *eccentricity* (Eric, 2003) of an ellipse, usually denoted by ε or e, is the ratio of the distance between the two foci to the length of the major axis. A line segment linking any two points on an ellipse is called *chord* (Eric, 2003). A line passing an ellipse and touching it at just one point is called *tangent* (Eric, 2003). A line that intersects an ellipse at two points is called *secant* (Eric, 2003). The author has derived necessary equations with analytic geometry and developed a new property for secant generated at a fixed point either anywhere on the *principal axes* (Bali, 2005) (major axis or minor axis) of an ellipse.

Derivation of Equations and Proof for the New Property

The new property is for secant generated at a fixed point either anywhere in the line of axis of major axis or anywhere in the line of axis of minor axis of ellipse. This is an advancement of similar existing property for circle.

Referring figure.1, we know already that the *equation of an ellipse* (Eric, 2003) with respect to centre of the ellipse 'O' is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad ------------------------------[1]$$

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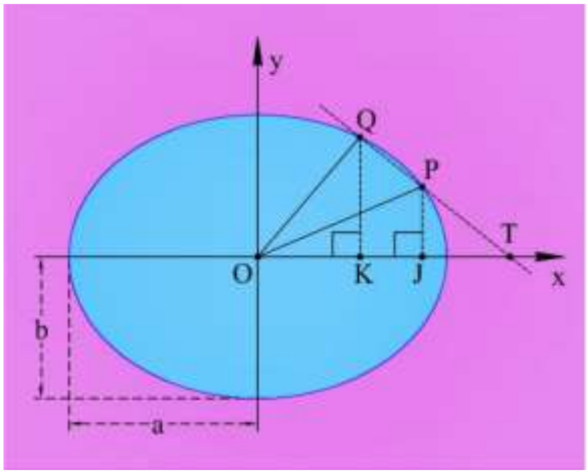

Figure 1: A secant passing through points 'P' and 'Q' of the ellipse from point 'T' on the line of x-axis

Let a secant passing through point 'T' which is anywhere in the line of major axis (x-axis) of the ellipse and intersecting the ellipse at 'P' and 'Q'. Let, co-ordinates of points 'P' & 'Q' are (x_1, y_1)&(x_2, y_2) respectively. The Points 'J' and 'K' are projections of point 'P' and 'Q' on x-axis respectively. Therefore, $PJ \perp OT$, $QK \perp OT$ and $OJ = x_1, JP = y_1, OK = x_2$ and $KQ = y_2$
In right-angled triangle OJP,

$$OP^2 = OJ^2 + JP^2$$

Substituting the values of co-ordinates of 'P', we get

$$OP^2 = {x_1}^2 + {y_1}^2 ---------------------------[2]$$

In right-angled triangle OKQ,

$$OQ^2 = OK^2 + KQ^2$$

Substituting the values of co-ordinates of 'Q', we get

$$OQ^2 = {x_2}^2 + {y_2}^2 ---------------------------[3]$$

We know that the *equation of line*(Eric, 2003) passing through two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) --------------------------[4]$$

Let, co ordinate of point 'T' $= (x_3, y_3)$. The point 'T' is on the line of x-axis. Therefore, $y_3 = 0$. Substituting in above equation,

we get, $$0 - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x_3 - x_1)$$

Therefore, $$x_3 = -y_1\left(\frac{x_2 - x_1}{y_2 - y_1}\right) + x_1$$

$$\therefore x_3 = \frac{-y_1(x_2 - x_1) + x_1(y_2 - y_1)}{y_2 - y_1}$$

$$\therefore x_3 = \frac{-x_2y_1 + x_1y_1 + x_1y_2 - x_1y_1}{y_2 - y_1}$$

Therefore, $x_3 = \dfrac{x_1y_2 - x_2y_1}{y_2 - y_1}$ ---------------------[5]

The co-ordinates of point T $(x_3, y_3) = \left(\dfrac{x_1y_2 - x_2y_1}{y_2 - y_1}, 0\right)$

$TP = \sqrt{(y_1 - y_3)^2 + (x_1 - x_3)^2}$
Substituting values of x_3, y_3 in above, we get

$$TP = \sqrt{(y_1 - 0)^2 + \left\{x_1 - \left(\frac{x_1y_2 - x_2y_1}{y_2 - y_1}\right)\right\}^2}$$

$$\therefore \; TP^2 = {y_1}^2 + \left\{x_1 - \left(\frac{x_1y_2 - x_2y_1}{y_2 - y_1}\right)\right\}^2$$

$$= {y_1}^2 + \left\{\frac{x_1(y_2 - y_1) - (x_1y_2 - x_2y_1)}{y_2 - y_1}\right\}^2$$

$$= {y_1}^2(y_2 - y_1)^2 + x_1^2(y_2 - y_1)^2 + (x_1y_2 - x_2y_1)^2 - 2x_1(y_2 - y_1)(x_1y_2 - x_2y_1) --- [6]$$

$$TQ = \sqrt{(y_2 - y_3)^2 + (x_2 - x_3)^2}$$

Substituting values of [5] and $y_3 = 0$ in above, we get

$$TQ = \sqrt{(y_2 - 0)^2 + \left[x_2 - \left(\frac{x_1y_2 - x_2y_1}{y_2 - y_1}\right)\right]^2}$$

$$\therefore \; TQ^2 = {y_2}^2 + \left[x_2 - \left(\frac{x_1y_2 - x_2y_1}{y_2 - y_1}\right)\right]^2$$

$$= \frac{{y_2}^2(y_2 - y_1)^2 + {x_2}^2(y_2 - y_1)^2 + (x_1y_2 - x_2y_1)^2 - 2x_2(y_2 - y_1)(x_1y_2 - x_2y_1)}{(y_2 - y_1)^2} --- [7]$$

Subtracting [7] − [6], we get

$$TQ^2 - TP^2 = \frac{(y_2 - y_1)^2({y_2}^2 - {y_1}^2) + (y_2 - y_1)^2({x_2}^2 - {x_1}^2) - 2(x_1 - x_2)(y_2 - y_1)(x_1y_2 - x_2y_1)}{(y_2 - y_1)^2}$$

$$= \left\{\frac{(y_2 - y_1)^2({y_2}^2 - {y_1}^2)}{(y_2 - y_1)^2}\right\} + \left\{\frac{(y_2 - y_1)^2({x_2}^2 - {x_1}^2)}{(y_2 - y_1)^2}\right\} - \left\{\frac{2(x_1 - x_2)(y_2 - y_1)(x_1y_2 - x_2y_1)}{(y_2 - y_1)^2}\right\}$$

$$= (y_2{}^2 - y_1{}^2) + (x_2{}^2 - x_1{}^2) - \left\{\frac{2(x_1 - x_2)(x_1y_2 - x_2y_1)}{(y_2 - y_1)}\right\}$$

$$= (x_2{}^2 + y_2{}^2) - (x_1{}^2 + y_1{}^2) + \left\{\frac{2(x_2 - x_1)(x_1y_2 - x_2y_1)}{(y_2 - y_1)}\right\}$$

$$= (x_2{}^2 + y_2{}^2) - (x_1{}^2 + y_1{}^2) + 2\left(\frac{x_1y_2 - x_2y_1}{y_2 - y_1}\right)(x_2 - x_1) ----------[8]$$

$$= (x_2{}^2 + y_2{}^2) - (x_1{}^2 + y_1{}^2) + 2(x_3)(x_2 - x_1) ----------[9]$$

The value x_3 is the horizontal coordinates of points 'T'. Substituting eqn.[2], [3] & [5] in above eqn., we get

$$TQ^2 - TP^2 = OQ^2 - OP^2 + 2\,(OT)(x_2 - x_1) -----------------[10]$$

In fig. 1, Let points 'J' & 'K' are the projection of points 'P' & 'Q' on major axis respectively. Therefore, x_1 and x_2 are the horizontal coordinates of points 'J' & 'K' respectively.
Therefore, $TQ^2 - TP^2 = OQ^2 - OP^2 + 2\,(OT)(JK)$

$$(TQ^2 - TP^2) - (OQ^2 - OP^2) - 2\,(OT)(JK) = 0 \qquad -----------------[11]$$

If the point 'T' is fixed, OT is also constant. Therefore,

$$(TQ^2 - TP^2) - (OQ^2 - OP^2) = 2\,(OT)(JK)$$

$$\therefore \frac{(TQ^2 - TP^2) - (OQ^2 - OP^2)}{JK} = (2 \times OT)$$

$$\therefore \frac{(TQ^2 - TP^2) - (OQ^2 - OP^2)}{JK} = \text{constant} -----------------[12]$$

New Theorem on Ellipse

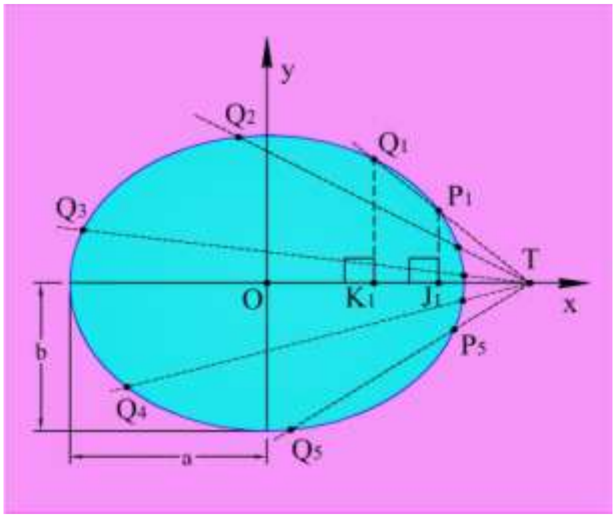

Figure 2: Several secants passing through the ellipse from fixed point 'T' on the line of x-axis

(i) If a secant drawn from a fixed point 'T' anywhere on the line of major axis at outside of an ellipse, intersecting at two points 'P' & 'Q', then

$$\frac{\left(TQ_1{}^2 - TP_1{}^2\right) - \left(OQ_1{}^2 - OP_1{}^2\right)}{J_1K_1} = \frac{\left(TQ_2{}^2 - TP_2{}^2\right) - \left(OQ_2{}^2 - OP_2{}^2\right)}{J_2K_2} = \ \dots \ = 2 \times OT \ \text{ is a constant}$$

Where, 'OP_1' 'OQ_1' are the distance of intersection of secant and ellipse from centre 'O' of the ellipse. J_1K_1 is the projection of P_1Q_1 on major axis and OT is the distance the fixed point from centre (Ref: fig.2). Similarly, (Referring fig.3) a line passing through any one of the points 'S' which is outside of the ellipse on minor axis and intersecting the ellipse at 'U' & 'V'.

$$(SV^2 - SU^2) - (OV^2 - OU^2) - 2\,(OS)(LM) = 0 \qquad ----------------[13]$$

Where, 'OU' 'OV' are the distance of intersecting points 'U' & 'V' from geometric centre 'O' of the ellipse. 'L' & 'M' are the projection of points 'U' & 'V' on minor axis respectively (Ref: fig. 3).

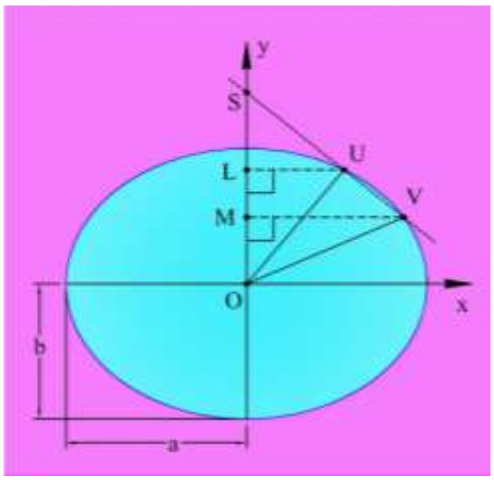

Figure 3: A secant passing through points 'U' and 'V' of the ellipse from point 'S' on the line of y-axis

If the point 'S' is fixed, OS is also constant, therefore

$$(SV^2 - SU^2) - (OV^2 - OU^2) = 2\,(OS)(LM)$$

$$\therefore \frac{(SV^2 - SU^2) - (OV^2 - OU^2)}{LM} = 2\,(OS)$$

$$\therefore \frac{(SV^2 - SU^2) - (OV^2 - OU^2)}{LM} = 2 \times OS = \text{constant} \qquad ---------------[14]$$

The eqn. [12] & [14] are the necessary mathematical expression of the property.
A theorem with respect to minor axis may be written as
If a secant drawn from a fixed point 'S' anywhere on the line of minor axis at outside of an ellipse, intersecting at two points 'U' & 'V', then

$$\frac{\left(SV_1{}^2 - SU_1{}^2\right) - \left(OV_1{}^2 - OU_1{}^2\right)}{L_1M_1} = \frac{\left(SV_2{}^2 - SU_2{}^2\right) - \left(OV_2{}^2 - OU_2{}^2\right)}{L_2M_2} = \cdots = 2 \times OS \ \text{ is a constant}$$

Where, 'OU_1' 'OV_1' are the distance of intersection of secant on ellipse from centre 'O' of the ellipse. L_1M_1 is the projection of UV on minor axis and OS is the distance the fixed point from centre (Ref: Fig.4).

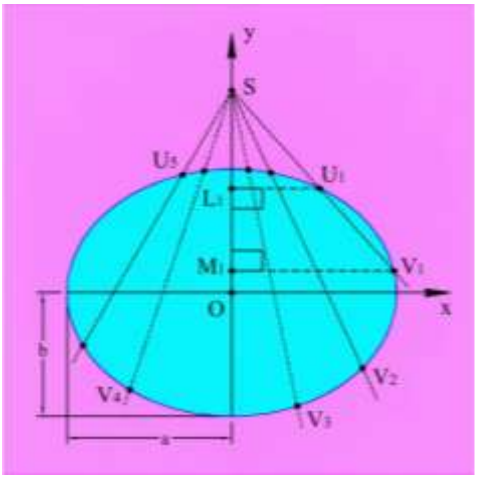

Figure 4: Several secants passing through the ellipse from fixed point 'S' on the line of y-axis

RESULTS AND DISCUSSION

Example-1

An ellipse of semi major axis (a) = 4 units and semi minor axis (b) = 3 units is drawn. There are five secants have been drawn from a fixed point on line of major axis at T. The distance of T from centre of the ellipse is 5.3382 units. These secants are drawn such that $\angle TOP_1$ = 38.4861°, $\angle TOP_2$ = 26.6669°, $\angle TOP_3$ = 6.7844° , $\angle TOP_4$ = -14.4688° and $\angle TOP_5$ = -31.5139°. The dimensions are measured from the figure. 3 and the same have been tabulated in table -1.

$$\frac{(TQ^2 - TP^2) - (OQ^2 - OP^2)}{JK} = \text{constant}$$

Let it be the constant in the above equation $= c$
As per eqn. [12], $c = 2 \times OT$
Therefore, $c = 2 \times 5.3382 = 10.6764$
The results which are calculated by the eqn. [12] have also been shown in the same table for main theorem. The results are found as same for all secant and it is equal to $2 \times OT$

Table 1: The result of equation 12

S. No	TQ	TP	OQ	OP	JK	c
1	4.0528	2.3697	3.3245	3.7827	1.3175	10.67645
2	6.6154	1.6332	3.0239	3.9474	4.4523	10.67632
3	9.1345	1.3534	3.8851	3.9955	7.7266	10.67448
4	8.4485	1.4107	3.5404	3.9879	6.8146	10.67649
5	5.6978	1.8089	3.0168	3.9121	3.3153	10.67664

Example-2

An ellipse of semi major axis (a) = 4 units and semi minor axis (b) = 3 units is drawn. There are five secants have been drawn from a fixed point on line of major axis at T. The distance of T from centre of the ellipse is 4.9189 units. These secants are drawn such that $\angle SOU_1$ = 41.8842°, $\angle SOU_2$ = 25.1973°, $\angle SOU_3$ = 11.6983°, $\angle SOU_4$ = -18.786° and $\angle SOU_5$ = -30.2537°. The dimensions are measured from the figure. 4 and the same have been tabulated in table 2.

Table 2: The result of equation 14

S. No	SV	SU	OV	OU	LM	c
1	5.8978	3.2007	3.9728	3.3163	2.0080	9.837784
2	7.4543	2.2141	3.6615	3.0641	4.7416	9.837708
3	7.8349	1.9750	3.1787	3.0117	5.7382	9.837795
4	7.6852	2.0713	3.4176	3.0323	5.3148	9.837975
5	7.1782	2.3814	3.8370	3.1032	4.1435	9.837763

$$\frac{(\mathrm{SV}^2 - \mathrm{SU}^2) - (\mathrm{OV}^2 - \mathrm{OU}^2)}{\mathrm{LM}} = \text{constant}$$

Let it be the constant in the above equation $= c$
As per eqn. [12], $c = 2 \times \mathrm{OS}$
Therefore, $c = 2 \times 4.9189 = 9.8378$
The results which are calculated by the eqn. [14] have also been shown in the same table for main theorem. The results are found as same for all secant and it is equal to $\mathbf{2} \times \mathbf{OS}$

Conclusion

In this article the author has derived necessary equations with analytic geometry and developed a new theorem for a property of a secant drawn at a fixed point either anywhere in the line of axis of major axis or anywhere in the line of axis of minor axis of ellipse and defined very clearly along with relevant drawings and proved with appropriate examples. It will be very useful for those doing research or further study in the geometry especially in ellipse.

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