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CYCLIC GROUPS IN TERMS OF VAGUE GROUP

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ABSTRACT
In this paper we defines Vague groups (VG) and some characterization of them with some numerical examples and cyclic groups are characterized as groups which admit a particular type of vague group.

Keywords: *Vague Set, Vague Group, Vague-Cut, Vague-Cut Group, Order of a Vague Group, Cyclic Group*

INTRODUCTION
Gau and Buehrer (1993) initiated the study of vague sets as an improvement over the theory of fuzzy sets to interpret and solve real life problems which are in general vague. Biswas (2006) defines Vague groups. The objectives of this paper are to characterize cyclic groups among the class of groups as groups admitting a particular type of vague group. Assad (1991) has characterized the cyclic groups of prime power order in terms of fuzzy sub groups. In fact Ramakrishna (2009) improved the result of Assad Md. by characterizing the cyclic groups in terms of L-fuzzy groups. In this paper we show that "A group G is a cyclic group iff G posses a vague group A such that image A has least element" Also a group G is a cyclic group of prime power order iff it admits a particular type of vague group A such that image A is finite chain.

Preliminaries
In this section, we present some preliminaries on the theory of vague sets (VS). In his pioneer work, Zadeh (1965) proposed the theory of fuzzy sets. And fuzzy sets theory has been applied in wide varieties of fields like Computer Science, Management Science, Medical Sciences, Engineering problems etc. to list a few only.

Let $U = \{x_1, x_2, x_3, \ldots\ldots, x_n\}$ be the universe of discourse. The membership function for fuzzy sets can take any value from the closed interval $[0, 1]$. Fuzzy set A is defined as the set of ordered pairs $A = \{(x, \mu_A(x)) \text{ where } x \in U\}$ where $\mu_A(x)$ is the grade of membership of elements x in set A. The greater $\mu_A(x)$ the greater is the truth of the statement that 'the element x belong to the set A '.

Definition 2.1
A vague set (or in short VS) A in the universe of discourse U is characterized by two functions given by:

1) A membership function $\mu_A : U \to [0 ,1]$ and
2) A non membership function $\nu_A : U \to [0\,1]$

Where $\mu_A(x)$ is a lower bound of the grade of membership of x derived from the 'evidence for x', and $\nu_A(x)$ is the lower bound on the negation of x derived from the 'evidence against x' and $\mu_A(x) + \nu_A(x) \le 1$. Thus the grade of membership of x in the vague set A is bounded by a subinterval [$\mu_A(x),\ 1-\nu_A(x)$] of [0,1]. This indicate that if the actual grade of membership is $\eta(x)$, then $\mu_A(x) \le \eta(x) \le 1-\nu(x)$.

The vague set is written as $A = \{< x, [\mu_A(x), \nu(x)] >: x \in U\}$.

Definition 2.2
The interval [$\mu_A(x),\ 1-\nu_A(x)$] is called the vague value of x in A and is denoted by $V_A(x)$. i.e $V_A(x) = [\mu_A(x), 1-\nu_A(x)]$

For example, consider a universe $U = \{TEACHER, DOCTER, ENGINEAR\}$.
A vague set A of U could be
$A = \{< TEACHER,[.6,.3] >, < DOCTER,[.3,.5], < ENGINEAR,[.4,.6] >\}$

Definition 2.3
Let $(G,*)$ be a group. A vague set A of G is called vague group (VG) of G if the following conditions are true:
$\forall x, y \in X, V_A(xy) \geq \min\{V_A(x), V_A(y)\}, i.e$
1) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$,
$1 - \nu_A(xy) \geq \min\{1-\nu_A(x), 1-\nu_A(y)\}$, and
2) $\mu_A(x^{-1}) \geq \mu_A(x)$,and $1-\nu_A(x^{-1}) \geq 1-\nu_A(x)$ here the element *xy* is stand for $x * y$

Definition 2.4 $(\alpha, \beta) - cut$ or vague-cut.
Let A be a vague set of universe X with membership function μ_A and the non membership function ν_A .The $(\alpha, \beta) - cut$ of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set X given by $A_{(\alpha,\beta)} = \{x : x \in X, V_A(x) \geq [\alpha, \beta]\}$ Clearly $A_{(0,0)} = X$. The $(\alpha, \beta) - cut$ are also called vague-cuts of the vague set A.

Definition 2.5 $\alpha, -cut$ of a vague set.
The $\alpha, -cut$ of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha,\alpha)}$.
Thus $A_0 = X$ and if $\alpha \geq \beta$ then $A_\beta \subseteq A_\alpha$ *and* $A_{(\alpha,\beta)} = A_\alpha$
Equivalently, we can define the $\alpha, -cut$ as $A_\alpha = \{x : x \in X, \mu_A(x) \geq \alpha\}$

Definition 2.6:
Let A be a vague group of a group G, and $H = \{x \in G : V_A(x) = V_A(e)\}$ then the order of A is defined as the order of the crisp sub group H and it is denoted by o(A).

Notations Let $I\,[0,1]$ denote the family of all the closed subintervals of [0 ,1] . If $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ be two elements of $I\,[0,1]$, we call $I_1 \geq I_2$ if , $a_1 \geq a_2$ and $b_1 \geq b_2$, Similarly we define the relation $I_1 \leq I_2$ if $a_1 \leq a_2$ and $b_1 \leq b_2$ *.and* $I_1 = I_2$ if $a_1 = a_2$ and $b_1 = b_2$. The relation $I_1 \geq I_2$ does not necessarily imply that $I_1 \supseteq I_2$ and conversely. We define the term 'imax' to mean the maximum of two intervals as $i\max(I_1, I_2) = [\max(a_1, a_2), \max(b_1, b_2)]$ and $i\min(I_1, I_2) = [\min(a_1, a_2), \min(b_1, b_2)]$
The concept of 'imax' and 'imin' could be extended to define 'isup' and 'iinf' of infinite number of elements of $I\,[0,1]$.
It is obvious that $L = \{I[0,1], i\sup, i\inf, \leq\}$ is a lattice with universal bounds [0,0] and [1,1].

Theorem 2.1: If A is the vague group of a group G, then $\forall x \in G, V_A(x^{-1}) = V_A(x), i.e$
$\mu_A(x^{-1}) = \mu_A(x)$ and $1-\nu_A(x^{-1}) = 1-\nu_A(x)$.

Theorem 2.2 A necessary and sufficient conditions for a vague set of group G to be a vague group of G is that $V_A(xy^{-1}) \geq \min\{V_A(x), V_A(y)\}$.

Proof: Let A be a vague group of the group G. Then
$\mu_A(xy^{-1}) \geq \min\{\mu_A(x), \mu_A(y^{-1})\}$,
$\geq \min\{\mu_A(x), \mu_A(y)\}$
Similarly

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$1-\nu_A(xy^{-1}) \geq \min\{1-\nu_A(x), 1-\nu_A(y)\}$

For the converse part, suppose that A be a V S of the group G of which e is the identity element.

Now $\mu_A(yy^{-1}) \geq \min\{\mu_A(y), \mu_A(y)\}$

$\mu_A(e) \geq \mu_A(y)$. (1)

Now $\mu_A(ey^{-1}) \geq \min\{\mu_A(e), \mu_A(y)\}$

$\mu_A(y^{-1}) \geq \mu_A(y)$. (2)

From equation (1) and (2) we get $\mu_A(y^{-1}) = \mu_A(y)$

Also

$\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y^{-1})\}$,

$\geq \min\{\mu_A(x), \mu_A(y)\}$

Similarly it can be proved that

$1-\nu_A(x^{-1}) \geq 1-\nu_A(x)$ and

$1-\nu_A(xy) \geq \min\{1-\nu_A(x), 1-\nu_A(y)\}$.

Theorem2.3 If A and B are two vague groups of a group G. Then $A \cap B$ is also a vague group of G.

Proof : $\mu_{A\cap B}(xy^{-1}) = \min\{\mu_A(xy^{-1}), \mu_B(xy^{-1})\}$

$\geq \min\{\min\{\mu_A(x),, \mu_A(y)\}, \min\{\mu_B(x), \mu_B(y)\}$

$= \min\{\mu_{A\cap B}(x), \mu_{A\cap B}(y)\}$

Theorem 2.4 If $A = (x, \mu_A, \nu_A)$ is a vague group of group G, then the following holds.

1) μ_A is a fuzzy group of G ;

2) $1-\nu_A$ is a fuzzy group of G ;

Theorem 2.5 A necessary and sufficient condition for a Vague set A of a group G to be a vague group is that μ_A and $1-\nu_A$ are fuzzy groups of G.

3. Vague Group and Cyclic Groups of Prime Power Order

We begin with the following theorem.

Theorem 3.1: Let A be a vague group of a group G. Then $V_A(xy^{-1}) = V_A(e)$

Implies $V_A(x) = V_A(y)$ for any x and y in G.

Proof: Suppose that $V_A(xy^{-1}) = V_A(e)$. Consider

$V_A(x) = V_A(x.e)\, V_A(x.y^{-1}.y)$

$\geq \min.\{V_A(x.y^{-1}), V_A(y)\}$

$= \min.\{V_A(e), V_A(y)\} = V_A(y)$ since $V(e) \geq V(y)$ for all y.

This gives us $V_A(x) \geq V_A(y)$, Since $V_A(z) = V_A(z^{-1})$, we get $V_A(yx^{-1}) = V_A(e)$ and now interchanging the x and y then we get $V_A(y) \geq V_A(x)$. This implies $V_A(x) = V_A(y)$.

Remark: The following example shows that the converse implication is not true.

Example 3.1. Consider the group $G = \{1, \omega, \omega^2\}$ with respect to the binary operation complex number multiplication where ω is the imaginary cube root of unity. Clearly the vague set

$A = \{(1,.9,.1), (\omega,.6,.2), (\omega^2,.6,.2)\}$ is a vague group of the group G.

But $V_A(1) = [\mu_A(1), 1-\nu_A(1)] = [.9,.9]$

$V_A(\omega) = [\mu_A(\omega), 1-\nu_A(\omega)] = [.6,.8]$

$V_A(\omega^2) = [\mu_A(\omega^2), 1-\nu_A(\omega^2)] = [.6,.8]$ we observe that $V_A(\omega) = V_A(\omega^2)$.

But $V_A(\omega(\omega^2)^{-1}) \neq V_A(1)$.

Example 3.2 consider the group $Z_4 = \{0,1,2,3,+_4\}$ where binary operation is addition modulo 4.

Define vague set $B = \{(0,.9,.1),(1,.6,.2),(2,.5,.3),(3,.6,.2)\}$.

The above set B satisfy all the property of vague group hence B is vague group of Z_4.

But $V_B(0) = [\mu_B(0), 1-\nu_B(0)] = [.9,.9]$

$V_B(1) = [\mu_B(1), 1-\nu_B(1)] = [.6,.8]$

$V_B(2) = [\mu_B(2), 1-\nu_B(2)] = [.5,.7]$

$V_B(3) = [\mu_B(3), 1-\nu_B(3)] = [.6,.8]$ we observe that $V_B(1) = V_B(3)$.

But $V_B(1 +_4 (3)^{-1}) = V_B(1 +_4 1) = V_B(2) = [.5,.7]$. And hence $V_B(1 +_4 (3)^{-1}) \neq V_B(0)$.

Theorem 3.2 : Let A be a vague group of a group G and if for a fixed y in G,and if for all x in G, $V_A(x) \leq V_A(y)$ then $V_A(xy) = V_A(x) = V_A(yx)$.

Proof : $V_A(xy) \geq \min\{V_A(x), V_A(y)\} = V_A(x)$ implies $V_A(xy) \geq V_A(x)$,

Since by hypothesis

$V_A(y) \geq V_A(x)$ for all x, we can take xy in place of x and hence $V_A(x) \geq V_A(xy)$

In particular $V_A(x) = V_A(x.e) = V_A(xy.y^{-1})$

$\geq \min.\{V_A(x.y), V_A(y^{-1})\}$

$= \min.\{V_A(x.y), V_A(y)\} = V_A(xy)$.

Which implies $V_A(x) \geq V_A(xy)$.

Thus $V_A(xy) = V_A(x)$.In a similar fashion, we can prove that $V_A(yx) = V_A(x)$.

This complete the theorem.

Theorem 3.3 : Let A be a vague group of group G and $x \in G$.Then $V_A(xy) = V_A(y)$ for all y in G iff $V_A(x) = V_A(e)$.

Proof: Suppose $V_A(xy) = V_A(y)$ for all $y \in G$. Choose $y = e$.in this equality then we have $V_A(x.e) = V_A(e)$ implies $V_A(x) = V_A(e)$. Conversely , suppose $V_A(x) = V_A(e)$, for any $y \in G$ $V_A(y) \leq V_A(e)$ implies $V_A(y) \leq V_A(x)$.Now $V_A(xy) \geq \min\{V_A(x), V_A(y)\} = V_A(y)$ and this implies $V_A(xy) \geq V_A(y)$ for all $y \in G$.

But $V_A(y) = V_A(e.y) = V_A(x^{-1}x.y)$

$\geq \min.\{V_A(x^{-1}, V_A(xy)\}$

$= \min.\{V_A(e), V_A(xy)\} = V_A(xy)$.

Which implies $V_A(y) \geq V_A(xy)$. And thus $V_A(xy) = V_A(y)$.

Theorem 3.4 : Let A be a Vague group of finite group G. Then $o(A)/o(G)$. This theorem is also known as Lagrange's Theorem .For any x in G, <x> denotes the cyclic group generated by x.

Theorem 3.5: Let G be a Group and A be a vague group of G such that $V_A(x) \leq V_A(y) \Rightarrow \langle x \rangle \supseteq \langle y \rangle$.Then G is cyclic iff Image A has a least element.

Proof: Image of $A = \text{Im}.A = \{V_A(x) : x \in G\}$.Suppose G is cyclic group generated by x. For any y in G, $y = x^m$,for some m in Z, so that

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$V_A(y) = V_A(x^m)$
$\geq \min.\{V_A(x), V_A(x), \ldots\ldots.. V_A(x)\ m\ times\}$
$= V_A(x) \Rightarrow V_A(x) \leq V_A(y)$ for all y in G, which implies $V_A(x)$ is the least element of Im.A. Thus Im.A has a least element.

Conversely let Im.A have least element, say $V_A(x)$.This implies $V_A(x) \leq V_A(y)$ for all $y \in G$,which implies $< x > \supseteq < y >$, for all $y \in G$. So that $y \in < x >$,which implies that $G = < x >$ is a cyclic group.

Theorem 3.6: let A be a vague group of a group G, such that image set of A is given by $\text{Im}.A = \{I_0 > I_1 > I_2, \ldots\ldots.. I_n\}$ and such that (i) $V_A(x) = V_A(y) \Rightarrow < x > = < y >$
(ii) $V_A(x) < V_A(y) \Rightarrow < x > \supset < y >$. Then G is a cyclic group of prime power order.

Proof: We can find $x_1, x_2, \ldots. x_n$ in G such that $V_A(e) = I_0, V_A(x_1) = I_1, \ldots\ldots., V_A(x_n) = I_n$.Let $y \in G$ then $V_A(y)$ must be equal to one of $V_A(e), V_A(x_1) \ldots\ldots\ldots, V_A(x_n)$. Let $V_A(y) = V_A(x_r)$ for some r, implies $< y > = < x_r > \Rightarrow y \in (x_r)$.But $V_A(x_r) \geq V_A(x_n)$ implies $y \in (x_n)$.Then G is cyclic and therefore G is isomorphic to $Z\ orZ_n$. If G is isomorphic to Z , without loss of generality, we may assume $V_A(2) > V_A(3) \Rightarrow < 2 > \subset < 3 >$. Which is imposible. So $V_A(2)$ is not greater than $V_A(3)$. Similarily $V_A(3)$ is not greater than $V_A(2)$ and $V_A(2) \neq V_A(3)$. Therefore G is isomorphic to Z_n for some n. Suppose p,q are distinct prime factor of n, then $V_A(p)$ is not greater than $V_A(q)$, $V_A(q)$ is not greater than $V_A(p)$, $V_A(p) \neq V_A(q)$. Thus n is a prime power order.

Theorem 3.7: Let G be a cyclic group of prime power order then there is a vague group A of G such that for $x, y \in G$, (i) $V_A(x) = V_A(y) \Rightarrow < x > = < y >$
(ii) $V_A(x) > V_A(y) \Rightarrow < x > \subset < y >$ and Im A is a finite chain.

Proof: Suppose G is a cyclic group of order p^n (p is prime).Define a vague set $A = (\mu_A, \nu_A)$ such that $\mu_A(x) = a_i, \nu_A(x) = b_i$.Where $a_i, b_i \in [0,1], a_i + b_i \leq 1, V_A(x) = I_i$ .if $o(x) = p^i$, if $i = 1,2,3,\ldots,n$ if $V_A(e) = I_o$. Choose Intervals $I_0, I_1, I_2, \ldots\ldots.. I_n in\ [0,1]$ such that $I_0 > I_1 > I_2, \ldots\ldots.. I_n$ where $I_i = [a_i, 1 - b_i]$ .Let $x, y \in G$ since G is a cyclic group of prime power order. We have $< x > \subseteq < y >\ or\ < y > \subseteq < x >$ so $< xy > \subseteq < x >\ or\ < xy > \subseteq < y >$ implies $o(xy) < o(x)\ or\ o(xy) = o(x)$ .Let $o(xy) = p^i, o(x) = p^i \Rightarrow p^i < p^j \Rightarrow i < j \Rightarrow I_i > I_j$

Let $V_A(x) = I_j, V_A(y) = I_k$.
$V_A(xy) = I_i = \min\{I_i, I_i\} \geq \min.\{I_j, I_j\} = \min\{V_A(x), V_A(y)\}$ also $o(x) = o(x^{-1}) \Rightarrow V_A(x) = V_A(x^{-1})$ implies A is a vague group of G.
(ii)Suppose $V_A(x) > V_A(y) \Rightarrow V_A(x) = I_i > I_j = V_A(y)$ for $i < j \Rightarrow p^i < p^j \Rightarrow o(x) < o(y) \Rightarrow < x > \subset < y >$, since G is a cyclic group of order p^n.

Remark: The two conditions (i) $V_A(x) = V_A(y) \Rightarrow < x > = < y >$
(ii) $V_A(x) > V_A(y) \Rightarrow < x > \subset < y >$ can be replaced by the single condition
$V_A(x) \geq V_A(y) \Rightarrow < x > \subseteq < y >$.

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This is substantiated by the observation. $< x > \subseteq < y > \Rightarrow V_A(x) \geq V_A(y)$, is always true. This can be seen because $y \in < x > \Rightarrow y = x^m$ for some $m \in Z^+$ *so that* $V_A(y) = V_A(x^m) \geq \min\{V_A(x), V_A(x), \ldots\ldots, V_A(x)\} = V_A(x)$

Conclusion

Group theory has many applications in physics, chemistry and computer science problems. In this paper we define vague groups and studied some properties of vague groups. The concept is analogous to notion of fuzzy groups introduced by Rosenfeld (1998). Also in this paper we have studied Characterization of cyclic groups in terms of vague groups. We too observe that the definition of 'Vague group' by Dimirici in 1999 is completely different concept of 'Vague Group' by Biswas (2006). We are Following Biswas (2006) concepts.

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