International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jpms.htm
2014 Vol. 4 (1) January-March, pp. 183-187/Natarajan

Research Article

A MADDOX TYPE AND A BUCK TYPE THEOREMS

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ABSTRACT
In this note, entries of infinite matrices, sequences and series are real or complex numbers. We prove two results which are the versions for series of well-known theorems by Maddox (1970a) and Buck (1943) for sequences.

2000 Mathematics Subject Classification: 40, 46.

Keywords: *Regular Matrix, Maddox Type Theorem, Buck Type Theorem, Lower Triangular Matrix*

INTRODUCTION

Throughout this short note, entries of infinite matrices, sequences and series are real or complex numbers. To make the note self-contained, we recall the following. Given an infinite matrix $A \equiv (a_{nk})$, n, k = 1, 2, ... and a sequence $x = \{x_k\}$, k = 1, 2, ..., by the A-transform of $x = \{x_k\}$, we mean the sequence $A(x) = \{(Ax)_n\}$,

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k, \quad n = 1,2,...,$$

where we suppose that the series on the right converge. If $\lim_{n\to\infty} (Ax)_n = \ell$, we say that $x = \{x_k\}$ is summable A or A-summable to ℓ.

If X, Y are sequence spaces, we write $A \in (X, Y)$ if $\{(Ax)_n\} \in Y$, whenever $x = \{x_k\} \in X$. We make use of the following sequence spaces:

$$\ell_\infty = \{x = \{x_k\} : \sup_{k\geq 1} |x_k| < \infty\};$$

$$c = \{x = \{x_k\} : \lim_{k\to\infty} x_k = \ell, \text{for some } \ell\};$$

$$\gamma_\infty = \{x = \{x_k\} : s = \{s_k\} \in \ell_\infty, s_k = \sum_{i=1}^{k} x_i, \quad k = 1,2,...\};$$

$$\gamma = \{x = \{x_k\} : s = \{s_k\} \in c\}.$$

If $A \in (c, c)$, we say that A is conservative. If $A \in (c, c)$ and $\lim_{n\to\infty} (Ax)_n = \lim_{k\to\infty} x_k$, $x = \{x_k\} \in c$, we say that A is regular. The set of all regular matrices is denoted by (c, c; P), P denoting "preservation of limit".

The following result, which gives a characterization of a conservative and a regular matrix in terms of its entries, is well-known (see, for instance, (Hardy, 1949), (Maddox, 1970b)).

Theorem 1.1. $A \equiv (a_{nk})$ is conservative if and only if

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| < \infty; \tag{1.1}$$

$$\lim_{n\to\infty} a_{nk} = \delta_k, \quad k = 1,2,...; \tag{1.2}$$

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and

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk} = \delta. \quad (1.3)$$

Further, A is regular if and only if (1.1), (1.2), (1.3) hold with $\delta_k = 0$, k = 1, 2, ... and $\delta = 1$.

We write $A \in (\gamma, \gamma; P)$ if $A \in (\gamma, \gamma)$ and $\sum_{n=1}^{\infty}(Ax)_n = \sum_{k=1}^{\infty} x_k$, $x = \{x_k\} \in \gamma$.

The following results are due to Maddox 1967.

Theorem 1.2. $A \equiv (a_{nk}) \in (\gamma_\infty, \gamma)$ if and only if

$$\sum_{k=1}^{\infty}|\Delta g_{nk}| \text{ converges uniformly in n;} \quad (1.4)$$

$$\sum_{r=1}^{\infty} a_{rk} = \alpha_k, \quad k = 1, 2, \ldots; \quad (1.5)$$

and

$$\lim_{k\to\infty} g_{nk} = 0, \quad n = 1, 2, \ldots, \quad (1.6)$$

where $\Delta g_{nk} = g_{nk} - g_{n,k+1}$, $g_{nk} = \sum_{r=1}^{n} a_{rk}$, $n, k = 1, 2, \ldots$.

Theorem 1.3. $(\gamma, \gamma; P) \cap (\gamma_\infty, \gamma) = \phi$.

RESULTS AND DISCUSSION

Maddox 1970a proved that $A \in (\ell_\infty, c)$ if and only if there exists a sequence $x = \{x_k\} \in \ell_\infty \setminus c$ such that A sums every subsequence of x. It is easily deduced from this result that a bounded sequence $x = \{x_k\}$ is convergent if and only if there exists a matrix $A \in (c, c; P)$ which sums every subsequence of x (see (Buck, 1943)). We now prove a characterization of the matrix class (γ_∞, γ) similar to Maddox's. We then deduce a characterization of sequences in γ among sequences in γ_∞ similar to Buck's.

Theorem 2.1. (Maddox type). Let $A \equiv (a_{nk})$ be such that $\lim_{k\to\infty} a_{nk} = 0$, n = 1, 2, ... Then $A \in (\gamma_\infty, \gamma)$ if and only if there exists a sequence $x = \{x_k\} \in \gamma_\infty \setminus \gamma$ such that every subsequence of $s = \{s_k\}$ is summable B, where $B \equiv (\Delta g_{nk})$, $s_k = \sum_{i=1}^{k} x_i$, k = 1, 2, ...

Proof. Necessity. Let $A \in (\gamma_\infty, \gamma)$ and $b_{nk} = \Delta g_{nk}$, n, k = 1, 2, ... Using Theorem 1.2, $\sum_{k=1}^{\infty}|b_{nk}|$ converges uniformly in n and $\lim_{n\to\infty} b_{nk} = \lim_{n\to\infty}(g_{nk} - g_{nk+1}) = \alpha_k - \alpha_{k+1}$, k = 1, 2, ... So $B \in (\ell_\infty, c)$ (see (Maddox, 1970b), p. 169, Theorem 6). If $x = \{x_k\} \in \gamma_\infty \setminus \gamma$, then $s = \{s_k\} \in \ell_\infty \setminus c$ so that B sums every subsequence of x.

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Sufficiency. Let there exist a sequence $x = \{x_k\} \in \gamma_\infty \setminus \gamma$ such that every subsequence of $s = \{s_k\}$ is summable B. Now $s = \{s_k\} \in \ell_\infty \setminus c$. By the result due to Maddox stated in the beginning of this section, $B \in (\ell_\infty, c)$. Let $p = \{p_k\} \in \gamma_\infty$, $q_k = \sum_{i=1}^{k} p_i$, $k = 1, 2, \ldots$. So $q = \{q_k\} \in \ell_\infty$. For $m = 1, 2, \ldots$,

$$\sum_{k=1}^{m} a_{nk} p_k = \sum_{k=1}^{m} a_{nk}(q_k - q_{k-1}) \quad (\text{where } q_0 = 0)$$

$$= a_{nm} q_m + \sum_{k=1}^{m-1} (a_{nk} - a_{n,k+1}) q_k. \qquad (2.1)$$

Since $B \in (\ell_\infty, c)$ and $q = \{q_k\} \in \ell_\infty$,

$$\sum_{k=1}^{\infty} (b_{nk} - b_{n-1,k}) q_k$$

converges, $n = 1, 2, \ldots$. Now,

$$b_{nk} - b_{n-1,k} = \Delta g_{nk} - \Delta g_{n-1,k}$$
$$= (g_{nk} - g_{n,k+1}) - (g_{n-1,k} - g_{n-1,k+1})$$
$$= (g_{nk} - g_{n-1,k}) - (g_{n,k+1} - g_{n-1,k+1})$$
$$= a_{nk} - a_{n,k+1}.$$

So

$$\sum_{k=1}^{\infty} (a_{nk} - a_{n,k+1}) q_k$$

converges, $n = 1, 2, \ldots$. Since $\{q_k\} \in \ell_\infty$ and $\lim_{m\to\infty} a_{nm} = 0$, $n = 1, 2, \ldots$, $\lim_{m\to\infty} a_{nm} q_m = 0$, $n = 1, 2, \ldots$. Taking limit as $m \to \infty$ in (2.1), we note that $\sum_{k=1}^{\infty} a_{nk} p_k$ converges and

$$y_n = \sum_{k=1}^{\infty} a_{nk} p_k = \sum_{k=1}^{\infty} (a_{nk} - a_{n,k+1}) q_k, \quad n = 1, 2, \ldots .$$

Now,

$$\sum_{k=1}^{\infty} b_{nk} q_k = \sum_{k=1}^{\infty} (g_{nk} - g_{n,k+1}) q_k$$
$$= \sum_{k=1}^{\infty} \left(\sum_{r=1}^{n} (a_{rk} - a_{r,k+1}) \right) q_k$$
$$= \sum_{r=1}^{n} \left(\sum_{k=1}^{\infty} (a_{rk} - a_{r,k+1}) q_k \right)$$
$$= \sum_{r=1}^{n} y_r$$
$$= t_n.$$

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Since B ∈ (ℓ_∞, c) and $\{q_k\} \in \ell_\infty$, $\left\{\sum_{k=1}^{\infty} b_{nk} q_k\right\}_{n=1}^{\infty} \in c$, i.e., $\{t_n\} \in c$, which implies that $\{y_n\} \in \gamma$. In other words, $A \in (\gamma_\infty, \gamma)$, completing the proof of the theorem. □

Using Theorem 2.1, we now deduce a Buck type result.

Theorem 2.2. (Buck type). A γ_∞ sequence $x = \{x_k\}$ is in γ if and only if there exists a matrix $A \equiv (a_{nk}) \in (\gamma, \gamma; P)$ with $\lim_{k\to\infty} a_{nk} = 0$, n = 1, 2, ... such that B sums every subsequence of $s = \{s_k\}$.

Proof. Sufficiency. Suppose there exists $A \in (\gamma, \gamma; P)$ with $\lim_{k\to\infty} a_{nk} = 0$, n = 1, 2, ... such that B sums every subsequence of $s = \{s_k\}$. We claim that $x \in \gamma$. Suppose not. Then $x \in \gamma_\infty \setminus \gamma$ such that B sums every subsequence of s. In view of Theorem 2.1, $A \in (\gamma_\infty, \gamma)$, which contradicts Theorem 1.3. Thus $x \in \gamma$.

Necessity. Let $x \in \gamma$. Then $s \in c$. Let $A \in (\gamma, \gamma; P)$ with $\lim_{k\to\infty} a_{nk} = 0$, n = 1, 2, ... Let $\{t_k\} \in c$. Define $y_k = t_k - t_{k-1}$, k = 1, 2, ..., where $t_0 = 0$. Then $\{y_k\} \in \gamma$ and so, by hypothesis,

$$\left\{\sum_{k=1}^{\infty} a_{nk} y_k\right\}_{n=1}^{\infty} \in \gamma.$$

Let n be a positive integer. For r = 1, 2, ...,

$$\begin{aligned}\sum_{k=1}^{n}(a_{rk} - a_{r,k+1})t_k &= \sum_{k=1}^{n}(a_{rk} - a_{r,k+1})\left(\sum_{j=1}^{k} y_j\right)\\ &= \sum_{j=1}^{n}\left(\sum_{k=j}^{n}(a_{rk} - a_{r,k+1})\right) y_j\\ &= \sum_{j=1}^{n} a_{rj} y_j - a_{r,n+1}\sum_{j=1}^{n} y_j\\ &= \sum_{j=1}^{n} a_{rj} y_j - a_{r,n+1} t_n. \end{aligned} \tag{2.2}$$

Since $A \in (\gamma, \gamma; P)$, $\sum_{j=1}^{\infty} a_{rj} y_j$ converges, r = 1, 2, ... Since $\lim_{n\to\infty} a_{rn} = 0$ and $\{t_n\} \in c$, it follows that $\lim_{n\to\infty} a_{r,n+1} t_n = 0$. So taking limit as $n \to \infty$ in (2.2), we see that

$$\sum_{k=1}^{\infty}(a_{rk} - a_{r,k+1})t_k$$

converges, r = 1, 2, ... and

$$\sum_{k=1}^{\infty}(a_{rk} - a_{r,k+1})t_k = \sum_{j=1}^{\infty} a_{rj} y_j,$$

r = 1, 2, ...

Now,

$$\sum_{k=1}^{\infty} b_{nk}t_k = \sum_{k=1}^{\infty}(g_{nk} - g_{n,k+1})t_k$$
$$= \sum_{k=1}^{\infty}\left(\sum_{r=1}^{n}(a_{rk} - a_{r,k+1})\right)t_k$$
$$= \sum_{r=1}^{n}\left(\sum_{k=1}^{\infty}(a_{rk} - a_{r,k+1})t_k\right)$$
$$= \sum_{r=1}^{n}\left(\sum_{j=1}^{\infty}a_{rj}y_j\right).$$

So

$$\sum_{k=1}^{\infty} b_{nk}t_k = \sum_{r=1}^{n}\left(\sum_{k=1}^{\infty}a_{rk}y_k\right) \in c,$$

since $\left\{\sum_{k=1}^{\infty} a_{nk}y_k\right\}_{n=1}^{\infty} \in \gamma$. Thus $B \in (c, c)$ and consequently B sums every subsequence of s. This completes the proof of the theorem. □

Remark 2.3. Note that Theorem 2.1 and 2.2 hold for lower triangular matrices, i.e., for matrices (a_{nk}) for which $a_{nk} = 0$, $k > n$, $n, k = 1, 2, \ldots$.

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