International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jpms.htm
2014 Vol. 4 (1) January-March, pp. 144-148/Durge
Research Article

NATURE OF ALGEBRAIC STRUCTURE

$A = \{a_0 + a_1 i + a_2 j \ / \ a_0, a_1, a_2 \ \in F(field)\}$

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ABSTRACT
This is my sincere efforts towards realization of Unchanging Truth. This work is dedicated to my spiritual teacher Sri SriRamakrishana. In the Present work first I proved that $(A, +, .)$ is a ring. Second I proved that $(A, +, *)$ Is a commutative ring with unity. Third I proved that $(A, +, \blacksquare)$ is a semi ring, a commutative ring with unity, where $A = \{a_0 + a_1 i + a_2 j \ / \ a_0, a_1, a_2 \ \in F(field)\}$

Keywords: *Binary Operation, Abelian Group, Semigroup, Ring, Field*

INTRODUCTION
Herstein cotes in 1992
Definition: A nonempty set of elements G is said to form a group if in G there is defined a binary operation, called the product and defined by *, such that
1 a, b ∈ G implies that a*b ∈ G
2 a, b, c ∈ G implies that (a*b)*c = a * (b*c)
3 There exist an element e ∈ G such that a*e = e*a = a for all a ∈ G
4 For every a ∈ G there exist an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a \ = e$
Definition: A group G is said to be abelian (or Commutative) if for every a, b ∈ G,

$$a * b = b * a.$$

Definition: A nonempty set R is said to be an associative ring if in R there are defined two operations, defined by + and * respectively, such that for all a,b,c in R:
1 a+b is in R.
2 a+b = b+a.
3 (a+b)+c = a+(b+c).
4 There is an element 0 in R such that a+0 = a, ∀ a ∈ R
5 There exist an element –a in R such that a + (-a) = 0.
6 a*b is in R
7 a*(b*c) = (a*b)*c.
8 a * (b+c) = a * b + a * c and (b+c) * a = b*a + c*a.
It may very well happen, or not happen, that there is an element 1 in R such that a*1 = 1*a = a for every a in R; if there is such we shall describe R as a ring with unit element.
If the multiplication of R is such that a*b = b*a for every a, b in R, then we call R a commutative ring.

DISCUSSION
Let $A = \{a_0 + a_1 i + a_2 j \ / \ a_0, a_1, a_2 \ \in F(field)\}$,
and
Let $= a_0 + a_1 i + a_2 j$; $y = b_0 + b_1 i + b_2 j$ be any two elements in A
Here first binary operation on A as + defined as
$x + y = (a_0 + a_1 i + a_2 j) + (b_0 + b_1 i + b_2 j)$
$= \ (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j$ ………… (1)

⇨ $(A,+)$ is an abelian group.

Case 1:
Second binary operation on A as. defined as
$x.y = (a_0 + a_1 i + a_2 j).(b_0 + b_1 i + b_2 j)$
$= (a_0 + a_1 + a_2)b_0 + (a_0 + a_1 + a_2)b_1 i$
$+ (a_0 + a_1 + a_2)b_2 j$ ....(2)

⇨ $(A,.)$ is a semi group.
Also
$x.(y+z) = x.y + x.z, \forall\, x,y,z \in A$
$(x+y).z = x.z + y.z, \forall\, x,y,z \in A$
.... (3)
From (1), (2) & (3) we get

$(A, +,.)$ is a ring.
Case 2:
Second binary operation on A as * defined as
$x * y = (a_0 + a_1 i + a_2 j) * (b_0 + b_1 i + b_2 j)$
$= (a_0 b_0 + a_1 b_2 + a_2 b_1) + (a_0 b_1 + a_1 b_0 + a_2 b_2)\, i$
$+ (a_0 b_2 + a_1 b_1 + a_2 b_0)\, j$ .... (4)

⇨ $x * y = y * x, \forall\, x,y \in A$
⇨ $x * (y * z) = (x * y) * z, \forall\, x,y,z \in A$

$=> 1 = 1 + 0i + 0j$ is the identity (multiplicative) of A, which is the multiplicative identity of F.

Let $x^{-1} = b_0 + b_1 i + b_2 j$, Where $x = a_0 + a_1 i + a_2 j$

⇨ $x * x^{-1} = x^{-1} * x = 1 = 1 + 0i + 0j$
⇨ $(a_0 b_0 + a_1 b_2 + a_2 b_1) + (a_0 b_1 + a_1 b_0 + a_2 b_2)i + (a_0 b_2 + a_1 b_1 + a_2 b_0)j$
$= 1 + 0i + 0j$
Hence
$a_0 b_0 + a_1 b_2 + a_2 b_1 = 1$ .... (5)
$a_0 b_1 + a_1 b_0 + a_2 b_2 = 0$ .... (6)
$a_0 b_2 + a_1 b_1 + a_2 b_0 = 0$ .... (7)

Rewriting equation (6) & (7)
$a_1 b_0 + a_0 b_1 + a_2 b_2 = 0$ .... (8)
$a_2 b_0 + a_1 b_1 + a_0 b_2 = 0$ .... (9)

⇨ $$\frac{b_0}{\begin{vmatrix} a_0 & a_2 \\ a_1 & a_0 \end{vmatrix}} = \frac{-b_1}{\begin{vmatrix} a_1 & a_2 \\ a_2 & a_0 \end{vmatrix}} = \frac{b_2}{\begin{vmatrix} a_1 & a_0 \\ a_2 & a_1 \end{vmatrix}} = constant = k$$

⇨ $$\frac{b_0}{a_0^2 - a_1 a_2} = \frac{-b_1}{a_1 a_0 - a_2^2} = \frac{b_2}{a_1^2 - a_0 a_2} = k$$
⇨ $b_0 = k(a_0^2 - a_1 a_2)$ .... (10)

$b_1 = -k(a_1 a_0 - a_2^2)$ .... (11)

$b_2 = k(a_1^2 - a_0 a_2)$ .... (12)
From equations (5), (10), (11) & (12) one obtains,

$a_0(a_0^2 - a_1a_2)k + a_2(-a_1a_0 + a_2^2)k + a_1(a_1^2 - a_0a_2)k = 1$
$[(a_0^3 + a_1^3 + a_2^3) - 3a_0a_1a_2]k = 1$

$$=> \ k = \frac{1}{[(a_0^3 + a_1^3 + a_2^3) - 3a_0a_1a_2]}$$

Hence

$$b_0 = \frac{(a_0^2 - a_1a_2)}{[(a_0^3 + a_1^3 + a_2^3) - 3a_0a_1a_2]}$$

$$b_1 = \frac{(a_2^2 - a_1a_0)}{[(a_0^3 + a_1^3 + a_2^3) - 3a_0a_1a_2]}$$

$$b_2 = \frac{(a_1^2 - a_0a_2)}{[(a_0^3 + a_1^3 + a_2^3) - 3a_0a_1a_2]}$$

⇨ $x^{-1} = b_0 + b_1 i + b_2 j$ be the inverse of $x = a_0 + a_1 i + a_2 j$
if $a_0 + a_1 + a_2 \neq 0$

⇨ $(A,*)$ Is an abelian monoid.
⇨ $(A,+,*)$ Is a commutative ring with unity.

Case 3:
Second binary operation on A as ■ defined as

$Let\ x = a_0 + a_1 i + a_2 j$, $y = (b_0 + b_1 i + b_2 j)$ be any two elements of A
binary operation on A as ■ defined as

$x ■ y = a_0b_0 + \{a_1b_0 + (a_1 + a_2)b_1 + a_1b_2\}\, i$
$+\{a_2b_0 + a_1b_1 + (a_0 + a_2)b_2\}\, j$ ………… (13)
$=> \ x ■ y = y ■ x, \forall\, x, y \in A$
$x ■ (y ■ z) = (x ■ y) ■ z, \forall\, x, y, z \in A$
$x ■ 1 = 1 ■ x$, $\forall\, x \in A$

Let $x = a_0 + a_1 i + a_2 j$ be any element of A and assume that $b_0 + b_1 i + b_2 j$ is the inverse of x.
Hence by definition

$x ■ x^{-1} = x^{-1} ■ x = 1$
$=> \ (a_0 + a_1 i + a_2 j) ■ (b_0 + b_1 i + b_2 j) = 1$
$=> \ a_0b_0 + \{a_1b_0 + (a_1 + a_2)b_1 + a_1b_2\}\, i$
$+\{a_2b_0 + a_1b_1 + (a_0 + a_2)b_2\}\, j$
$= 1 + 0i + 0j$
$=> \ a_0b_0 = 1$ …………. (14)
$a_1b_0 + (a_1 + a_2)b_1 + a_1b_2 = 0$…………. (15)
$a_2b_0 + a_1b_1 + (a_0 + a_2)b_2 = 0$…………. (16)

Solving (15) and (16) for b_0, b_1, b_2

$$\frac{b_0}{\begin{vmatrix}(a_1+a_2) & a_1\\ a_1 & (a_0+a_2)\end{vmatrix}} = \frac{-b_1}{\begin{vmatrix}a_1 & a_1\\ a_2 & (a_0+a_2)\end{vmatrix}} = \frac{b_2}{\begin{vmatrix}a_1 & (a_1+a_2)\\ a_2 & a_1\end{vmatrix}} = k$$

$$b_0 = k\begin{vmatrix}(a_1 + a_2) & a_1\\ a_1 & (a_0 + a_2)\end{vmatrix}$$

$$= k\{(a_1 + a_2)(a_0 + a_2) - {a_1}^2\}$$

$b_0 = k(a_0a_1 + a_1a_2 + a_0a_2 + {a_2}^2 - {a_1}^2)$ …… (17)

$$b_1 = -k\begin{vmatrix} a_1 & a_1 \\ a_2 & (a_0 + a_2) \end{vmatrix}$$

$$= -k\{a_1(a_0 + a_2) - a_1a_2\}$$

$b_1 = -k(a_0a_1)$ ……….. (18)

$$b_2 = k\begin{vmatrix} a_1 & (a_1 + a_2) \\ a_2 & a_1 \end{vmatrix}$$

$$= k\{a_1^2 - a_1a_2 - a_2^2\}$$

$b_2 = k\{a_1^2 - a_2^2 - a_1a_2\}$ ……….. (19)

From eq^n (14)
$b_0 = {}^1/_{a_0}$
eq^n (17) gives

$k = {}^1/_{a_0\{a_0a_1 + a_1a_2 + a_0a_2 + a_2^2 - a_1^2\}}$ …… (20)

Substituting the values of k in eq^n(18) & (19) one obtains
$b_1 = -{}^{a_1}/_{\{a_0a_1 + a_1a_2 + a_0a_2 + a_2^2 - a_1^2\}}$ ….. (21)

$$b_2 = \frac{\{a_1^2 - a_2^2 - a_1a_2\}}{a_0\{a_0a_1 + a_1a_2 + a_0a_2 + a_2^2 - a_1^2\}}$$
……….. (22)
Substituting the values of b_0, b_1 & b_2 in LHS of eq^n (15) one obtains

$$a_1b_0 + (a_1 + a_2)b_1 + a_1b_2$$

$$= a_1\frac{1}{a_0} + (a_1 + a_2)\left\{\frac{-a_1}{a_0a_1 + a_1a_2 + a_0a_2 + a_2^2 - a_1^2}\right\}$$

$$+ \frac{a_1\{a_1^2 - a_2^2 - a_1a_2\}}{a_0\{a_0a_1 + a_1a_2 + a_0a_2 + a_2^2 - a_1^2\}}$$

$$= \frac{\{a_0a_1 + a_1a_2 + a_0a_2 + a_2^2 - a_1^2\} - (a_1 + a_2)a_0a_1 + a_1\{a_1^2 - a_2^2 - a_1a_2\}}{a_0\{a_0a_1 + a_1a_2 + a_0a_2 + a_2^2 - a_1^2\}}$$

$$= \frac{[a_0a_1 + a_1a_2 + a_0a_2 + a_2^2 - a_1^2 - a_1^2a_0 - a_0a_1a_2 + a_1^3 - a_2^2a_1 - a_1^2a_2]}{a_0\{a_0a_1 + a_1a_2 + a_0a_2 + a_2^2 - a_1^2\}}$$
$\neq 0$
Hence value of b_0, b_1 & b_2 are not unique i.e. x^{-1} not unique
=> x^{-1} does not exist for each $(a_0 + a_1i + a_2j)$ in A.
=> Inverse is not exist for each non zero element in A.
=> $(A, \blacksquare)$ is a commutative monoid.

$=> \ (A, +, \blacksquare)$ is a semi ring, a commutative ring with unity.

Conclusion

From the above discussion, I come to the following conclusions first I proved that $(A,\ +,.)$ is a ring. Second I proved that $(A, +,*)$ Is a commutative ring with unity. Third I proved that $(A, +, \blacksquare)$ is a semi ring, a commutative ring with unity. Where $A = \{a_0 + a_1 i + a_2 j \ / \ a_0, a_1, a_2 \ \in F(field)\}$

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