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Research Article

NATURE OF SEQUENCE $\left\{\frac{n^m}{n^m+k} \;/\; m, k \in N\right\}$

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ABSTRACT
This is my sincere efforts towards realization of Unchanging Truth. This work is dedicated to my spiritual teacher Sri SriRamakrishana. In the Present work first I proved that any sequence monotonic and bounded need not be a Convergent sequence. Second I proved that any sequence S is convergent then its reciprocal (1/S) need not be a convergent sequence. Third I proved that any sequence T is divergent though it is a finite monotonic and bounded then its reciprocal (1/T) be a convergent sequence. Fourth I proved that$\left\{\frac{n}{n+k}\right\}$ is not a Cauchy Sequence, $\left\{\frac{n}{n+k}\right\}$ is not a convergent Sequence.
Fifth I proved Sequence $\left\{\frac{n^m}{n^m+k} \;/\; m, k \in N\right\}$ is not Cauchy Sequence.
Sixth I proved Sequence $\left\{\frac{n^m}{n^m+k} \;/\; m, k \in N\right\}$ is not Convergent Sequence.
Lastly I proved that reciprocal of every Convergent sequence need not be a Cauchy sequence though it is a finite monotonic and bounded.

Keywords: *Cauchy Sequence, Convergent Sequence, Monotone Sequence, Bounded Sequence*

INTRODUCTION
Kreyszig (1978) cotes that1.4-1 Definition (Convergence of a sequence, limit).
A sequence (x_n) in a metric space X = (X, d) is said converge or to be convergent if there is an x ϵ X such that

$$\lim_{n \to \propto} d(x_n, x) = 0.$$

X is called the limit of (x_n) and we write

$$\lim_{n \to \infty} x_n = x.$$

Or, simply,

$$\mathbf{X_n \to x}$$

Convergence of sequences and related concepts in normed spaces follow readily from the corresponding definitions 1.4-1 and 1.4-3 for metric spaces and the fact that now $d(x,y) = \|x - y\|$:

(i) A sequence (x_n) in normed space X is convergent if $\lim_{n \to \infty} \|x_n - x\| = 0$.

Then we write $x_n \to x$ and call x the limit of (x_n).

(1) A sequence (x_n) in a normed space X is Cauchy if for every $\epsilon > 0$ there is an N such that $\|x_m - x_n\| < \epsilon$ for all m, n > N.

Simmons cotes in (1) that
We say that $\{x_n\}$ is convergent if there exists a point x in X such that either
(1) for each $\epsilon > 0$, there exists a positive integer n_0 such that $n \geq n_0 \Rightarrow d(x_n,x) < \epsilon$;
or equivalently,
(2) for each open sphere $S_\epsilon(x)$ centered on x, there exists a positive integer n_0 such that x_n is in $S_\epsilon(x)$ for all $n \geq n_0$.

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Karade and Bendre (No date) cotes that

Limit of a sequence

Definition. A real number ∫ is said to be a limit of a real sequence s = ‹s_n› if for any ε >0, there is a positive number M depending on εsuch that

$$n>M \Rightarrow |s_n - \int| < \varepsilon \quad (3.1)$$

We write

$$\int = \lim s \text{ or } \int = \lim \langle s_n \rangle \text{ or } \lim_{n\to\infty} s_n = \int \text{ or } s_n \to \int.$$

Convergent sequence. If the limit of a sequence exists, the sequence is said to be *convergent*. If the sequence has no limit it is *divergent.*

Theorem 17. (Monotone convergence theorem)
A monotone sequence of real numbers is convergent if and only if it is bounded.

Cauchy sequence

A sequence ‹s_n› is called a Cauchy sequence if for any $\varepsilon > 0, \exists a M \epsilon N$such that

$$|s_m - s_n| < \epsilon, \forall m, n \geq M. \quad (6.1)$$

Theorem 21. Every convergent sequence of real numbers is a Cauchy sequence.

Walter Rudin (no date) cotes

3.1 Definition A sequence {p_n} in a metric space X is said to converge if there is a point p $\varepsilon X$with the following property: For every $\varepsilon > 0$ there is an integer N such that n ≥ N implies that d(p_n,p) <ε. (Here d denotes the distance in X.)
In this case we also say that {p_n} converges to p, or that p is the limit of
(p_n) [see Theorem 3.2(b)], and we write $p_n \to p$, or

$$\lim_{n\to\infty} p_n = p.$$

If {p_n} does not converge, it is said to *diverge.*

Cauchy Sequences

3.8 Definition A sequence {p_n} in a metric space x is said to be a Cauchy sequence if for every $\varepsilon >$ 0there is an integerN such d(p_n, p_m)< $\varepsilon if n \geq N$
and m ≥ N.
In our discussion ofCauchy sequences, as well as in other situations which will arise later, the following geometric concept willbe useful.

3.14 Theorem Suppose {s_n) is monotonic. Then {s_n} is converges if and only if it is bounded.

Hardy (no date) cotes

The meaning of the above equation, expressed roughly, is that by adding more and more of the u's together we get nearer and nearer to the limit s. More precisely, if any small positive number $\partial$is chosen, we can choose n_0 (∂) so that the sum of the first n_0 (∂) terms, or of any greater number of terms, lies between
s -$\partial$and s + ∂; or in symbols

$$s - \partial < s_n < s + \partial,$$

if n ≧n_0(∂). In these circumstances we shall call the series

$$u_1 + u_2 + \ldots$$

A convergent infinite series and we shall call s the sum of the series, or the sum of all the terms of the series.
Khanna and Varshey (1995) cotes

2.7 Limit of a Sequence:
Definition. Assume $<s_n: n \in N>$is a sequence of real numbers. Then s_n approaches the limit '$\int$' as n approaches infinity, if for each $\in > 0$ there exists a positive integer m such that

$$n \geq m \Rightarrow | s_n - \int < \in$$

We observe that $| s_n - \int | < \in$ means

$$\int - \in < s_n < \int + \in$$

or equivalently $s_n$belongs to the open interval $]\int - \in, \int + \in[$ containing '$\int$'.
If:$_n$ approaches the limit '$\int$', we write

$$\lim_{n \to \infty} s_n = \int$$

$$\text{or } n \to \infty \Rightarrow s_n \to \int$$

2.11. Divergent Sequences:
Definition. Assume$<s_n>$ is a sequence of real numbers.
Then $<s_n>$ is said to diverge to ∞or is said to be divergent to ∞if for a real number r > 0 there exists a positive integer m(ϵ) such that

$$n \geq m \Rightarrow s_n > r$$

In this case, we write$n \to \infty \Rightarrow s_n \to \infty$
(i1) A sequence $<s_n>$ is said to diverge to - ∞ if for a real number r < 0, there exists a positive integer m(ϵ) > 0 such that

$$n \geq m \Rightarrow s_n > r$$

We then write $n \to \infty \Rightarrow s_n \to -\infty$.

Theorem11. (Monotone Convergence Theorem).
A monotone sequence which is bounded is convergent.
Equivalently, a necessary and sufficient condition for convergence of a monotone sequence is that it is bounded (Bihar 1980)

2.21 Cauchy (Fundamental Sequence).
Definition. Assume $<s_n: n \in N <$ is a sequence of real numbers. Then $<s_n>$ is called a Cauchy sequence if for any $\epsilon > 0$ there exists a positive integer p such that

$$m, n \geq p \Rightarrow | s_m - s_n | < \epsilon.$$

Roughly, $<s_n>$ is Cauchy if s_m and s_n are close together when m and n are large.

DISCUSSION

A. $\left\{\frac{n}{n+k}\right\}$ is monotonic increasing.

$$Let S_n = \frac{n}{n+k}$$

$$S_{n+1} = \frac{n+1}{n+1+k}$$

$$S_{n+1} - S_n = \frac{n+1}{n+1+k} - \frac{n}{n+k}$$

$$= \frac{k}{(n+1+k)(n+k)} \geq 0 \ \forall n, k \in N$$

$Hence S_{n+1} \geq S_n, \forall n \in N$

$\left\{\frac{n}{n+k}\right\}$is monotonic increasing.

B. $\left\{\frac{n}{n+k}\right\}$ is bounded.

$$\left|\frac{n}{n+k}\right| \leq 1 \ \forall n \in N$$

$|S_n| \leq 1 \ \forall n \in N$

$\left\{\frac{n}{n+k}\right\}$is bounded.

C. $\left\{\frac{n}{n+k}\right\}$ is not convergent.

$Let S_n = \frac{n}{n+k}$& lbe the limit of sequence$\left\{\frac{n}{n+k}\right\}$

∴By definition for each $\varepsilon > 0, \exists$a positive integer Msuch that

$|S_n - l| < \varepsilon, \forall n > M$

⇨ $\left|\frac{n}{n+k} - l\right| < \varepsilon, \forall n > M$

⇨ $-\varepsilon < \frac{n}{n+k} - l < \varepsilon$

i.e.$l - \varepsilon < \frac{n}{n+k} < l + \varepsilon$

$l - \varepsilon < -\frac{k}{n+k} + 1 < l + \varepsilon$

$(l - 1 - \varepsilon) < -\frac{k}{n+k} < (l - 1 + \varepsilon)$

$$\frac{1}{l-1-\varepsilon} > -\left(\frac{n+k}{k}\right) > \left(\frac{1}{l-1+\varepsilon}\right)$$

$$-\frac{1}{l-1-\varepsilon} < \left(\frac{n+k}{k}\right) < -\left(\frac{1}{l-1+\varepsilon}\right)$$

$$-\frac{1}{l-1-\varepsilon} < \frac{n}{k} + 1 < -\frac{1}{l-1+\varepsilon}$$

$-1 - \frac{1}{l-1-\varepsilon} < \frac{n}{k} < -\frac{1}{l-1+\varepsilon} - 1$

$- \left(1 + \frac{1}{l-1-\varepsilon}\right) < \frac{n}{k} < -\left(1 + \frac{1}{l-1+\varepsilon}\right)$

$- \quad \frac{k}{1}\left(1+\frac{1}{l-1-\varepsilon}\right) < n < -\frac{k}{1}\left(1+\frac{1}{l-1+\varepsilon}\right), \forall n, k \in N \& n, k > 0$
$Hence M$
$\neq +ve integer, for each + ve epsilon (\varepsilon)$

⇨ $\left\{\frac{n}{n+k}\right\}$ is not convergent.

D. $\left\{\frac{n}{n+k}\right\}$ is not Cauchy sequence.

Let $S_n = \frac{n}{n+k} \& S_m = \frac{m}{m+k}$

$$S_m - S_n = \frac{m}{m+k} - \frac{n}{n+k}$$

$= \frac{-(m-n)k}{(m+k)(n+k)}$

$|S_m - S_n| = \frac{(m-n)k}{(m+k)(n+k)}$
Let $\left\{\frac{n}{n+k}\right\}$ is a Cauchy sequence.
∴ By definition for each $\varepsilon > 0, \exists$ a positive integer M such that

⇨ $|S_m - S_n| < \varepsilon, \forall m, n > M$

Hence $\left|\frac{(m-n)k}{(m+k)(n+k)}\right| < \varepsilon, \forall m, n > M$

Let m = M+2, n = M+1

⇨ $\left|\frac{k}{(M+k+2)(M+k+1)}\right| < \varepsilon$

⇨ $\frac{k}{[M+(k+2)][M+(k+1)]} < \varepsilon$

$$[M + (k + 2)][M + (k + 1)] > \frac{k}{\varepsilon}$$

i.e. $M^2 + (2k + 3)M + (k + 2)(k + 1) - \frac{k}{\varepsilon} > 0$

i.e. $M^2 + (2k + 3)M + \left(k^2 + 3k + 2 - \frac{k}{\varepsilon}\right) > 0$

i.e. $M^2 + (2k + 3)M + k^2 + \left(3 - \frac{1}{\varepsilon}\right) k + 2 > 0$

⇨ $$M = \frac{-(2k+3) \pm \sqrt{-(2k+3)^2 - 4\left[k^2 + \left(3 - \frac{1}{\varepsilon}\right)k + 2\right]}}{2}$$

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⇨ $M = \frac{-(2k+3) \pm \sqrt{\frac{4}{\varepsilon}k+1}}{2}$

$M = \frac{-(2k+3) - \sqrt{\left(\frac{4}{\varepsilon}\right)k+1}}{2}$ is not a +ve integer for every $\varepsilon > 0$

OR

$$M = \frac{-(2k+3) + \sqrt{\left(\frac{4}{\varepsilon}\right)k+1}}{2}$$

$$M = -k - \frac{3}{2} + \sqrt{\frac{k}{\varepsilon} + \frac{1}{4}}$$

⇨ $M = not + veinteger for \varepsilon = \frac{1}{k}, \forall k \in N$

Hence $\left\{\frac{n}{n+k}\right\}$ is not Cauchy sequence.

E. $\left\{\frac{A_n}{A_n+k}\right\}$ *has similar behaviour as that of* $\left\{\frac{n}{n+k}\right\}$,

$\forall A_n \geq n.$

$i.e., \left\{\frac{A_n}{A_n+k}\right\}$ is monotonic, bounded, not convergent &
not Cauchy sequence.

Hence, $\left\{\frac{n^m}{n^m+k} \;/\; m, k \in N\right\}$ is again a monotonic, bounded and not convergent and not Cauchy sequence ∀ m≥ 1& $m, k \in N$.

(F) { $\frac{n+k}{n}$ } Is Convergent.
Hence Sequence $\left\{\frac{n^m+k}{n^m} \;/\; m, k \in N\right\}$ is convergent

But $\{1/(\frac{n^m+k}{n^m})\} = \left\{\frac{n^m}{n^m+k}\right\}$ is divergent.
Hence Reciprocal of Convergent sequence is divergent though it is finite, monotonic and bounded.

Conclusion
From the discussion (A), (B), (C), (D), (E), (F) and method of Counter example in mathematical analysis I come to the following conclusions-
First I proved that any sequence monotonic and bounded need not be a Convergent sequence.
Second I proved that any sequence S is convergent then its reciprocal (1/S) need not be a convergent sequence.
Third I proved that any sequence T is divergent every terms are finite monotonic and bounded then its reciprocal (1/T) be a convergent sequence.

Fourth I proved that $\left\{\frac{n}{n+k}\right\}$ is not a Canchy Sequence, $\left\{\frac{n}{n+k}\right\}$ is not a convergentSequence**.**
Fifth I proved Sequence $\left\{\frac{n^m}{n^m+k} \ / \ m, k \in N\right\}$ is not Cauchy Sequence.
Sixth I proved Sequence $\left\{\frac{n^m}{n^m+k} \ / \ m, k \in N\right\}$ is not Convergent Sequence.

Lastly I proved that reciprocal of every Convergent sequence need not be a Cauchy sequence though it is a finite monotonic and bounded.

REFERENCES

Simmons GF (2008). Introduction to topology and modern analysis (Tata McGraw – Hill Edition Eighth) Reprint 2008, ISBN – 13: 978-0-07-059784-6, ISBN – 10: 0-07-07-059784-7 70-71.
Hardy GH (No date). A Course of Pure Mathematics, Cambridge University Press, ISBN -13 – 978-0-521-17014-7, pp.146.
Kreyszig (No date). Introductory Functional Analysis with Applications, Wiley Student Edition, ISBN: 978–81–265-1191-4, 25-67.
Rudin Walter (No date). Principles of Mathematical Analysis (Third Edition), McGraw-Hill International Editions, ISBN 0-07- 054234 – X, (47-55).
Khanna ML & Varshey LS (1995) Real Analysis, Jai Prakash Nath & Co. Meerut 250002 India, 23-60.
Karade TM & Bendre Maya S (No date). Differential Equation and Analysis, SONU NILU, ISBN 978-81-88763-34-4 91-124.