International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jpms.htm
2014 Vol. 4 (1) January-March, pp. 94-101/Kanyeki et al.

Research Article

MEAN-FIELD INSTABILITY OF TRAPPED DILUTE BOSON-FERMION MIXTURE

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ABSTRACT
The influence of boson-boson and boson-fermion interactions on the stability of a binary mixture of bosonic and fermionic atoms is investigated. The effect of different combinations of signs of the boson-boson and boson-fermion scattering lengths is studied on the stability of the mixture of boson-fermion atoms.

Keywords: *Scattering Length, Trapped System, Oscillator Length, Quantum Degeneracy*

INTRODUCTION

A trapped system will mean that the system is in a small container with a finite potential. The nature of the potential may be positive or negative. A positive potential means repulsive interaction, while a negative potential means attractive interaction between the particles. The number of particles will be finite and not very large; hence the system may be dilute. The numbers of bosons, N_B, and fermions, N_F, could be equal or unequal. The type of interactions in the trapped system could be boson-boson, fermion-fermion and boson-fermion. The boson-boson and boson-fermion interaction may be the kind of interaction that may contribute to the implications for the stability of the mixture. Due to Pauli's exclusion principle, the fermion-fermion interaction is negligible (Ospelkaus *et al.*, 2006)

Recent experimental successes in the trapping and cooling of mixtures of bosonic and fermionic atoms constitute a new branch in the field of trapped ultra-cold gases. Similar to the purely bosonic gases boson-fermion mixtures offer unique possibilities to study fundamental quantum phenomena. Moreover they appear as a promising candidate to realize a BCS transition to a super fluid phase of the fermionic component (Ospelkaus *et al.*, 2006). One of the most appealing features of these systems is that the strength of the interaction between the atoms can be tuned in a wide range by utilizing a Feshbach resonance (Papp *et al.*, 2006). For the sympathetic cooling of a Fermi gas in binary boson-fermion mixtures the collapse caused by attractive interactions is responsible for a severe limitation of the lowest achievable temperature (Greiner *et al.*, 2005)

When the force between the particles is enough, the mixture may be stable to withstand the trap length. If the trap length falls below a certain value and the force is not large, the mixture collapses or, it becomes unstable. The occurrence of the mean-field instability when the particle number exceeds a critical value for purely bosonic systems only has been studied.

The interplay between boson-boson and boson-fermion interactions and the implications for the instability of the trapped dilute boson-fermion mixture has been investigated. The effects of different combinations of signs of the boson-boson and boson-fermion scattering lengths using the expressions for critical particle numbers as a function of the scattering lengths has been studied.

Stability, critical number and scattering length

Instability of the boson-fermion mixture against collapse induced by attractive boson-boson or fermion-boson interactions is investigated. In order to keep the discussion simple, we restrict ourselves to spherically symmetric systems with equal numbers of bosons and fermions i.e., $N_B = N_F$. We assume equal masses for the two species $m = m_B = m_F = m_{BF}$ and identical parabolic trapping potentials $U_B(x) = U_F(x)$ (Zhang *et al.*, 2004)

Repulsive boson-boson and attractive boson-fermion interactions

First we consider the case of repulsive boson-boson and attractive boson-fermion interactions.

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Here, the attractive interaction between the species induces a mean-field collapse if the densities or particle numbers exceed a critical value. If the strength of the boson-fermion attractive interaction exceeds a critical value, then the mixture collapses towards high densities.In this case, the attractive mean-field is not stabilized by the positive kinetic energy contribution or the repulsive boson-boson interaction any more, i.e. the gas can lower its energy by contracting and increasing the density in the central region. The critical boson-fermion scattering length, a_{BF}, can be determined for which the collapse occurs with a numerical procedure.The quantum diffusion algorithm used to obtain the solution of the Gross-Pitaevskii equation diverges if the mean-field instability occurs, i.e. the change of the boson density in the trap centre increases for successive imaginary time steps.Thus by observing the convergence behaviour of the central density during the imaginary time evolution, one can decide whether the mixture is stable or collapses.

To obtain a simple measure for the stability, we proceeded in two steps.

First, the critical boson-fermion scattering length is determined numerically for a set of particle numbers, $N_F = N_B$, and boson-boson scattering lengths a_B / ℓ where a_B is the scattering length due to boson-boson interaction and ℓ is the trap oscillator length. Secondly, we fit parameterization which connects the particle number with the two scattering lengths to this data set. This leads to an expression for the critical particle number N_{cr} as a function of the scattering lengths $a_B / \ell \geq 0$ and $a_{BF} / \ell < 0$, such that

$$N_{cr}(a_B, a_{BF}, \ell) = \frac{0.283}{|a_{BF} / \ell|^{1.78}} + \frac{0.374(a_B / \ell)^{2.36}}{|a_{BF} / \ell|^{5.69}} \quad (1)$$

Any mixture with $N_B = N_F > N_{cr}$ is unstable against mean-field induced collapse.

Attractive boson-boson and repulsive boson-fermion interactions

Secondly, we consider mixtures with attractive boson-boson and repulsive boson-fermion interactions$(a_B < 0, a_{BF} \geq 0)$.Infact, for very weak boson-fermion repulsion, the two species separate spatially. The bosons occupy the central region of the trap (boson core) and the fermions constitute a shell around it.. This structure may have interesting implications for the mean-field instability of the bosons. The fermionic shell compresses the boson core. This could promote the mean-field collapse in the presence of attractive boson-boson interactions and lower the critical particle number .The dependency of N_{cr} on the scattering lengths, which is obtained from the direct numerical solution of the coupled problem, is for $a_B < 0$ and $a_{BF} \geq 0$, thus,

$$N_{cr}(a_B, a_{BF}, \ell) = \frac{0.575 - 0.230(a_{BF} / \ell)^{0.333}}{|a_B / \ell|} \quad (2)$$

The influence of the repulsive boson-fermion interaction on the critical particle number is marginal. The critical particle number reduces slightly if a_{BF} / ℓ is increased. This can be attributed to the compression of the boson core mentioned before. Although the boson-fermion repulsive interaction has a strong influence on the density profiles, its influence on the critical particle number is negligible.

RESULTS AND DISCUSSION

Repulsive boson-boson and attractive boson-fermion interactions

^{23}Na -^{6}Li Mixture

Here ^{23}Na is a boson and ^{6}Li is a fermion. Assuming $\ell = 1\mu m$, a_B= 2.75nm and a_{BF} = -14.5nm, the value of N_{cr} can be calculated using eq. (1). Different N_{cr} values can be obtained by taking values of $\ell = 0.25\mu m$ to $\ell = 1\mu m$.

It should be clearly understood that for given particle numbers N_B (number of bosons), N_F (number of fermions), and given boson-boson scattering length a_B, there exists a maximum strength of the attractive boson-fermion interaction up to which the mixture is stable. Physically, this stability limit is governed by the balance between the kinetic energy of bosons and fermions and the mutually attractive mean-field

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generated by the boson-fermion interaction. If boson-fermion attraction becomes too strong and likewise, if the number of boson or fermion numbers becomes too large, the attractive mean-field cannot be stabilized by the kinetic energy anymore and the mixture can lower its energy by increasing the boson and fermion density. Both density distributions collapse simultaneously within the overlap region.
To determine the stability limit in the presence of attractive boson-fermion interactions, it is most useful to vary a_{BF}/ℓ for fixed values of N_B, NF and a_B/ℓ until the onset of instability is reached.

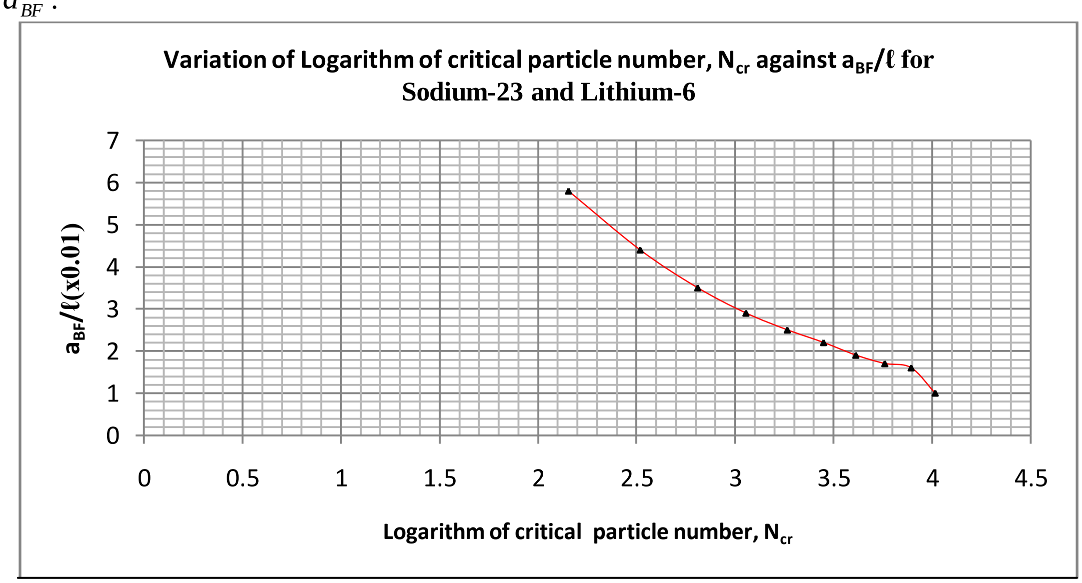
Calculations of critical particle number for ^{23}Na -^{6}Li mixture

Assuming a_B = 2.75nm and a_{BF} = -14.5nm, the values of N_{cr} are calculated for values of l = 0.25μm to l = 1μm using eq.(1).
The table 1 summarises the calculations for ^{23}Na -^{6}Li mixture.

Table 1: Trap length, critical particle number, scattering length and logarithm of the critical number for ^{23}Na -^{6}Li mixture

Trap length, l ($\times 10^{-6}$m)	Critical particle number, Ncr ($\times 10^{2}$)	a_{BF}/l ($\times 10^{-2}$)	Log N_{cr}
0.25	1.42	5.8	2.152
0.33	3.28	4.4	2.516
0.42	6.43	3.5	2.808
0.50	11.3	2.9	3.053
0.58	18.33	2.5	3.263
0.67	27.99	2.2	3.447
0.75	40.8	1.9	3.611
0.83	57.27	1.7	3.758
0.92	77.93	1.6	3.892
1.00	103.36	1.0	4.014

The graph 1.below depicts the critical particle number N_{cr} as a function of the boson-fermion scattering length a_{BF} .



Graph 1: Variation of a_{BF}/ℓ and log N_{cr} for ^{23}Na- ^{6}Li mixture

It is observed that a moderate boson-fermion attraction causes a severe limitation of the particle number of the stable mixture and the inclusion of a repulsive boson-boson interaction leads to significant stabilization, i.e. an increase of the critical particle number. The particle number increases as a_{BF} increases.

^{87}Rb -^{40}K Mixture

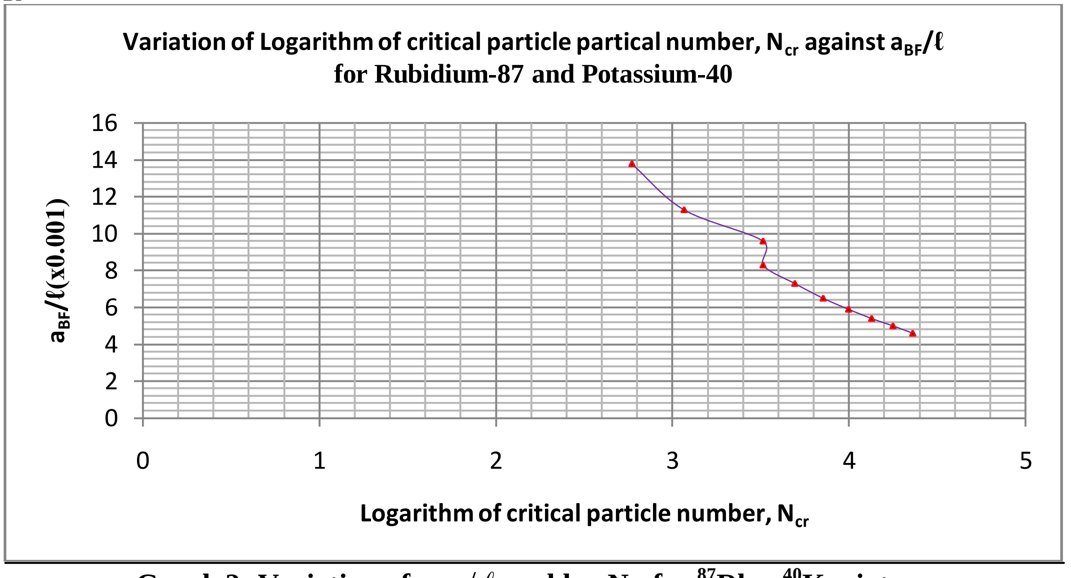
It is a mixture of bosonic ^{87}Rb and fermionic ^{40}K.This system is particularly interesting since here we have large negative boson-fermion scattering length and large a_B.For this,a_B = 5.25nm and a_{BF} = -13.82 nm and the expression for the critical particle number N_{cr} as a function of scattering length is given by eq. (1).For $\ell = 1.00\mu m$ the value of N_{cr} can be calculated. The inclusion of the boson-boson repulsion interaction leads to a significant stabilization, i.e., increase of the critical particle number. Calculations for N_{cr} can also be done by taking different value for $\ell = 1.00\mu m$ to $\ell = 3.00\mu m$.

The table 2 summarises the calculations for ^{87}Rb- ^{40}K mixture.

Table 2: Trap length, critical particle number, scattering length and logarithm of the critical number for ^{87}Rb- ^{40}K mixture

Trap length, l ($\times10^{-6}$m)	Critical particle number, Ncr ($\times10^{3}$)	a_{BF}/l ($\times10^{-3}$)	Log N_{cr}
1.00	0.589	13.8	2.77
1.22	1.165	11.3	3.066
1.44	2.028	9.6	3.514
1.67	3.263	8.3	3.514
1.89	4.947	7.3	3.694
2.11	7.160	6.5	3.855
2.33	9.987	5.9	3.999
2.56	13.517	5.4	4.13
2.78	17.838	5.0	4.251
3.00	23.043	4.6	4.363

The graph below depicts the critical particle number N_{cr} as a function of the boson-fermion scattering length a_{BF}.



Graph 2: Variation of a_{BF}/ℓ and log N_{cr} for ^{87}Rb – ^{40}K mixture

The particular combination of scattering lengths of the ^{87}Rb - ^{40}K mixture opens interesting perspectives for the implementation of a sympathetic cooling scheme. The strong boson-fermion attraction generates an enhancement of the boson and fermion density in the overlap region which is-due to the boson-boson repulsion-expanded over a large volume. Therefore a large fraction of the fermionic cloud overlaps with the bosonic distribution and enables a very efficient interspecies thermalization. At the same time the boson-boson repulsion stabilizes the system against simultaneous collapse for all experimentally relevant particle numbers.

Density profiles

Some of the basic properties of mixtures with attractive boson-fermion interactions become evident from the shape of the density profiles. For the non-interacting case, $a_{BF} / \ell = 0$, reveals a fundamental property of these mixtures: The fermionic density distribution has a much larger spatial extension than a bosonic distribution with the same particle number. This is a direct consequence of the Pauli principle that forces the fermions to occupy "excited" single-particle states of the external potential which have a larger radial extension whereas the bosons in a Bose-Einstein condensate all occupy the ground state of the trap. This manifestation of the so called Fermi pressure was observed experimentally..

The inclusion of an attractive boson-fermion interaction leads to an enhancement of the boson and the fermion density within the central overlap region. The bosonic profile narrows and the central density increases moderately. The effect on the fermionic component is more pronounced within the overlap region where the fermion density exhibits a high-density bump on top of the low-density profile. This structure can be understood by considering the mean-field potential experienced by the fermions: In addition to a shallow trapping potential the boson-fermion attraction generates a tight potential well with the shape of the bosonic density distribution which causes a localized enhancement of the fermion density profile. The boson-boson interaction has a strong influence on this density enhancement. The presence of boson-boson repulsion broadens the bosonic distribution and reduces the maximum density significantly. This in turn strongly reduces the enhancement of the fermion density in the overlap region. Boson-fermion attractions are required to generate a similar increase of the fermion density in the presence of repulsive boson-boson interactions. However, the overlap volume is much larger in these cases; thus a larger fraction of the fermions is contained in the high-density region.

Simultaneous mean-field collapse

For given particle numbers N_B and N_F and given boson-boson scattering length a_B, there exist a maximum strength of the attractive boson-fermion interaction up to which the mixture is stable. Physically, this stability limit is governed by the balance between the kinetic energy of bosons and fermions and the mutually attractive mean-field generated by the boson-fermion interaction. If the boson-fermion attraction becomes too strong, likewise if the boson or fermion numbers become too large the attractive mean-field cannot be stabilized by the kinetic energy anymore and the mixture can lower its energy by increasing the boson and fermion density; both density distributions collapse simultaneously within the overlap region.

In the numerical treatment of the coupled Gross-Pitaevskii problem, the instability is indicated by a divergence of the central boson and fermion density during the imaginary time propagation. By monitoring the central density, one can determine whether the system is stable or collapses, i.e., whether the density converges or diverges for a given set of parameters N_B, N_F, a_B / ℓ, and a_{BF} / ℓ.The boson-boson repulsive interaction has a very strong influence on the structure and stability of the mixture. If weak boson-boson repulsion with a fraction of the strength of the boson-fermion attraction is included, the system is stabilized significantly, i.e., the critical number of bosons N_B^{cr} is increased.

For reasonably strong boson-fermion attraction the mutual mean-field couples the boson and fermion density strongly to each other such that both densities collapse simultaneously in the overlap region. Ifthe boson-fermion attractiveinteraction is reduced, this coupling weakens until bosons and fermions decouple for $a_{BF} / \ell = 0$. In this case, the boson density may still collapse due to the boson-fermion attraction. The fermion density, however, is not affected and remains stable. The number of fermions in the system has a rather weak influence on the stability.

Attractive boson-boson and repulsive boson-fermion interactions

^{7}Li -^{6}Li Mixture

Here ^{7}Li is a boson and ^{6}Li is a fermion. Quantum degeneracy in a magnetically trapped dilute boson-fermion mixture was obtained by using a mixture of ^{7}Li atoms and fermionic ^{6}Li. For this particular combination, the boson-boson interaction is attractive and the boson-fermion interaction repulsive with scattering lengths,

$a_B = a_{BB} = -1.46$nm; $a_{BF} = 2.16$nm.

The attractive boson-boson interaction can cause the collapse of the bosonic component and lead to a severe limitation of the number of bosons. This is a major hindrance for the implementation of the scheme for the sympathetic cooling of the fermionic component. Ideally, one would like to have a large bosonic cloud at a very low temperature to act as a coolant for the fermions. The attractive boson-boson interaction, however, restricts the number of particles in the Bose-Einstein condensate and hence sets a lower bound to the temperature of the bosonic cloud. Consequently, the lowest achievable temperature for the fermionic component is also restricted.

The boson number above the stability limit of the bosonic component is unstable against mean-field collapse. Narrowing of the trapping potential, which means reducing the oscillator length ℓ_B, enhances the effect of the interactions. It should be emphasized that the stability of the mixture depends mainly on the boson number, N_B, and the critical boson number is to a good approximation given by the corresponding value of the pure Bose gas. The fermionic component may have no influence on the stability since the boson-fermion interaction which is repulsive may be weak.

The dependency of the critical particle number $N_{cr}(a_B, a_{BF}, \ell)$ for stability of the mixture is given by eq.(2) . Here the bosons will be assumed to occupy the central region of the trap (boson core) and the fermions constitute a shell around it. The fermionic shell compresses the boson core, i.e., increase the maximum boson density and this could promote the mean-field collapse in the presence of attractive boson-boson interaction and lower the critical particle number. Eq. (2), can now be used to calculate N_{cr} for different values of ℓ , and the values of a_B and a_{BF} given above.

Calculations of critical particle number for ^{7}Li -^{6}Li mixture

The critical number N_{cr} is calculated for $\ell = 1.2\ \mu m$ to $\ell = 3.5\ \mu m$ with a_B = -1.46 nm, a_{BF} = 2.16 nm using eq.(2)

The table 3 summarises the calculations for ^{7}Li -^{6}Li mixture.

Table 3: Trap length, critical particle number, scattering length and logarithm of the critical number for ^{7}Li- ^{6}Li mixture

Trap length, *l* ($\times 10^{-6}$m)	Critical particle number, Ncr ($\times 10^3$)	a_{BF}/l ($\times 10^{-3}$)	Log N_{cr}
1.23	0.461	1.75	2.664
1.48	5.572	1.46	3.746
1.73	6.536	1.25	3.815
1.99	7.502	1.09	3.875
2.24	8.468	0.96	3.923
2.49	9.436	0.87	3.975
2.74	10.404	0.79	4.017
3.00	11.373	0.72	4.056
3.25	12.343	0.66	4.091
3.50	13.314	0.62	4.124

The critical particle number reduces if a_{BF}/ℓ is increased. This can be attributed to the compression of the boson core mentioned before. Although the boson-fermion repulsive interaction has a strong influence on the density profiles, its influence on the critical particle number is negligible.

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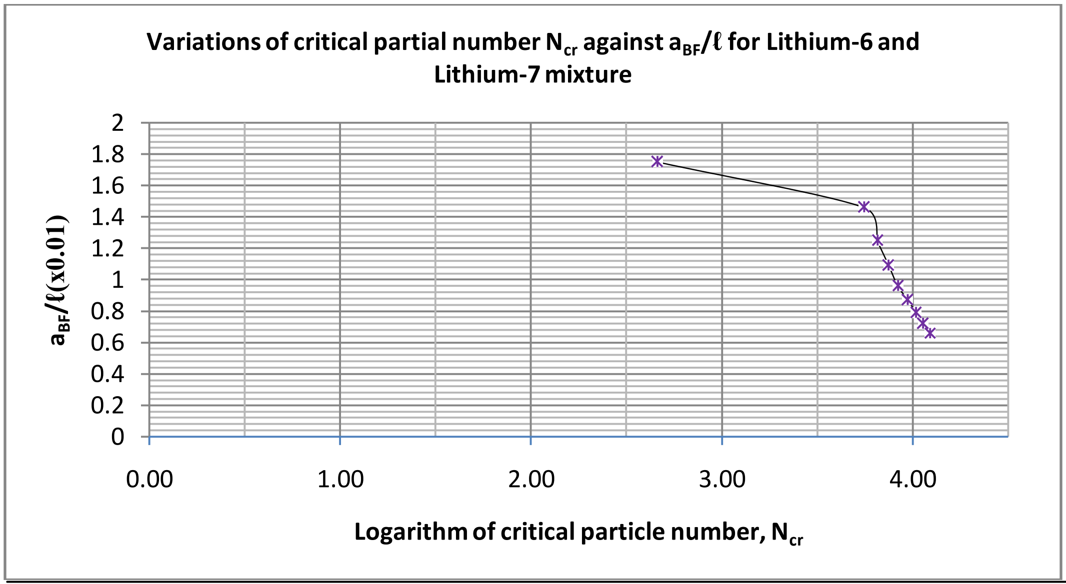

Graph 3: Variation of a_{BF}/ℓ and log N_{cr} for ^{7}Li – ^{6}Li mixture

Density profiles

Obviously a repulsive boson-fermion interaction will tend to reduce the overlap between the two species in contrary to the attractive boson-fermion interactions discussed in the previous section. Hence the structural transition characteristic for this class of interactions is the spatial separation of the two species. The compact boson distribution in the trap center repels the fermionic cloud from the overlap region. With increasing boson-fermion scattering length $a_{BF}/\ell > 0$, the fermionic distribution gets more and more depleted until N_F reaches zero. For even larger a_{BF}/ℓ the fermions are expelled farther from the trap center and the overlap of both species is reduced continuously. The boson cloud in return is slightly compressed by the outer shell of fermions. The boson-boson repulsive interaction has a very strong influence on the density profiles of bosons and fermions. Repulsion reduces the influence of the boson-fermion attractive interaction. Much larger boson-fermion scattering lengths are required to cause phase separation in the presence of repulsive boson-boson interactions. Because the bosonic distribution is broadened due to the boson-boson repulsion, the fermion density is depleted in a significantly larger volume. A peculiar structure appears for interactions with $a_B/\ell = a_{BF}/\ell > 0$. The fermion density is nearly constant within the whole overlap. Depending on the trap geometry the separated phase can exhibit different structures. If the fermionic species experiences a tighter confinement than the bosonic component (requires an appropriate combination of magnetic moments) an inversion of the separated configuration can appear: a central core of fermions surrounded by a thin boson shell which compresses the fermions. In these cases there exists a second structural transition from the fermion-core configuration at smaller N_B to the usual boson-core configuration at large N_B.

Collapse of the bosonic component

If the repulsive boson-fermion interaction is supplemented by a boson-boson attraction ($a_B<0$) then a competition between component separation and mean-field collapse occurs. For vanishing boson-fermion scattering length,a_{BF} = 0, the two species decouple. The bosonic component may undergo a mean-field collapse just like an isolated Bose-Einstein condensate if the boson-boson interaction is attractive. The critical particle number for this collapse of the bosonic component in a decoupled mixture can be parameterized. This coincides with the result for the collapse of a pure Bose gas which was basically confirmed by experiment. One should notice that the character of the collapse induced by a boson-boson attraction differs from that of the collapse caused by an attractive boson-fermion interaction discussed earlier. In the latter case the boson-fermion interaction generates a mutual attractive mean-field that acts

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on both, fermions and bosons, and causes a simultaneous collapse of both density distributions. In the case of a boson-boson attraction only the bosonic component experiences an attractive mean-field and may become unstable. For $a_{BF}>0$, the fermionic density profile will be influenced by a collapse of the bosonic component but it will not collapse itself. The collapse of the bosonic component may, for example, generate a collective excitation of the fermionic cloud.

Although the collapse for $a_B<0$ concerns the bosonic density alone, the boson-fermion repulsive interaction, $a_{BF} > 0$, modifies the stability properties significantly. We will concentrate on the case $a_{BF}>0$; the collapse in the presence of attractive boson-fermion interactions was discussed earlier.

Due to the boson-fermion repulsion the outer fermionic cloud tends to compress the compact bosonic core. The boson density and thus the attractive mean-field are enhanced which eventually promotes thecollapse of the bosonic component. However, the boson-fermion scattering length a_{BF}/ℓ has to be significantly larger than $|a_B/\ell|$ to have a noticeable effect. Evidently, component separation due to a boson-fermion repulsion has a significant effect on the stability of the mixture with respect to collapse induced by the boson-boson attraction. For weak attractive boson-boson interactions, separation happens at lower particle numbers than the collapse of the bosonic component.

Conclusions

In summary, we have investigated the mean-field instability of binary boson-fermion mixtures with equal particle numbers and the following are the conclusions.

1. Boson-boson and the boson-fermion interactions have very different effects on the collapse of the mixture.
2. In the presence of attractive boson-fermion interactions, ($a_{BF}< 0$) the system is stabilized by weak repulsive boson-boson interactions $(a_B > 0)$
3. Repulsive boson-fermion interactions ($a_{BF}> 0$) have significant effect on the stability of the mixture with respect to collapse induced by the boson-boson attraction $(a_B < 0)$.

REFERENCES

Donley EA, Claussen NR, Thompson ST and Carl EW (2002). Atom-molecule coherence in a Bose-Einstein condensate. *Nature* **417** 529-33. doi:10.1038/417529a.

Greiner M, Regal C A and Jin D S (2005). Probing the Excitation Spectrum of a FermiGas in the BCS-BEC Crossover Regime. *Physical Review Letters* **94** 070403. doi:10.1103/*PhysRevLett*.**94**.070403.

Ospelkaus C, Ospelkaus S, Sengstock K and Bongs K (2006). Interaction-Driven Dynamics of ^{40}K-^{87}Rb Fermion-Boson Gas Mixtures in the Large-Particle-Number Limit. *Physical Review Letters* **96** 020401:10.1103/*Phys Rev Lett*.96.020401.

Ospelkaus S, Ospelkaus C, Humbert L, Sengstock K and Bongs K (2006). Tuning of Heteronuclear Interactions in a Degenerate Fermi-Bose Mixture. *Physical Review Letter* **97** 120403.

Ospelkaus S, Ospelkaus C, Dinter R, Fuchs J, Nakat M, Sengstock K and Bongs K(2006). Degenerate K-Rb Fermi-Bose gas mixtures with large particle numbers. *Journal of Modern Optics Preview*. doi:10.1080/09500340600777763.

Papp SB and Wieman CE (2006). Observation of Heteronuclear Feshbach Moleculesfrom a^{85}Rb -^{87} Rb Gas. *Physical Review Lette*r **97** 180404.

Zhang J, van Kampen EGM, Bourdel T, Khykovich L, Cubizores j, Chevy F, Teichman M, Ta L Kokkelmans SJJMF and Solomon C (2004). P-wave Feshbach resonance of ultra cold ^{6}Li. *Physics Review A* **70**, 030702. 101103.