International Journal of Physics and Mathematical Sciences ISSN: 2277-2111 (Online)
An Open Access, Online International Journal Available at http://www.cibtech.org/jpms.htm
2014 Vol. 4 (1) January-March, pp.34-40/Dange
Research Article

DETERMINATION OF THERMAL STRESSES OF A THREE DIMENSIONAL TRANSIENT THERMOELASTIC PROBLEM OF A SQUARE PLATE

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ABSTRACT
The present paper deals with the determination of the temperature distribution, unknown temperature at any point of the square plate, thermoelastic displacement function , displacement components and thermal stresses of square plate occupying the space $D: 0 \le x \le a,\ 0 \le y \le a,\ 0 \le z \le h$, with known boundary conditions by applying finite Fourier sine transform , Fourier cosine transform and Laplace transform techniques. Numerical calculations are carried out for a particular case of square plate made of copper (pure) metal by assigning suitable values to the parameters and functions in the equations and results are depicted graphically.

***Keywords**: Unsteady State, Thermo Elastic Problem, Thermal Stresses, Fourier Sine Transform, Fourier Cosine Transform, Laplace Transform*

INTRODUCTION
Tanigawa and Komatsubara (1997), Vihak et al., (1998), and Adams and Bert (1999) have studied the direct problem of thermoelasticity in a rectangular plate under thermal shock. Khobragade and Wankhede (2003) have studied the inverse steady state thermoelastic problem to determine the temperature displacement function and thermal stresses at the boundary of a thin rectangular plate. They have used the finite Fourier sine transform technique. Dange and Khobragade (2009) have studied three dimensional inverse steady-state thermoelastic problem of a thin rectangular plate. Lamba and Khobragade (2012) have studied thermoelastic problem of a thin rectangular plate due to partially distributed heat supply. They have used the Marchi – Fasulo transform and Laplace transform technique. In the present paper an attempt is made to determine temperature distribution ,unknown temperature at any point of a square plate, thermoelastic displacement function, displacement components and thermal stresses of square plate occupying the space $D: 0 \le x \le a,\ 0 \le y \le a,\ 0 \le z \le h$, with known boundary conditions by applying finite Fourier sine transform and Fourier cosine transform and Laplace transform techniques. Numerical calculations are carried out for a particular case of square plate made of copper (pure) metal by assigning suitable values to the parameters and functions in the equations and results are depicted graphically.

Statement of the problem
Consider a square plate occupying the space $D: 0 \le x \le a, 0 \le y \le a, 0 \le z \le h$. The displacement components u_x, u_y and u_z in the X, Y, Z direction respectively are in the integral form as

$$u_x = \int_0^a \frac{1}{E}\left(\frac{\partial^2 \cup}{\partial y^2} + \frac{\partial^2 \cup}{\partial z^2} - \nu \frac{\partial^2 \cup}{\partial x^2} + \alpha T\right)\, dx \quad (1)$$

$$u_y = \int_0^a \frac{1}{E}\left(\frac{\partial^2 \cup}{\partial z^2} + \frac{\partial^2 \cup}{\partial x^2} - \nu \frac{\partial^2 \cup}{\partial y^2} + \alpha T\right)\, dy \quad (2)$$

$$u_z = \int_0^h \frac{1}{E}\left(\frac{\partial^2 \cup}{\partial x^2} + \frac{\partial^2 \cup}{\partial y^2} - \nu \frac{\partial^2 \cup}{\partial z^2} + \alpha T\right)\, dz \quad (3)$$

Where E, ν and α are the Young's modulus, Poison's ratio and the linear coefficient of thermal expansion of the material of the plate respectively, $\cup(x, y, z, t)$ is the Airy's stress function which satisfies the differential equation namely,

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$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)^2 \cup(x, y, z, t)=-\alpha E\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) T(x, y, z, t) \quad (4)$$

Where $T(x, y, z, t)$ denotes the temperature of square plate satisfying the following differential equation.

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{k}\frac{\partial T}{\partial t} \quad (5)$$

Where, k is thermal diffusivity of the material.
The initial condition is

$$T(x, y, z, t)\big|_{t=0}=0 \quad (6)$$

The boundary conditions are

$$\left[T(x, y, z, t)\right]_{x=0}=0 \quad (7)$$

$$\left[T(x, y, z, t)\right]_{x=a}=0 \quad (8)$$

$$\left[\frac{\partial T}{\partial y}\right]_{y=0}=0 \quad (9)$$

$$\left[\frac{\partial T}{\partial y}\right]_{y=a}=0 \quad (10)$$

$$\left[T(x, y, z, t)\right]_{z=0}=u(x, y, t) \quad (11)$$

$$\left[T(x, y, z, t)\right]_{z=h}=g(x, y, t) \quad (12)$$

The interior condition

$$\left[T(x, y, z, t)\right]_{z=\xi}=f(x, y, t) \quad (13)$$

The stresses components in terms are given by

$$\sigma_{xx}=\left(\frac{\partial^2 \cup}{\partial y^2}+\frac{\partial^2 \cup}{\partial z^2}\right) \quad (14)$$

$$\sigma_{yy}=\left(\frac{\partial^2 \cup}{\partial z^2}+\frac{\partial^2 \cup}{\partial x^2}\right) \quad (15)$$

$$\sigma_{zz}=\left(\frac{\partial^2 \cup}{\partial x^2}+\frac{\partial^2 \cup}{\partial y^2}\right) \quad (16)$$

The equations (1) to (16) constitute the mathematical formulation of the problem under consideration.

Solution of the problem

Applying Fourier sine transform over x to the equation (5) (7)and (8). Applying Fourier Cosine transform over y variables to the equations (9) (10) , taking Laplace transform and then their inverses one obtains the expression for temperature and unknown temperature gradient $f(x, y, t)$ as

$$T(x, y, z, t)=\frac{8k\pi}{h^2a^2}\sum_{m-1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\left[\frac{l}{\cos l\pi}\right]\left\{\left[\sin\frac{l\pi}{h}(z-h)\int_0^t \bar{\bar{u}}(m,n,t')e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](t-t')}dt'\right]\right.$$

$$\left.-\left[\sin\left(\frac{l\pi}{h}\right)z\int_0^t \bar{\bar{g}}(m,n,t')e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](t-t')}dt'\right]\right\}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{a} \quad (17)$$

$$f(x,\ y,\ t) = \frac{8k\pi}{h^2 a^2} \sum_{m-1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \left[\frac{l}{\cos l\pi}\right] \left\{ \left[\sin\frac{l\pi}{h}(\xi - h) \int_0^t \bar{\bar{u}}(m,n,t') e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](t-t')} dt' \right] \right.$$
$$\left. - \left[\sin\left(\frac{l\pi}{h}\right) \xi \int_0^t \bar{\bar{g}}(m,n,t') e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](t-t')} dt' \right] \right\} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{a} \quad (18)$$

Determination of thermoelastic displacement function:

Substituting the values of $T(x, y, z, t)$ from equation (17) in equation (4) one obtains

$$U(x,y,z,t) = \frac{8\alpha k\pi E}{h^2 a^2} \sum_{m-1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \left[\frac{l}{\cos l\pi \left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right]} \right] \left\{ \left[\sin\frac{l\pi}{h}(z-h) \int_0^t \bar{\bar{u}}(m,n,t') e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](t-t')} dt' \right] \right.$$
$$\left. - \left[\sin\frac{l\pi}{h} z \int_0^t \bar{\bar{g}}(m,n,t') e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](t-t')} dt' \right] \right\} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{a}$$

(19)

Determinations of displacement components

Substituting the values (19) in the equation (1) to (3) one obtains

$$u_x = \frac{8\alpha k\pi}{h^2 a^2} \sum_{m-1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \left[\frac{l(1+\nu)\frac{m^2\pi^2}{a^2}\left[(-1)^{m+1}+1\right]}{\cos l\pi \left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right]} \right] \left\{ \left[\sin\frac{l\pi}{h}(z-h) \int_0^t \bar{\bar{u}}(m,n,t') e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](t-t')} dt' \right] \right.$$
$$\left. - \left[\sin\left(\frac{l\pi}{h}\right) z \int_0^t \bar{\bar{g}}(m,n,t') e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](t-t')} dt' \right] \right\} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{a} \quad (20)$$

$$u_y = 0 \quad (21)$$

$$u_z = \frac{8\alpha k\pi}{h^2 a^2} \sum_{m-1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \left[\frac{l}{\cos l\pi}\right] \left[\frac{(1+\nu)\frac{l^2\pi^2}{h^2}}{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}} \right] \left\{ \left[(-1)^l - 1\right] \int_0^t \bar{\bar{u}}(m,n,t') e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](t-t')} dt \right.$$
$$\left. - \left[(-1)^{l+1} + 1\right] \int_0^t \bar{\bar{g}}(m,n,t') e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](t-t')} dt' \right\} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{a} \quad (22)$$

Determination of stress function:

Substituting values of (19) in equations (14) to (16) one obtains

$$\sigma_{xx} = \frac{-8\alpha k\pi E}{h^2 a^2} \sum_{m-1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \left[\frac{l}{\cos l\pi}\right] \left[\frac{\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}}{\cos l\pi \left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right]} \right] \left\{ \left[\sin\frac{l\pi}{h}(z-h) \int_0^t \bar{\bar{u}}(m,n,t') e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](t-t')} dt' \right] \right.$$

$$-\left[\sin\frac{l\pi}{h} z\int_0^{t} \bar{\bar{g}}(m,n,t')e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](t-t')}dt'\right]\Bigg\}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{a} \quad (23)$$

$$\sigma_{yy} = \frac{-8\alpha k\pi E}{h^2a^2}\sum_{m-1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\left[\frac{l}{\cos l\pi}\right]\left[\frac{\frac{m^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}}{\cos l\pi\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right]}\right]\Bigg\{\left[\sin\frac{l\pi}{h}(z-h)\int_0^{t}\bar{\bar{u}}(m,n,t')e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](t-t')}dt'\right]$$

$$-\left[\sin\frac{l\pi}{h} z\int_0^{t} \bar{\bar{g}}(m,n,t')e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](t-t')}dt'\right]\Bigg\}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{a} \quad (24)$$

$$\sigma_{zz} = \frac{-8\alpha k\pi E}{h^2a^2}\sum_{m-1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\left[\frac{l}{\cos l\pi}\right]\left[\frac{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}}{\cos l\pi\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right]}\right]\Bigg\{\left[\sin\frac{l\pi}{h}(z-h)\int_0^{t}\bar{\bar{u}}(m,n,t')e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](t-t')}dt'\right]$$

$$-\left[\sin\frac{l\pi}{h} z\int_0^{t} \bar{\bar{g}}(m,n,t')e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](t-t')}dt'\right]\Bigg\}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{a} \quad (25)$$

Special case and numerical results and discussion:

Setting $u(x, y,t) = (1-e^t)x(a-x)y^2(3a-2y)$

$g(x, y,t) = (1-e^t)x(a-x)y^2(3a-2y)e^h$

In the equation (17) and(18) ,one obtains

$$T(x, y, z, t) = \frac{8k\pi}{h^2a^2}\sum_{m-1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\left[\frac{l}{\cos l\pi}\right]\left[\frac{4(3a-2)a^6}{m^3n^2\pi 4}\right]\left[(-1)^n+(-1)^{m+n}\right]$$

$$\left[\frac{k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](1+e^t)-e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right]t}+1}{k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right]\left[k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right]+1\right]}\right]$$

$$\left[\sin\frac{l\pi}{h}(z-h)-e^h\sin\left(\frac{l\pi}{h}\right)z\right]\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{a} \quad (26)$$

$$f(x, y, t) = \frac{8k\pi}{h^2a^2}\sum_{m-1}^{\infty}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\left[\frac{l}{\cos l\pi}\right]\left[\frac{4(3a-2)a^6}{m^3n^2\pi 4}\right]\left[(-1)^n+(-1)^{m+n}\right]$$

$$\left[\frac{k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right](1+e^t)-e^{-k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right]t}+1}{k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right]\left[k\left[\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{a^2}+\frac{l^2\pi^2}{h^2}\right]+1\right]}\right]$$

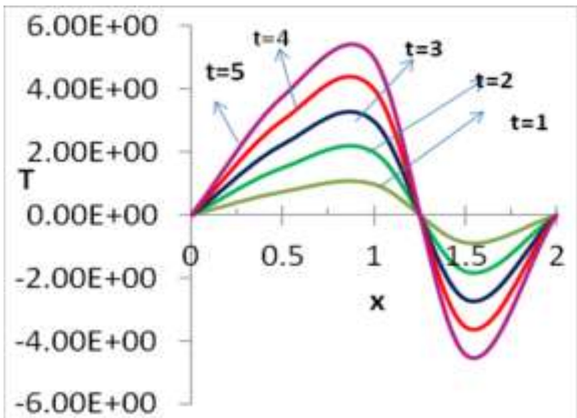
$$\left[\sin\frac{l\pi}{h}(\xi-h)-e^{h}\sin\left(\frac{l\pi}{h}\right)\xi\right]\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{a} \quad (27)$$

Dimensions of the square plate
Length, breadth of square plate a = 2m
Thickness of the square plate, h=0.5m
$\xi = 0.05m$

To interpret the numerical computations, we consider material properties of copper (pure) square plate with the material properties.

Poisson ratio, υ = 0.35
Thermal expansion coefficient, α(cm/cm-^{0}C) = 16.5×10^{-6}
Thermal diffusivity, κ (cm^2/sec) = 112.34×10^{-6}
Youngs Modulus. E=120GPa

DISCUSSION



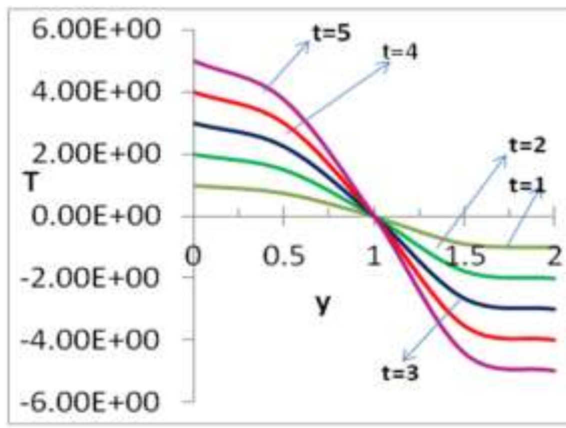
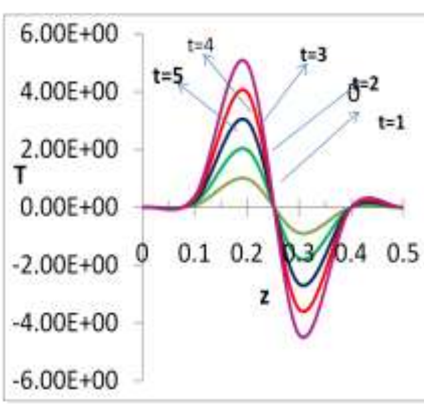


Figure 1: Graph of T versus x for different values t **Figure 2: Graph of T versus y for different values t**

Figure 1: Represents the graphs of T versus x for different values of t.It is obeserved that $T(x,y,z,t)$ develops tensile stress from $x=0$ to $x=1.25$ and compressive stresses in the region $x=1.25$ to $x=2$ in the square region for different values of t.

Figure 2: Represents the graphs of $T(x,y,z,t)$ versus y for different values of t.It is obeserved that $T(x,y,z,t)$ goes on decreasing from y=0 to y=2 and $T(x,y,z,t)$ is zero at point y=1 in the square region for different values of t.



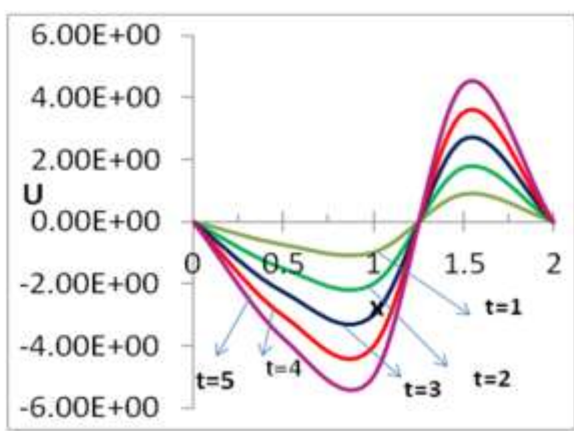
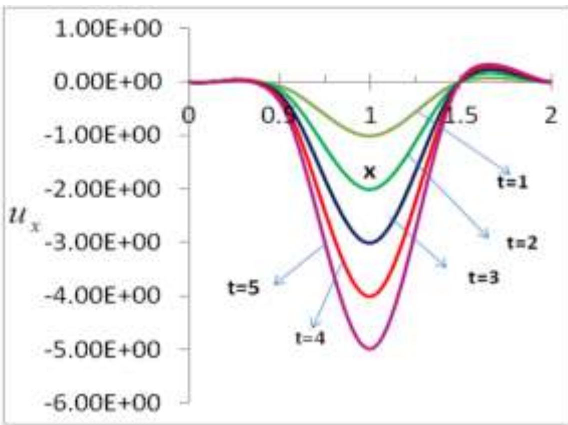


Figure 3: Graph of T versus z for different values t Figure 4 : Graph of U versus x for different values t

Figure 3: Represents the graphs of $T(x,y,z,t)$ versus z for different values of t.It is obeserved that the temperature $T(x,y,z,t)$ approximately zero from z=0 to z=0.1 . Also $T(x,y,z,t)$ develops tensile stress from $z=0.1$ to $z=0.25$ and compressive stresses from $z=0.25$ to $z=0.4$ and then T is zero from T=0.4 to 0.5 in the square region.

Figure 4: Represents the graphs of $U(x,y,z,t)$ versus x for different values of t.It is obeserved that $U(x,y,z,t)$ develops tensile stress from $x=1.25$ to $x=2$ and compressive stresses in the region $x=0$ to $x=1.25$ in the square region for different values of t.



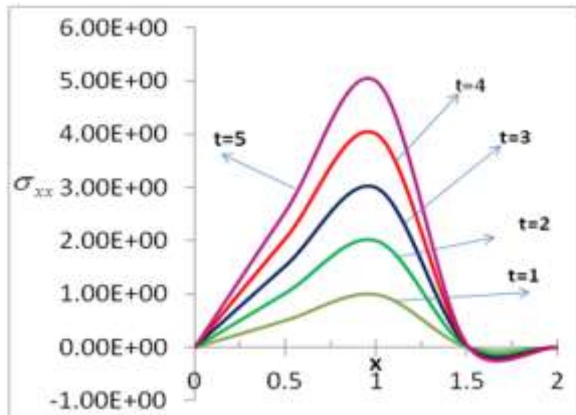


Figure 5: Graph of u_x versus x for different values t Figure 6: Graph of σ_{xx} versus x for different values t

Figure 5: Represents the graphs of u_x versus x for different values of t.It is obeserved that u_x is approximately zero for the region $x=0$ to the $x=0.5$. Then it goes on decreasing from $x=0.5$ to $x=1$ after this u_x is goes on increasing from x=1 to x=2 in the square region for different values of t.

Figure 6: Represents the graphs of σ_{xx} versus x for different values of t.It is obeserved that σ_{xx} is goes on increasing from x=0 to x=1 It is seen that σ_{xx} has maximum value f or x=1 and from x=1 to x=1.5 σ_{xx} is goes on decreasing. From to x=1.5 to x=2 it is approximately zero in the square region for different values of t.

Conclusion

In this study we treated the three dimensional unsteady state thermoelastic problem of square plate with stated boundary conditions. Under these condition the temperature distribution $T(x,y,z,t)$,the unknown temperature $f(x,y,t)$ at any point of the square plate ,The thermoelastic displacement $U(x,y,z,t)$, displacement components u_x , u_y ,u_z in X, Y, Z axes respectively and thermal stresses σ_{xx}, σ_{yy}, σ_{zz} have been determined with the help of finite Fourier Sine transform ,Fourier Cosine transform and Laplace transform techniques. Any particular case can be derived by assigning suitable values to the parameters and functions in the expressions .We may conclude that the system of equations proposed in this study can be adopted to design of useful structures or machines in engineering application in the determination of thermoelastic behavior and illustrated graphically.

ACKNOWLEDGEMENT

The author is thankful to prof. Anand Raipure for kind help and guidance in the preparation of this paper.

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