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Research Article

DISPERSIVE STANDING WAVES IN PERIDYNAMIC MODEL

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ABSTRACT
The paper presents a mathematical model. Interaction of elastic standing wave with Peridynamic nonlocal spring material model, the model was uniformly discretization with certain grid spacing. The waves are dispersive which is due to the size of the horizon and the size of the discretization. Special attention is given to the evaluation of the dispersion relation and group velocity to the standing waves within the system. Analytical estimates are accompanied by numerical simulations and analysis of dispersion surfaces.

Key Words: *Peridynamic Theory, Standing Wave, Dispersion Relation and Group Velocity*

INTRODUCTION
The theory of non-local model of force interaction suggested in Silling (2000) reformulated the classical continuum mechanics equations in Peridynamic equation of motion which is an integro-differential form and it is free of any spatial derivative (Emmrich and Weckner, 2006). Integration instead of differentiation used to compute the force on a material particle by summing up interactions with other near-by particles. The nonlocal interactions between the particles located at x and $x^/$ are called a bond. The concept of bond that extends over a finite distance is a fundamental difference between the Peridynamic model and classical models for materials which are based on the idea of contact forces. i.e., interactions between particles that is in direct contact with each other. In this paper Taylor’s theorem applied to the integral part of Peridynamic equation for one dimension infinite bar (Chen and Max, 2011). In this paper we will study the standing waves on one dimension Peridynamic particles, which oscillated under external force and discontinuous discretization, are conforming for the model. The dispersion relation provides a fundamental characterization of the nature of wave propagation in an elastic solid. Its derivation for various solid configurations, such as beams, plates, surfaces etc., which has been key to the development of the field of elastic wave propagation, The paper will discuss one dimension Peridynamic homogenous model for dispersive wave equations are discussed, the homogeneous influence functions may also be used for systems where the Peridynamic horizon has fixed value for different points in a body. The dispersion analysis of obtained model together with numerical solutions of typical initial and boundary value problems have been considered.

Peridynamic Model
Peridynamic model (Silling and Lehoucq, 2010) will be introduced the material particles interact with each other directly through finite distances, the acceleration of any particle at x in the reference configuration at time *t* is given by:

$$\rho \ddot{u}(x,t) = \int_{Hx} f(u(x^/,t) - u(x,t), x^/ - x) dv_{x^/} + b(x,t) \quad (1)$$

where H_x denotes a neighborhood of x. u The displacement vector field, b a prescribed body force density field, ρ the mass density in the reference configuration, and f a pairwise force function which the value is the force vector (per unit volume squared) that the particle located at x′ (in the reference configuration) exerts on the particle located at the point x (also in the reference configuration). The relative position ξ of these two particles in the reference configuration is given by:

$$\xi = x' - x \quad (2)$$

and their relative displacement ς by

$$\eta = u(x^{/},t) - u(x,t) \tag{3}$$

Note that $\varsigma+\xi$ is the relative position vector between the two particles in the deformed configuration.
One dimensional peridynamic model of an infinitely long bar with constant cross-section (A) parallel with x_1-axis, integrate Eq.(1) over the cross-section A located at any x_1 which x coincided x_1 and divided by A to obtain :-

$$\rho\ddot{u} = \int_{-\infty}^{\infty} f(x, x^{/}, t)dv^{/} + b(x,t) \tag{4}$$

Where:

$$dv^{/} = dA^{/}dx^{/} \tag{5}$$

$$u(x_1, t) = \frac{1}{A}\int_A u_1(x,t)\, dA \tag{6}$$

$$b(x_1, t) = \frac{1}{A}\int_A b_1(x,t) dA \tag{7}$$

$$f(x_1, x_1^{/}, t) = \frac{1}{A}\int_A \int_{A^{/}} f_1(x, x^{/}, t) dA dA^{/} \tag{8}$$

When the bond respond independently for each other

$$f(x, x^{/}, t) = \hat{f}(u(x^{/}, t) - u(x,t), x^{/} - x) \tag{9}$$

Where $\hat{f}$ related to the force between the two particles x′ and x and depend only on the deformation of that particular bond, from Newton third

$$\hat{f}(\eta, \xi) = -\hat{f}(\eta, \xi) \tag{10}$$

For linear peridynamic material (Etienne and Olaf, 2005):

$$\hat{f}(x, x^{/}, \eta) = f_0(x, x^{/}) + C(x, x^{/}) \cdot \eta \tag{11}$$

Where f_0 denoting forces in the reference configuration (Emmrich and Weckner, 2007). Without loss of generality, we may assume $f_0 \equiv 0$ then:

$$\hat{f}(\xi, \eta) = C(\xi) \cdot \eta \tag{12}$$

Where C is the material micromodulus function is time independent and an even function:

$$C(\xi) = C(-\xi) \qquad \text{for} \quad -\infty < \xi < \infty \tag{13}$$

The second order tensor of micromodulus:

$$C(\xi) = \frac{\partial f}{\partial \eta}(0, \xi) \qquad \forall \xi \tag{14}$$

In one dimension:

$$C(\xi) = c\frac{\xi \times \xi}{|\xi|^3} \tag{15}$$

Where c is constant depend on the material properties and the dimension of the model. From Eqs. (1), (12), (15) and H_x denote the spherical neighborhood of x in the body *R* with radius *δ*:

$$\rho\ddot{u}(x,t) = \int_{H_x} c\frac{(x' - x)\otimes(x' - x)}{|x' - x|^3}(u(x',t) - u(x,t))dv_{x'} + b(x,t) \tag{16}$$

In one dimension (Emmrich and Weckner, 2006), $c=18k/5\delta^2$, k is the bulk modulus with Poisson ratio $\nu=1/4$, for convenience, we set $\rho=1$ and $k=5/18$ then Eq. (16) will be:

$$\rho\ddot{u}(x,t) = \frac{1}{\delta^2}\int_{x-\delta}^{x+\delta}\frac{u(x,t) - u(x',t)}{|x - x'|}dx' + b(x,t) \tag{17}$$

Applying Taylor's theorem to:

$$\frac{1}{\delta^2}\int_{x-\delta}^{x+\delta}\frac{u(x,t) - u(x',t)}{|x - x'|}dx' = -\frac{1}{2}u_{xx}(x,t) - \frac{1}{48}u_{xxxx}(x,t)\delta^2 - \frac{1}{2160}u_{xxxxxx}(x,t)\delta^4 + ... \tag{18}$$

Then Eq. (17) with the first two terms of Taylor series:

$$\rho\ddot{u}(x,t) = -\frac{1}{2}u_{xx}(x,t) - \frac{1}{48}u_{xxxx}(x,t)\delta^2 + b(x,t) \tag{19}$$

The manufactured solution is a technique used for the verification of computational models in this paper the method of manufactured solutions will be used to define the problem of one dimension spring model, of which the exact solutions are known (Chen and Max, 2011), if each particle vibrate as standing wave:

$$u(x,t) = 2u_0\cos(k_\omega x)e^{j\omega t}$$

Where k_ω is the wave number, substitute in Eq. (17) and using Eq. (18) then to compute the body force density:

$$\rho(-2u_0\omega^2\cos(k_\omega x)e^{j\omega t}) = (\frac{k_\omega^2}{2} - \frac{k_\omega^2}{48}\delta^2)(2u_0\cos(k_\omega x)e^{j\omega t}) + b(x,t) \tag{20}$$

$$b(x,t) = [-2u_0(\frac{k_\omega^2}{2} - \frac{k_\omega^2}{48}\delta^2 + \rho\omega^2)]\cos(k_\omega x)e^{j\omega t} \tag{21}$$

Computational Results

The purpose of numerical computation is to analyze the propagation characteristics of the particles vibrated as standing wave in one dimension spring model. Let consider a set of discretization nodes equally spaced (Δx) as in Figure 1,

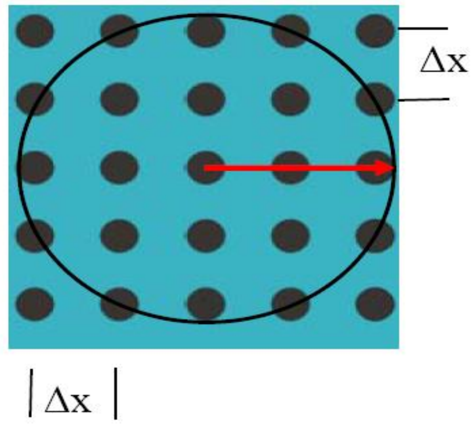

Figure 1: Computational spacing

Equation (21) for body force density will be:

$$\mathbf{b(x,t) = -2u_0(\frac{k_\omega^2}{2} - \frac{k_\omega^2}{48}\delta^2 + \rho\omega^2)]\cos(k_\omega r\Delta x)e^{j\omega t}}, \text{ r=1, 2, 3, 4,………} \qquad 22$$

The solution for body force , the horizons δ of Δx,2Δx,3Δx, 4Δx are selected when is the horizon smaller than 1Δx , in this case would have to no Peridynamic bonds within the horizon. Let the time (0..5 , step time 0.1), ρ=1, $\mathbf{k_\omega}$ = 3π metres^{-1}, u_0 =0.5 , ω = 3 radians/second and Δx =0.5, for homogenous Peridynamic model the horizon with fixed value. Equation (22) examined graphically using Maple software package in Figure 2 the solutions for different nodes of length (Δx,2Δx,3Δx, 4Δx) will represent a standing wave in which the amplitude depend on the wave number, horizon, angular frequency and the density. .

The dispersion relation for standing wave model will be derived, assuming b(x, t)=0 , for two terms of Taylor's expand in Eq. (21) we obtain:-

$$\boldsymbol{\omega^2 = -\frac{1}{\rho}(\frac{k_\omega^2}{2} - \frac{k_\omega^4}{48}\delta^2)} \qquad \text{……….} \qquad 23$$

$$\boldsymbol{\omega(k) = \sqrt{\frac{1}{\rho}(\frac{k_\omega^4\delta^2}{48} - \frac{k_\omega^2}{2})}} \qquad \text{………..} \qquad 24$$

In Figure 3, the angular frequency will depend not only on the wave number and density but also on the horizon (δ), in dispersion relation Eq. (24) have real part and imagery part depend on the values of ($\mathbf{\frac{\delta^2}{48}}$), the dispersion curve provide an exact fundamental description of how an elastic harmonic standing wave locally, and instantaneously, disperses in a Peridynamic model under certain conditions, when the wave number is taken to be zero, the frequency equal to zero that all points in Peridynamic model have same amplitude and move in phase. In imaginary part as the wave number increase the frequency will be increase too .this means all points will vibrate in same amplitude but out of phase until the peak then the frequency deceases to cutoff frequency then the real part of dispersion relation begin and as the wave number increase the frequency also increases.

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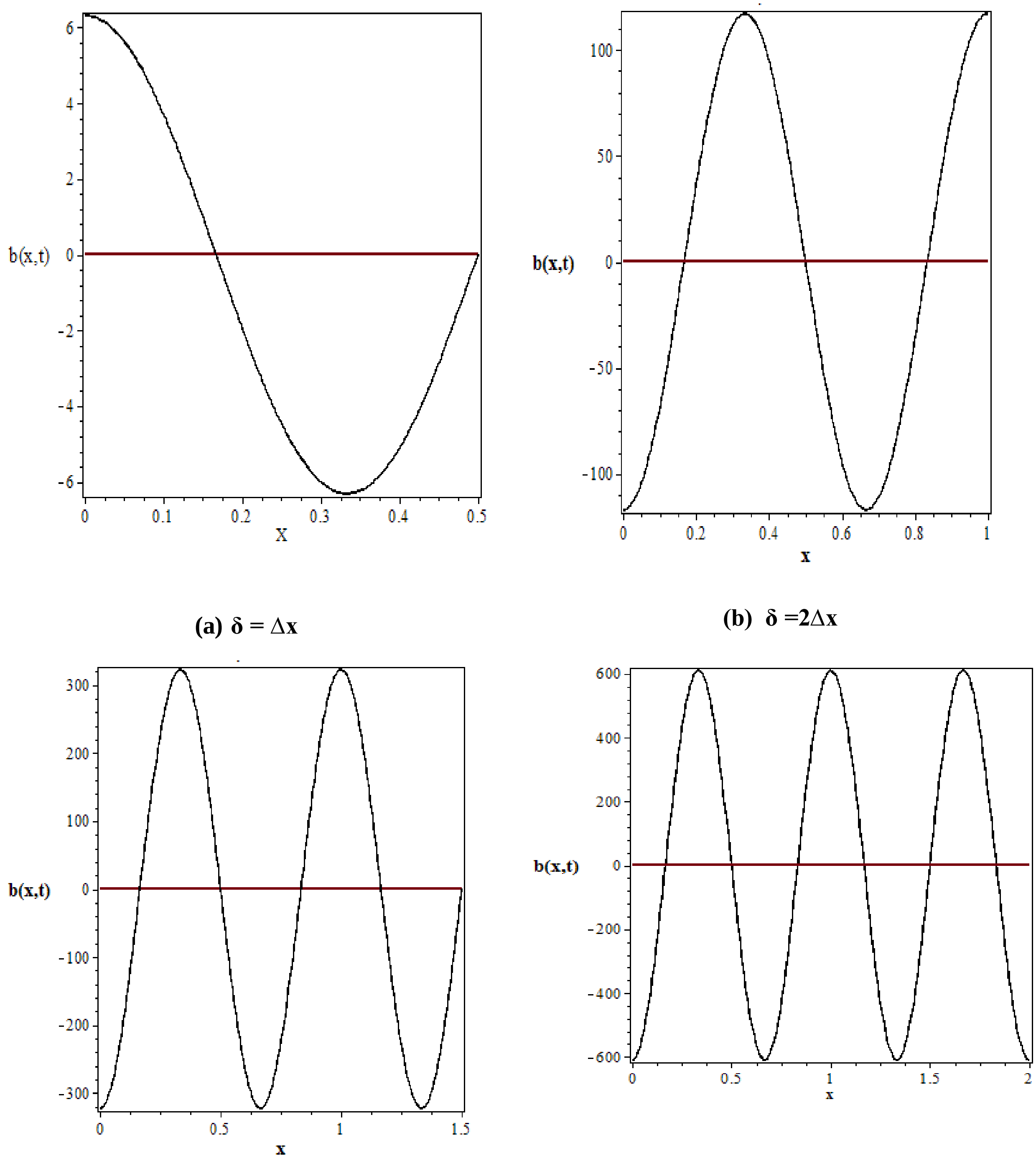


(a) δ = Δx

(b) δ =2Δx

(c) δ =3Δx

(d) δ =4Δx

Figure 2: The solution of equation (2-12) for standing wave.

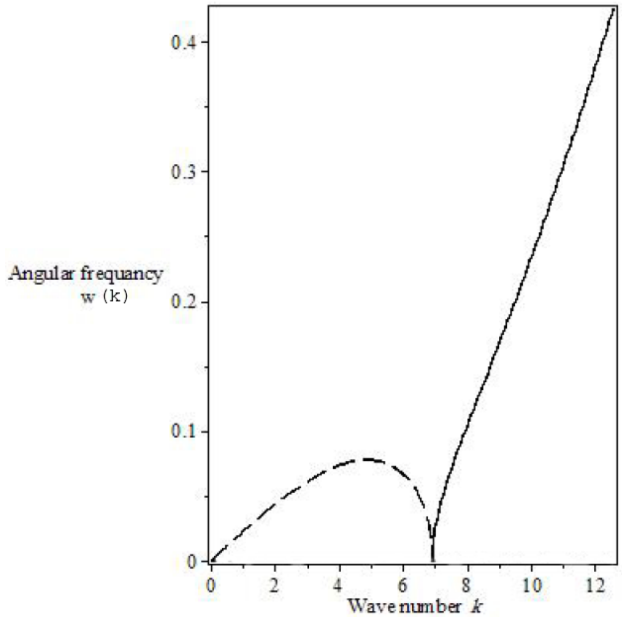
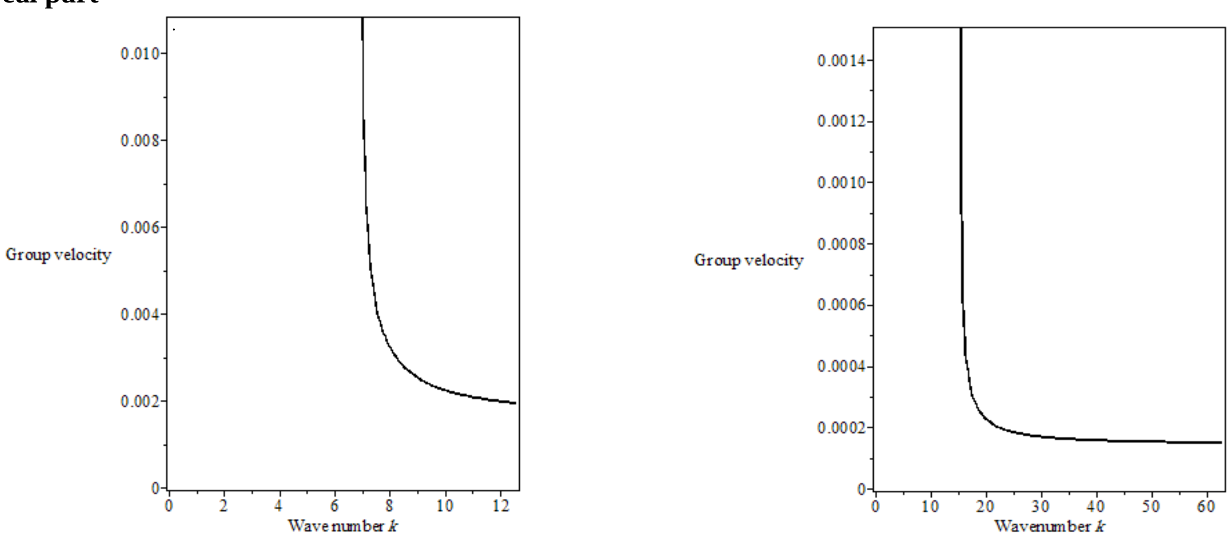


Figure 3: Dispersion relation for standing wave model, dash line for imaginary part, solid line for real part



a **b**

Figure 4: Group velocity for standing wave model (a) δ=0.5, (b) δ= 0.1

The group velocity is defined as $\mathbf{v_g = \frac{\partial \omega}{\partial k}}$, from Eq. 24:

$$\frac{\partial \omega}{\partial k} = \pm\left(\frac{k\delta^2}{48\rho}\right) / \sqrt{\frac{1}{\rho}\left(\frac{k^2\delta^2}{48} - \frac{1}{2}\right)} \quad \text{…………} \qquad 25$$

The group velocity, the velocity at which the energy of a wave packet travels considered in Figure 4.

REFERENCES

Chen X and Max Gunzburger (2011). Continuous and discontinuous finite element methods for a peridynamics model of mechanics, *Computer Methods in Applied Mechanics and Engineering Journal* **200** 1237–1250.

Chen X and Max Gunzburger (2011). Continuous and discontinuous finite element methods for a peridynamics model of mechanics. *Computer Methods in Applied Mechanics and Engineering Journal* **200** 1237–1250.

Emmrich E and Weckner O (2006). The peridynamic equation of motion in non-local elasticity theory; *in Proceedings of III European Conference on Computational Mechanics Solids, Structures and Coupled Problems in Engineering*, Lisbon.

Emmrich E and Weckner O (2006). The peridynamic equation of motion in non-local elasticity theory, edited by Mota Soares CA *et al*.,, III European Conference on Computational Mechanics. Solids, Structures and Coupled Problems in Engineering, Mota Soares CA *et al*.,, Lisbon, Portugal 5–8.

Emmrich E and Weckner O (2007). On the well-posedness of the linear peridynamic model and its convergence towards the Navier equation of linear elasticity, *Communications in Mathematical Sciences* **5** 851–864.

Etienne Emmricha and Olaf Weckner (2005). Analysis and numerical approximation of an integro-differential equation modelling non-local effects in linear elasticity. *Mathematics and Mechanics of Solids* **12** 363-384.

Graff KF (1991). Wave Motion in Elastic Solids, Dover Publications. New York,

Silling SA (2000). Reformulation of elasticity theory for discontinuities and long-range forces. *Journal of the Mechanics and Physics of Solids* **48** 175–209.

Silling SA (2000). Reformulation of elasticity theory for discontinuities and long-range forces", *Journal of the Mechanics and Physics of Solids* **48** 175–209.

Silling SA and Lehoucq RB (2010). Peridynamic Theory of Solid Mechanics, Sandia National Laboratories Albuquerque. New Mexico 87185-1322 USA.