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Research Article

HYERS- ULAM- RASSIAS STABILITY OF FUNCTIONAL EQUATION THAT HAVE QUADRATIC PROPERTY

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ABSTRACT
The aim of this paper is to prove the stability of functional Equation that has Quadratic property in sprit of Hyers- Ulam- Rassias.

Key Words: *Hyers- Ulam Rassias Stability, Quadratic functional Equation*

AMS Subject Classification (2000). Primary 39B52, 39B72.

INTRODUCTION
The problem of stability of homomorphism stemmed from the question posed by S.M. Ulam in 1940 in his lecture before the mathematical club of the University of Wisconsin. He demanded an answer to the following question of stability of homomorphism for metric groups.
Let G' be a group and let G" be a metric group with the metric d. Given
$\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping h: G' $\rightarrow$ G" satisfies the following inequality
$d[h(xy), h(x)h(y)] < \delta$ for all x and y in G',
then there exists a homomorphism H : G' $\rightarrow$ G" with
$d[h(x), H(x)] < \varepsilon$ for all x in G' ?
In 1941 Hyers answered his question considering the case of Banach spaces. Hyers (1941) proved the following result where E' and E" are Banach spaces.

1.1 Result
Let f: E' $\rightarrow$ E" be a mapping between Banach spaces. If f satisfies the following inequality
$\| f(x+y) - f(x) - f(y) \| \leq \delta$
for all x and y in E' and some $\delta > 0$ then the limit
$T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$
exists for all x in E' and T: E' $\rightarrow$ E" is a unique additive mapping such that
$\| f(x) - T(x) \| \leq \delta$ for all x in E'.
Moreover, if f(tx) is continuous in t for each fixed x in E', then the mapping T is linear.
In 1978 Rassias generalized the result of Hyers by proving the following result.

1.2 Result
Let f: E' $\rightarrow$ E" be a mapping between Banach spaces. If f satisfies the following inequality
$\| f(x+y) - f(x) - f(y) \| \leq \theta(\| x \|^p + \| y \|^p)$
for all x and y in E' and for some $\theta > 0$ and some p with $0 \leq p < 1$, then there exists a unique additive mapping T: E' $\rightarrow$ E" such that

$$\| T(x) - f(x) \| \leq 2\theta \left(\frac{\| x \|^p}{2 - 2^p} \right)$$

for all x and y in E'. In addition, if f(t x) is continuous in t for each fixed x in E', then the mapping T is linear.
The method adopted by Hyers (1941) is designated as *Direct Method* and the stability of any functional inequality which had an independent bound is termed as ***Hyers - Ulam Stability***.
Rassias (1978) and thereafter others like George Isac, Gauvruta, Forti, John A Baker, Skof and Jung etc. obtained several useful result on the stability of functional inequalities which had bounds dependant in some way on the elements in the domain of the function under consideration. Such type of stability is termed as ***Hyers - Ulam - Rassias Stability***
In 1992 Czerwik proved the Hyers - Ulam - Rassias stability for quadratic functional equation and stated the following result.

1.3 Result

Let δ and $\theta \geq 0$ be given and let E' and E'' be real normed spaces, if f: E' $\rightarrow$ E'' satisfied the following inequality

$$\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \| \leq \delta + \theta[\|x\|^p + \|y\|^p]$$

for all x and y are non zero and $p < 2$.
or

$$\| f(x+y) + f(x-y) - 2f(x) - 2f(y) \| \leq \theta[\|x\|^p + \|y\|^p]$$

for all x and y in E' and $p > 2$, then there exists a unique quadratic mapping Q: E' $\rightarrow$ E'' such that

$$\| Q(x) - f(x) \| \leq 2\theta \qquad (\delta + \|f(0)\|)$$

$$\left(\frac{\|x\|^p}{4 - 2^p} + \frac{1}{3}(\delta + \|f(0)\|) \right)$$

for all x and y in E' and $p < 2$.

In this paper Hyers - Ulam - Rassias stability of the following functional inequality

$$\| f(x-y+z)+f(x)+f(y) + f(z) - f(y-x) - f(y-z) - f(z+x) \| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

is obtained. This inequality is reduces to quadratic functional inequalities

$$\| 2f(x) + 2f(y) - f(x-y) - f(x+y) \| \leq \theta(2\|x\|^p + 3\|y\|^p)$$

Where upon the result 1.3 is used to prove the existence of a unique quadratic function Q: X $\rightarrow$ Y, such that f − Q is bounded on X.

2. Preliminaries

2.1 Quadratic Functional Equation

Let X and Y be a normed and a Banach space, respectively, if there is no specification. A mapping f: X $\rightarrow$ Y is called a ***quadratic mapping*** if f satisfies the following quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all x and y in X.

2.2 Definition of Metric Group

Let G be a group. A metric d on G is said to be left invariant if for every x, y, z $\in$ G, d(y, z) = d(xy, xz). Right invariant is defined similarly, and a metric is said to be bi-invariant if it is both left and right invariant. A group with a left-invariant metric such that the inversion function x $\rightarrow$ x^-1 is continuous is called a ***metric group***.
Very important **examples** of metric groups come from what are known as finitely generated groups.
In particular, the Euclidean spaces are **examples** of metric groups.

3. Hyers - Ulam - Rassias Stability of Functional Equation
f (x - y + z)+f (x) + f (y) + f (z) = f (y - x) + f (y - z) + f (z + x)

3.1 Theorem

Assume that a function f: X $\to$ Y satisfies the following inequality

$$\| f(x-y+z)+f(x)+f(y)+f(z)-f(y-x)-f(y-z)-f(z+x) \| \leq \theta (\|x\|^p + \|y\|^p + \|z\|^p) \quad (1)$$

for some $0 < p < 2$, $p \neq 2$, and for all x , y , z in X. Then there exists a unique quadratic function Q: X $\to$ Y such that

$$\| f(x) - Q(x) \| \leq 5\theta \left(\frac{\|x\|^p}{4 - 2^p} \right) \quad (2)$$

for all x in X. Moreover, f (t x) is continuous in t in R for each fixed x in X, then Q satisfies $Q(tx) = t^2 Q(x)$, for all x in X and t in R.

Proof

Put x = y = z = 0 in (1) to obtain

$\| f(0) \| \leq 0$ or $f(0) = 0$

Now put y = z = 0 in (1) to obtain

$$\| f(x) - f(-x) \| \leq \theta [\|x\|^p]$$

for all x in X.

Further set z = 0 in (1) to obtain

$$\| f(x-y) - f(y-x) \| \leq \theta [\|x\|^p + \|y\|^p]$$

for all x and y in X.

Now let z = y in (1) to obtain

$$\| 2f(x) + 2f(y) - f(y-x) - f(y+x) \| \leq \theta [\|x\|^p + 2\|y\|^p]$$

or

$$\| 2f(x)+2f(y)-f(y-x)-f(y+x)-f(x-y) + f(x-y) \| \leq \theta [\|x\|^p + 2\|y\|^p]$$

or

$$\| 2f(x)+2f(y)-f(x-y)-f(x+y) \| \leq \theta [2\|x\|^p + 3\|y\|^p] \quad (3)$$

for all x and y in X.

According to Czerwik (1992), it follows from (3) that there exists a quadratic function Q: X $\to$ Y, and with the use of direct method the definition of Q is

$$Q(x) = \lim_{n\to\infty} \left(\frac{f(2^n x)}{2^{2n}} \right)$$

for all x in X.

From (3) it follows that

$$\| f(x) - Q(x) \| \leq 5\theta \sum_{n=0}^{\infty} \left(\frac{\|x\|^p}{2^{2n-np+2}} \right) \text{ for } p < 2 .$$

or

or

$$\| f(x) - Q(x) \| \leq 5\theta \left(\frac{\|x\|^p}{4 - 2^p} \right) \text{ for } p < 2.$$

for all x in X . This is exactly (2).

Uniqueness of Q

Let T: X $\rightarrow$ Y be another quadratic mapping which satisfies inequality (1) and (2). Obviously
$Q(2^n x) = 4^n Q(x)$ and $T(2^n x) = 4^n T(x)$
for all x in X and n being positive. Therefore

$$\begin{aligned} \| Q(x) - T(x) \| &= 4^{-n} \| Q(2^n x) - T(2^n x) \| \\ &\leq 4^{-n} \| Q(2^n x) - f(2^n x) \| + 4^{-n} \| T(2^n x) - f(2^n x) \| \\ &\leq 10\theta \left(\frac{\|x\|^p}{4^n (4 - 2^p)} \right) \end{aligned}$$

for all x in X and p < 2. By letting n $\rightarrow \infty$ in the preceding inequality immediately there is a uniqueness of Q.

Expected Outcome of the Present Work

1. Generalization of this result to topological vector space can be further taken up.
2. In addition to these some open problems posed by Rassias (1978) in his Survey paper on the topic shall be taken for further investigation.

REFRENCES

Czerwik S (1992). On the stability of the quadratic mapping in normed spaces. *Abhandlungen Mathematisches Seminar der Universität Hamburg* **62** 59 - 64.

Hyers DH (1941). On the stability of the linear functional equation. *Proceedings of National Academy of Science of U.S.A.* **27** 222 - 224.

Hyers DH and Rassias THM (1992). Survey paper. Approximate homorphisms, *Aequationes Mathematicae* **44** 125 - 153.

Rassias THM (1978). On the stability of the linear mapping in Banach spaces. *Proceedings of the American Mathematical Society* **72** 297 - 300.

Rassias THM (1994). Problem 18. In report of the thirty-first Internet Symposium on Functional Equation, *Aequationes Mathematicae* **47** 263 - 327.