

## Research Article

# DEGENERATED BERNOULLI NUMBERS AND POLYNOMIALS

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## ABSTRACT

The degenerate Bernoulli numbers  $\beta_m(\lambda)$  can be defined by means of the exponential generating function

$t \left[ (1 + \lambda t)^{\frac{1}{\lambda}} - 1 \right]^{-1}$  L.Carltz proved an analogue of the Staudt- Clausen theorem for these numbers and he

showed that  $\beta_m(\lambda)$  is polynomials in  $\lambda$  of degree  $\leq m$ . As further applications we derive several identities, recurrences, and congruences involving the Bernoulli numbers, degenerate Bernoulli numbers and polynomials.

**Key Words:** Bernoulli Polynomial, Bernoulli Number, Degenerate Bernoulli Polynomial, Degenerate Bernoulli Number

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## INTRODUCTION

Carlitz (1956) defined the degenerate Bernoulli numbers  $\beta_m(\lambda)$  by means of the generating function

$$\frac{t}{\left[ (1 + \lambda t)^{\frac{1}{\lambda}} - 1 \right]} = \sum_{m=0}^{\infty} \beta_m(\lambda) \frac{t^m}{m!} \quad (1.1)$$

We have  $\beta_m(0) = \beta_m$ , the ordinary Bernoulli number In (Carlitz, 1956, 1979) Carlitz proved many properties of  $\beta_m(\lambda)$ , including an analogue of the staudt-clausentheorem. He also pointed out that  $\beta_m(\lambda)$  is a polynomials in  $\lambda$  with degree  $\leq m$ . we have

$$\beta_0(\lambda) = 1$$

$$\beta_1(\lambda) = \frac{-1}{2} + \frac{\lambda}{2}$$

$$\beta_2(\lambda) = \frac{1}{6} - \frac{\lambda^2}{6}$$

$$\beta_3(\lambda) = \frac{-\lambda}{4} + \frac{\lambda^3}{4}$$

$$\beta_4(\lambda) = \frac{-1}{30} + \frac{2}{3} \lambda^2 - \frac{19}{30} \lambda^4$$

$$\beta_5(\lambda) = \frac{1}{4} \lambda - \frac{5}{2} \lambda^3 + \frac{9}{4} \lambda^5 \text{ And so on.}$$

Carlitz (1956, 1979) also defined the degenerate Bernoulli polynomials  $\beta_m(\lambda, x)$  for  $\lambda \neq 0$  by means of the generating function.

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$$\frac{t}{[(1+\lambda t)^u - 1]} (1+\lambda t)^{\lambda x} = \sum_{m=0}^{\infty} \beta_m(\lambda, x) \frac{t^m}{m!} \quad (1.2)$$

Where  $\lambda u=1$ . These are polynomials in  $\lambda$  and  $x$  with rational coefficients. We often write  $\beta_m(\lambda)$  for  $\beta_m(\lambda, 0)$ , and refer to the polynomial  $\beta_m(\lambda)$  as a degenerate Bernoulli number. The first few are

$$\beta_0(\lambda, x) = 1$$

$$\beta_1(\lambda, x) = x - \frac{1}{2} + \frac{1}{2} \lambda$$

$$\beta_2(\lambda, x) = x^2 - x + \frac{1}{6} - \frac{1}{6} \lambda^2$$

$$\beta_3(\lambda, x) = x^3 - \frac{3}{2} x^2 + \frac{1}{2} x - \frac{3}{2} \lambda x^2 + \frac{3}{2} \lambda x + \frac{1}{4} \lambda^3 - \frac{1}{4} \lambda$$

And so on.

One combinatorial significance these polynomials have found is in expressing sums of generalized falling

factorials  $\left(\frac{i}{\lambda}\right)_m$  : specifically, we have

$$\sum_{i=0}^{a-1} \left(\frac{i}{\lambda}\right)_m = \frac{1}{m+1} [\beta_{m+1}(\lambda, a) - \beta_{m+1}(\lambda)] \quad (1.3)$$

For all integers  $a > 0$  and  $m \geq 0$  [2 Eq. (5.4)], where

$$\left(\frac{i}{\lambda}\right)_m = i(i-\lambda)(i-2\lambda) \dots (i-(m-1)\lambda).$$

The Bernoulli polynomials  $\beta_m(x)$  may be defined by the generating function,

$$\frac{t}{(e^t - 1)} e^{xt} = \sum_{m=0}^{\infty} \beta_m(x) \frac{t^m}{m!} \quad (1.4)$$

And their values at  $x=0$  are called the Bernoulli numbers and denoted  $\beta_m$ . Since  $(1+\lambda t)^u \rightarrow e^t$  as  $\lambda \rightarrow 0$  it is evident that  $\beta_m(0, x) = \beta_m(x)$  letting  $\lambda \rightarrow 0$  in (1.3) yields the familiar identity

$$\sum_{i=0}^{a-1} i^m = \frac{1}{m+1} [\beta_{m+1}(a) - \beta_{m+1}] \quad (1.5)$$

Expressing power sums in terms of Bernoulli polynomials.

### A Recurrence Relation of $\beta_m$

In (Howard, 1996), For any positive integer  $m$  and any positive integer  $n \geq 1$ , we have

$$\beta_m = \frac{1}{n(1-n^m)} \sum_{k=0}^{m-1} n^k \binom{m}{k} \beta_k \sum_{j=1}^{n-1} j^{m-k} \quad (2.1)$$

**Proof:** Let  $n$  be any positive integer greater than 1. Noticing that  $\frac{(1-e^{nx})}{(1-e^x)}$  is the sum of finite geometric series, we have

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$$\frac{(1-e^{nx})}{(1-e^x)} = \sum_{j=0}^{n-1} e^{jx} = \sum_{j=0}^{n-1} \sum_{m=0}^{\infty} \frac{j^m x^m}{m!} = \sum_{m=0}^{\infty} \sum_{j=0}^{n-1} \frac{j^m x^m}{m!}$$

Multiplying both sides by  $\frac{x}{(1-e^{nx})}$ , we obtain

$$\frac{x}{(1-e^{nx})} = \frac{1}{n} \left( \frac{nx}{1-e^{nx}} \right) = \sum_{m=0}^{\infty} \sum_{j=0}^{n-1} \frac{j^m x^m}{m!}$$

From the definition of the Bernoulli numbers, we now have

$$\sum_{m=0}^{\infty} \beta_m \frac{x^m}{m!} = \left( \frac{1}{n} \sum_{m=0}^{\infty} \frac{\beta_m \cdot n^m \cdot x^m}{m!} \right) \sum_{m=0}^{\infty} \sum_{j=0}^{n-1} \frac{j^m x^m}{m!}$$

By the Cauchy product rule, we get

$$\sum_{m=0}^{\infty} \beta_m \frac{x^m}{m!} = \frac{1}{n} \sum_{m=0}^{\infty} \sum_{k=0}^m \left( \frac{\beta_k n^k x^k}{k!} \sum_{j=0}^{n-1} \frac{j^{m-k} x^{m-k}}{(m-k)!} \right)$$

$$\sum_{m=0}^{\infty} \beta_m \frac{x^m}{m!} = \sum_{m=0}^{\infty} \left( \frac{1}{n} \sum_{k=0}^m \left( \frac{\beta_k n^k}{k! (m-k)!} \sum_{j=0}^{n-1} j^{m-k} \right) x^m \right)$$

Because a power series expansion is unique, we have

$$\beta_m = \frac{1}{n} \sum_{k=0}^m \left( n^k \binom{m}{k} \beta_k \sum_{j=0}^{n-1} j^{m-k} \right)$$

$$\beta_m = \frac{1}{n} \sum_{k=0}^{m-1} \left( n^k \binom{m}{k} \beta_k \sum_{j=0}^{n-1} j^{m-k} \right) + \frac{1}{n} \left( n^m \cdot \beta_m \sum_{j=0}^{n-1} 1 \right)$$

$$\beta_m = \frac{1}{n} \sum_{k=0}^{m-1} \left( n^k \binom{m}{k} \beta_k \sum_{j=0}^{n-1} j^{m-k} \right) + (n^m \cdot \beta_m)$$

Therefore

$$\beta_m = \frac{1}{n(1-n^m)} \sum_{k=0}^{m-1} \left( n^k \binom{m}{k} \beta_k \sum_{j=0}^{n-1} j^{m-k} \right) \text{ For all } m \geq 1$$

### A Recurrence Relation of $\beta_m(\lambda, x)$

In this section, we derive the following recurrence relation for  $\beta_m(\lambda, x)$

$$\beta_m(\lambda, x) = \sum_{k=0}^m \binom{m}{k} \beta_k(\lambda) \left( \frac{x}{\lambda} \right)_{m-k} \quad (3.1)$$

**Proof:-** We know that the degenerate Bernoulli polynomial

$$\frac{t}{[(1+\lambda t)^\mu - 1]} (1+\lambda t)^{\mu\alpha} = \sum_{m=0}^{\infty} \beta_m(\lambda, x) \frac{t^m}{m!}$$

By (1.1) we get

$$\sum_{m=0}^{\infty} \beta_m(\lambda) \frac{t^m}{m!} (1+\lambda t)^{\mu\alpha} = \sum_{m=0}^{\infty} \beta_m(\lambda, x) \frac{t^m}{m!}$$

By the Binomial expansion

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$$\sum_{m=0}^{\infty} \beta_m(\lambda) \frac{t^m}{m!} \sum_{m=0}^{\infty} \left( \frac{x}{\lambda} \right)_m \frac{t^m}{m!} = \sum_{m=0}^{\infty} \beta_m(\lambda, x) \frac{t^m}{m!}$$

By the Cauchy product rule

$$\beta_m(\lambda, x) = \sum_{k=0}^m \binom{m}{k} \beta_k(\lambda) \left( \frac{x}{\lambda} \right)_{m-k}$$

$$\text{Where } \left( \frac{x}{\lambda} \right)_m = [x(x-\lambda)(x-2\lambda)\dots(x-(m-1)\lambda)]$$

This is new recurrence relation.

### Properties of Degenerate Bernoulli Polynomial

In this section, some of well-known properties of Degenerate Bernoulli polynomials are derived from the generating function (1.2)

#### Property 1:

$$\beta_m(\lambda, x+y) = \sum_{k=0}^m \binom{m}{k} \beta_k(\lambda, x) \left( \frac{y}{\lambda} \right)_{m-k} \quad (4.1)$$

**Proof:** Now put  $x \rightarrow x+y$  in (1.2)

$$\frac{t}{[(1+\lambda t)^\mu - 1]} (1+\lambda t)^{\mu(x+y)} = \sum_{m=0}^{\infty} \beta_m(\lambda, x+y) \frac{t^m}{m!}$$

$$\frac{t}{[(1+\lambda t)^\mu - 1]} (1+\lambda t)^{\mu x} (1+\lambda t)^{\mu y} = \sum_{m=0}^{\infty} \beta_m(\lambda, x+y) \frac{t^m}{m!}$$

By the equation (1.2)

$$\sum_{m=0}^{\infty} \beta_m(\lambda, x) \frac{t^m}{m!} (1+\lambda t)^{\mu y} = \sum_{m=0}^{\infty} \beta_m(\lambda, x+y) \frac{t^m}{m!}$$

By the help of Binomial expansion

$$(1+\lambda t)^{\mu y} = \sum_{m=0}^{\infty} \left( \frac{y}{\lambda} \right)_m \frac{t^m}{m!}$$

Therefore

$$\sum_{m=0}^{\infty} \beta_m(\lambda, x) \frac{t^m}{m!} \sum_{m=0}^{\infty} \left( \frac{y}{\lambda} \right)_m \frac{t^m}{m!} = \sum_{m=0}^{\infty} \beta_m(\lambda, x+y) \frac{t^m}{m!}$$

By the Cauchy product rule

$$\left( \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \right) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}$$

$$c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k}$$

Where

$$\beta_m(\lambda, x+y) = \sum_{k=0}^m \binom{m}{k} \beta_k(\lambda, x) \left( \frac{y}{\lambda} \right)_{m-k}$$

Here  $y=1$  then,

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$$\beta_m(\lambda, x+1) = \sum_{k=0}^m \binom{m}{k} \beta_k(\lambda, x) \left(\frac{1}{\lambda}\right)_{m-k} \quad (4.2)$$

### Property 2:

$$\beta'_m(\lambda, x) = \lambda^{-1} \beta_m(\lambda, x) \quad (4.3)$$

**Proof:** By the generating function of degenerate Bernoulli polynomials

$$\frac{t(1+\lambda t)^{\lambda x}}{[(1+\lambda t)^{\lambda} - 1]} = \sum_{m=0}^{\infty} \beta_m(\lambda, x) \frac{t^m}{m!}$$

Differentiate above equation with respect to x

$$\frac{\mu t(1+\lambda t)^{\lambda x}}{[(1+\lambda t)^{\lambda} - 1]} = \sum_{m=0}^{\infty} \frac{d}{dx} \beta_m(\lambda, x) \frac{t^m}{m!}$$

$$\mu \sum_{m=0}^{\infty} \beta_m(\lambda, x) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \beta'_m(\lambda, x) \frac{t^m}{m!}$$

Equating the coefficients

$$\beta'_m(\lambda, x) = \mu \beta_m(\lambda, x)$$

$$\beta'_m(\lambda, x) = \frac{1}{\lambda} \beta_m(\lambda, x) \text{ Where } \mu\lambda=1$$

$$\beta'_m(\lambda, x) = \lambda^{-1} \beta_m(\lambda, x)$$

### Property 3:

$$\beta_m(\lambda, 1-x) = (-1)^m \beta_m(\lambda, x) \quad (4.4)$$

Proof:- By equation (1.2)

$$\frac{t(1+\lambda t)^{\lambda x}}{[(1+\lambda t)^{\lambda} - 1]} = \sum_{m=0}^{\infty} \beta_m(\lambda, x) \frac{t^m}{m!}$$

$x \rightarrow 1-x$  in above equation

Put

$$\frac{t(1+\lambda t)^{\mu(1-x)}}{[(1+\lambda t)^{\mu} - 1]} = \sum_{m=0}^{\infty} \beta_m(\lambda, 1-x) \frac{t^m}{m!}$$

$$\frac{t(1+\lambda t)^{\mu} (1+\lambda t)^{-\mu x}}{[(1+\lambda t)^{\mu} - 1]} = \sum_{m=0}^{\infty} \beta_m(\lambda, 1-x) \frac{t^m}{m!}$$

$$\frac{t(1+\lambda t)^{\mu} (1+\lambda t)^{-\mu x}}{(1+\lambda t)^{\mu} [1 - (1+\lambda t)^{-\mu}]} = \sum_{m=0}^{\infty} \beta_m(\lambda, 1-x) \frac{t^m}{m!}$$

$$\frac{t(1+\lambda t)^{-\mu x}}{[1 - (1+\lambda t)^{-\mu}]} = \sum_{m=0}^{\infty} \beta_m(\lambda, 1-x) \frac{t^m}{m!}$$

$$-\frac{t(1+\lambda t)^{-\mu x}}{[(1+\lambda t)^{-\mu} - 1]} = \sum_{m=0}^{\infty} \beta_m(\lambda, 1-x) \frac{t^m}{m!}$$

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$$\sum_{m=0}^{\infty} (-1)^m \beta_m(\lambda, x) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \beta_m(\lambda, 1-x) \frac{t^m}{m!}$$

Equating the coefficients

$$\beta_m(\lambda, 1-x) = (-1)^m \beta_m(\lambda, x)$$

**Property 4:**

$$\beta_n(\lambda, x+1) - \beta_n(\lambda, x) = n \left( \frac{x}{\lambda} \right)_{n-1} \quad (4.5)$$

**Proof:** By equation (1.2)

$$\frac{t(1+\lambda t)^{\mu x}}{[(1+\lambda t)^{\mu} - 1]} = \sum_{n=0}^{\infty} \beta_n(\lambda, x) \frac{t^n}{n!} \quad (4.6)$$

$x \rightarrow x+1$  in above equation

Put

$$\frac{t(1+\lambda t)^{\mu(x+1)}}{[(1+\lambda t)^{\mu} - 1]} = \sum_{n=0}^{\infty} \beta_n(\lambda, x+1) \frac{t^n}{n!} \quad (4.7)$$

Subtracting (4.7)-(4.6)

$$\frac{t(1+\lambda t)^{\mu(x+1)}}{[(1+\lambda t)^{\mu} - 1]} - \frac{t(1+\lambda t)^{\mu x}}{[(1+\lambda t)^{\mu} - 1]} = \sum_{n=0}^{\infty} \beta_n(\lambda, x+1) \frac{t^n}{n!} - \sum_{n=0}^{\infty} \beta_n(\lambda, x) \frac{t^n}{n!}$$

$$t(1+\lambda t)^{\mu x} = \sum_{n=0}^{\infty} [\beta_n(\lambda, x+1) - \beta_n(\lambda, x)] \frac{t^n}{n!}$$

By the Binomial expansion

$$t \sum_{n=0}^{\infty} \left[ \frac{x}{\lambda} \right]_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} [\beta_n(\lambda, x+1) - \beta_n(\lambda, x)] \frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} \left[ \frac{x}{\lambda} \right]_n \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} [\beta_n(\lambda, x+1) - \beta_n(\lambda, x)] \frac{t^n}{n!}$$

$n \rightarrow n-1$  In L.H.S.

$$\sum_{n=0}^{\infty} \left[ \frac{x}{\lambda} \right]_{n-1} \frac{t^n}{(n-1)!} = \sum_{n=0}^{\infty} [\beta_n(\lambda, x+1) - \beta_n(\lambda, x)] \frac{t^n}{n!}$$

Multiply and divide by  $n!$

$$\sum_{n=0}^{\infty} \left[ \frac{x}{\lambda} \right]_{n-1} \frac{t^n}{n!} \cdot \frac{n!}{(n-1)!} = \sum_{n=0}^{\infty} [\beta_n(\lambda, x+1) - \beta_n(\lambda, x)] \frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} \left[ \frac{x}{\lambda} \right]_{n-1} \frac{t^n}{n!} \cdot \frac{n(n-1)!}{(n-1)!} = \sum_{n=0}^{\infty} [\beta_n(\lambda, x+1) - \beta_n(\lambda, x)] \frac{t^n}{n!}$$

Equating the coefficients

$$\beta_n(\lambda, x+1) - \beta_n(\lambda, x) = n \left( \frac{x}{\lambda} \right)_{n-1}$$

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### Property 4:

$$\left(\frac{x}{\lambda}\right)_m = \frac{1}{m+1} \cdot \sum_{k=0}^m \binom{m+1}{k} \beta_k(\lambda, x) \left(\frac{1}{\lambda}\right)_{m+1-k} \quad (4.8)$$

**Proof:** By equation (4.2)

$$\begin{aligned} \beta_m(\lambda, x+1) &= \sum_{k=0}^m \binom{m}{k} \beta_k(\lambda, x) \left(\frac{1}{\lambda}\right)_{m-k} \\ \beta_m(\lambda, x+1) &= \sum_{k=0}^{m-1} \binom{m}{k} \beta_k(\lambda, x) \left(\frac{1}{\lambda}\right)_{m-k} + \binom{m}{m} \beta_m(\lambda, x) \left(\frac{1}{\lambda}\right)_{m-m} \\ \beta_m(\lambda, x+1) - \beta_m(\lambda, x) &= \sum_{k=0}^{m-1} \binom{m}{k} \beta_k(\lambda, x) \left(\frac{1}{\lambda}\right)_{m-k} \end{aligned}$$

But we know that by (4.5)

$$\begin{aligned} \beta_n(\lambda, x+1) - \beta_n(\lambda, x) &= n \left(\frac{x}{\lambda}\right)_{n-1} \\ m \left(\frac{x}{\lambda}\right)_{m-1} &= \sum_{k=0}^{m-1} \binom{m}{k} \beta_k(\lambda, x) \left(\frac{1}{\lambda}\right)_{m-k} \\ m &\rightarrow m+1 \end{aligned} \quad (4.9)$$

Now put

$$\begin{aligned} (m+1) \left(\frac{x}{\lambda}\right)_m &= \sum_{k=0}^m \binom{m+1}{k} \beta_k(\lambda, x) \left(\frac{1}{\lambda}\right)_{m+1-k} \\ \left(\frac{x}{\lambda}\right)_m &= \frac{1}{m+1} \cdot \sum_{k=0}^m \binom{m+1}{k} \beta_k(\lambda, x) \left(\frac{1}{\lambda}\right)_{m+1-k} \end{aligned}$$

Now put  $x=0$  in equation (4.9), then

$$\sum_{k=0}^{m-1} \binom{m}{k} \beta_k(\lambda, 0) \left(\frac{1}{\lambda}\right)_{m+1-k} = 0 \quad (4.10)$$

By definition

$$\beta_m(\lambda, 0) = \beta_m(\lambda)$$

Therefore

$$\sum_{k=0}^{m-1} \binom{m}{k} \beta_k(\lambda) \left(\frac{1}{\lambda}\right)_{m+1-k} = 0 \quad (4.11)$$

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