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Research Article

MATHEMATICAL MODELING OF QUASI -STATIC THERMOELASTIC STEADY STATE BEHAVIOR OF THICK CIRCULAR PLATE WITH INTERNAL HEAT GENERATION

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ABSTRACT
The present paper deals with the determination of displacement and thermal stresses in a thick circular plate with internal heat generation. Arbitrary heat flux is applied at the upper surface of a thick circular plate, whereas lower surface at zero temperature and the fixed circular edge is thermally insulated. Here we compute the effects of internal heat generation of a thick circular plate in terms of stresses along radial direction. The governing heat conduction equation has been solved by the method of integral transform technique. The results are obtained in a series form in terms of Bessel's functions. The results for temperature change and stresses have been computed numerically and illustrated graphically.

Key Words: *Thermal Stresses, Internal Heat Generation, Steady State*

INTRODUCTION
During the second half of the twentieth century, nonisothermal problems of the theory of elasticity became increasingly important. This is due to their wide application in diverse fields. The high velocities of modern aircraft give rise to aerodynamic heating, which produces intense thermal stresses that reduce the strength of the aircraft structure.
The steady state thermal stresses in circular disk subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face and the circular edge has been considered by Nowacki (1957). Sharma *et al.,* (2004) studied the behavior of thermoelastic thick plate under lateral loads. Kulkarni and Deshmukh (2008) determined quasi-static thermal stresses in steady state thick circular plate. Deshmukh *et al.,* (2009) studied non homogeneous steady state heat conduction problem in a thin circular plate and discussed its thermal stresses due to its internal heat generation at a constant rate. Most recently Bhongade and Durge (2013) considered thick circular plate and discuss, effect of Michell function on steady state behavior of thick circular plate. In this paper thick circular plate is considered and discussed its thermoelasticity with the help of the Goodier's thermoelastic displacement potential function and the Michell's function. To obtain the temperature distribution integral transform method is applied. The results are obtained in series form in terms of Bessel's functions and the temperature change and stresses have been computed numerically and illustrated graphically. Here we compute the effects of internal heat generation in terms of stresses along radial direction. A mathematical model has been constructed of a thick circular plate with the help of numerical illustration by considering aluminum (pure) circular plate. No one previously studied such type of problem. This is new contribution to the field.
The direct problem is very important in view of its relevance to various industrial mechanics subjected to heating such as the main shaft of lathe, turbines and the role of rolling mill, base of furnace of boiler of a thermal power plant, gas power plant.

Formulation of the Problem
Consider a thick circular plate of thickness h defined by $0 \leq r \leq a, \frac{-h}{2} \leq z \leq \frac{h}{2}$. The initial temperature in a thick circular plate is zero. The arbitrary heat flux $-\frac{Q}{\lambda} f(r)$ is applied over the upper surface ($z = \frac{h}{2}$) and the lower surface ($z = -\frac{h}{2}$) is at temperature zero. The fixed circular edge ($r = a$) is thermally insulated.

Assume the circular boundary of a thick circular plate is free from traction. Under these prescribed conditions, the thermal steady state temperature, displacement and stresses in a thick circular plate with internal heat generation are required to be determined. The differential equation governing the displacement potential function $\phi(r,z)$ is given in (2003) as

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}= K\tau \tag{1}$$

where K is the restraint coefficient and temperature change $\tau = T - T_{i,}$ T_i is initial(ambient) temperature. Displacement function ϕ is known as Goodier's thermoelastic displacement potential.
The steady state temperature of the plate satisfies the heat conduction equation,

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}+\frac{q}{k}= 0 \tag{2}$$

with the boundary conditions

$$T = 0 \; at \; z = -\frac{h}{2}, \;\; 0 \le r \le a \tag{3}$$

$$\frac{\partial T}{\partial Z} = -\frac{Q}{\lambda}\, f(r) \;\; at\; z = \frac{h}{2}, \;\; 0 \le r \le a \tag{4}$$

$$\frac{\partial T}{\partial r} = 0 \; at \; r = a, \; \frac{-h}{2} \le z \le \frac{h}{2} \tag{5}$$

where *k* is the thermal conductivity of the material of the plate and *q* is the internal heat generation.
The Michell's function *M* must satisfy

$$\nabla^2\nabla^2 M = 0 \tag{6}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2} \tag{7}$$

The components of the stresses are represented by the thermoelastic displacement potential ϕ and Michell's function *M* as

$$\sigma_{rr} = 2G\left\{\frac{\partial^2\phi}{\partial r^2} - K\tau + \frac{\partial}{\partial z}\left[v\nabla^2 M - \frac{\partial^2 M}{\partial r^2}\right]\right\} \tag{8}$$

$$\sigma_{\theta\theta} = 2G\left\{\frac{1}{r}\frac{\partial\phi}{\partial r} - K\tau + \frac{\partial}{\partial z}\left[v\nabla^2 M - \frac{1}{r}\frac{\partial M}{\partial r}\right]\right\} \tag{9}$$

$$\sigma_{zz} = 2G\left\{\frac{\partial^2\phi}{\partial z^2} - K\tau + \frac{\partial}{\partial z}\left[(2-v)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right]\right\} \tag{10}$$

and

$$\sigma_{rz} = 2G\left\{\frac{\partial^2\phi}{\partial r\partial z} + \frac{\partial}{\partial r}\left[(1-v)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right]\right\} \tag{11}$$

where *G* and *v* are the shear modulus and Poisson's ratio respectively.
For traction free surface stress functions

$$\sigma_{rr} = \sigma_{zz} = \sigma_{rz} = 0 \; at \; z = -\frac{h}{2} \tag{12}$$

Solution

To obtain the expression for temperature $T(r, z)$, we introduce the finite Hankel transform over the variable *r* and its inverse transform defined by (1968)as

$$\bar{T}(\beta_m, z) = \int_{r'=0}^{a} r' K_0(\beta_m, r')\, T(r', z)\, dr' \tag{13}$$

$$T(r,z) = \sum_{m=1}^{\infty} K_0(\beta_m, r)\, \bar{T}(\beta_m, z) \tag{14}$$

where

$$K_0(\beta_m, r) = \frac{\sqrt{2}}{a}\frac{J_0(\beta_m r)}{J_0(\beta_m a)} \tag{15}$$

Eigen value β_m are the positive root of $J_0{}'(\beta_m a) = 0$ (16)
$and \;\beta_1, \beta_2 \ldots\ldots$ are roots of the transcendental equation
where $J_n(x)$ is Bessel function of the first kind of order *n*.
On applying the finite Hankel transform defined in the Eq. (13) and its inverse transform defined in Eq. (14) to the Eq. (2), one obtains the expression for temperature as

$$T(r,z) = \sum_{m=1}^{\infty} \frac{\sqrt{2}}{a} \frac{J_0(\beta_m r)}{J_0(\beta_m a)} \frac{-A(\beta_m,-\frac{h}{2})\cosh\left[\beta_m\left(z-\frac{h}{2}\right)\right]}{\cosh(h\beta_m)} + \left(\frac{-Q\,F(\beta_m)}{\beta_m\lambda} - \frac{1}{\beta_m}\frac{\partial A\left(\beta_m,\frac{h}{2}\right)}{\partial Z}\right)\frac{\sinh\left[\beta_m\left(z+\frac{h}{2}\right)\right]}{\cosh(h\beta_m)} + A(\beta_m, z)\Bigg\} \tag{17}$$

$A(\beta_m, z)$ is particular integral of differential Eq. (2).

Michells function M

Now suitable form of M which satisfy Eq. (6) is given by

$$M = \sum_{m=1}^{\infty} J_0(\beta_m r)\left\{B_m \cosh\left[\beta_m\left(z+\frac{h}{2}\right)\right] + C_m\,\beta_m\left(z+\frac{h}{2}\right)\sinh\left[\beta_m\left(z+\frac{h}{2}\right)\right]\right\} \tag{18}$$

where B_m *and* C_m are arbitrary functions.

Goodiers thermoelastic displacement potential $\phi(r,z)$

Assuming the displacement function $\phi(r,z)$ which satisfies Eq. (1) as

$$\phi(r,z) = \sum_{m=1}^{\infty} \frac{\sqrt{2}}{a} \frac{J_0(\beta_m r)}{J_0(\beta_m a)} \left\{\frac{-A(\beta_m,-\frac{h}{2})\cosh\left[\beta_m\left(z-\frac{h}{2}\right)\right]}{\cosh(h\beta_m)} + \left(\frac{-Q\,F(\beta_m)}{\beta_m\lambda} - \frac{1}{\beta_m}\frac{\partial A\left(\beta_m,\frac{h}{2}\right)}{\partial Z}\right)\frac{\sinh\left[\beta_m\left(z+\frac{h}{2}\right)\right]}{\cosh(h\beta_m)} + A\left(\beta_m,-\frac{h}{2}\right) e^{\beta_m\left(z+\frac{h}{2}\right)}\right\} \tag{19}$$

Now using Eqs. (17), (18) and (19) in Eqs. (8), (9), (10) and (11), one obtains the expressions for stresses respectively as

$$\begin{aligned}\frac{\sigma_{rr}}{K} = 2G \sum_{m=1}^{\infty} \Bigg\{ &\frac{-\sqrt{2}\beta_m{}^2 J_1{}'(\beta_m r)}{a\,J_0(\beta_m a)} \\ &\times\left[\frac{-A(\beta_m,-\frac{h}{2})\cosh\left[\beta_m\left(z-\frac{h}{2}\right)\right]}{\cosh(h\beta_m)} + \left(\frac{-Q\,F(\beta_m)}{\beta_m\lambda} - \frac{1}{\beta_m}\frac{\partial A\left(\beta_m,\frac{h}{2}\right)}{\partial Z}\right) \times \frac{\sinh\left[\beta_m\left(z+\frac{h}{2}\right)\right]}{\cosh(\beta_m h)} + A\left(\beta_m,-\frac{h}{2}\right)e^{\beta_m\left(z+\frac{h}{2}\right)}\right] \\ &- \frac{\sqrt{2}J_0(\beta_m r)}{a\,J_0(\beta_m a)} \times \left[\frac{-A(\beta_m,-\frac{h}{2})\cosh\left[\beta_m\left(z-\frac{h}{2}\right)\right]}{\cosh(h\beta_m)} + \left(\frac{-Q\,F(\beta_m)}{\beta_m\lambda} - \frac{1}{\beta_m}\frac{\partial A\left(\beta_m,\frac{h}{2}\right)}{\partial Z}\right) \times \frac{\sinh\left[\beta_m\left(z+\frac{h}{2}\right)\right]}{\cosh(\beta_m h)} + A(\beta_m, z)\right] \\ &+ \beta_m{}^2\left[2v\,J_0(\beta_m r)\,C_m + (C_m + B_m)\,J_1'(\beta_m r)\,\sinh\left[\beta_m\left(z+\frac{h}{2}\right)\right]\beta_m + C_m\,J_1{}'(\beta_m r)\left(z+\frac{h}{2}\right)\cosh\left[\beta_m\left(z+\frac{h}{2}\right)\right]\right]\Bigg\}\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{\sigma_{\theta\theta}}{K} = 2G \sum_{m=1}^{\infty} \Bigg\{ &\frac{-\sqrt{2}\beta_m\,J_1(\beta_m r)}{a\,rJ_0(\beta_m a)} \\ &\times\left[\frac{-A(\beta_m,-\frac{h}{2})\cosh\left[\beta_m\left(z-\frac{h}{2}\right)\right]}{\cosh(h\beta_m)} + \left(\frac{-Q\,F(\beta_m)}{\beta_m\lambda} - \frac{1}{\beta_m}\frac{\partial A\left(\beta_m,\frac{h}{2}\right)}{\partial Z}\right) \times \frac{\sinh\left[\beta_m\left(z+\frac{h}{2}\right)\right]}{\cosh(\beta_m h)} + A\left(\beta_m,-\frac{h}{2}\right)e^{\beta_m\left(z+\frac{h}{2}\right)}\right] \\ &- \frac{\sqrt{2}J_0(\beta_m r)}{a\,J_0(\beta_m a)} \times \left[\frac{-A(\beta_m,-\frac{h}{2})\cosh\left[\beta_m\left(z-\frac{h}{2}\right)\right]}{\cosh(h\beta_m)} + \left(\frac{-Q\,F(\beta_m)}{\beta_m\lambda} - \frac{1}{\beta_m}\frac{\partial A\left(\beta_m,\frac{h}{2}\right)}{\partial Z}\right) \times \frac{\sinh\left[\beta_m\left(z+\frac{h}{2}\right)\right]}{\cosh(\beta_m h)} + A(\beta_m, z)\right] \\ &+ \beta_m{}^2\,\sinh\left[\beta_m\left(z+\frac{h}{2}\right)\right]\left[2v\,\beta_m\,J_0(\beta_m r)\,C_m + \frac{J_1(\beta_m r)}{r}\,B_m\right] \\ &+ C_m\,\beta_m{}^2\,\frac{J_1(\beta_m r)}{r}\left[\sinh\left[\beta_m\left(z+\frac{h}{2}\right)\right] + \beta_m\left(z+\frac{h}{2}\right)\cosh\left[\beta_m\left(z+\frac{h}{2}\right)\right]\right]\Bigg\}\end{aligned} \tag{21}$$

$$\frac{\sigma_{zz}}{K} = 2G\sum_{m=1}^{\infty}\Bigg\{\frac{\sqrt{2}\,\beta_m{}^2\,J_0(\beta_m r)}{a\,J_0(\beta_m a)}$$

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$$\times\left[\begin{array}{c}\frac{-A\left(\beta_m,-\frac{h}{2}\right)\cosh\left[\beta_m\left(z-\frac{h}{2}\right)\right]}{\cosh(h\beta_m)}+\left(\frac{-Q\,F(\beta_m)}{\beta_m\lambda}-\frac{1}{\beta_m}\frac{\partial A\left(\beta_m,\frac{h}{2}\right)}{\partial Z}\right)\\ \times\frac{\sinh\left[\beta_m\left(z+\frac{h}{2}\right)\right]}{\cosh(\beta_m h)}+A\left(\beta_m,-\frac{h}{2}\right)e^{\beta_m\left(z+\frac{h}{2}\right)}\end{array}\right]$$

$$-\frac{\sqrt{2}J_0(\beta_m r)}{a\,J_0(\beta_m a)}\times\left[\begin{array}{c}\frac{-A\left(\beta_m,-\frac{h}{2}\right)\cosh\left[\beta_m\left(z-\frac{h}{2}\right)\right]}{\cosh(h\beta_m)}+\left(\frac{-Q\,F(\beta_m)}{\beta_m\lambda}-\frac{1}{\beta_m}\frac{\partial A\left(\beta_m,\frac{h}{2}\right)}{\partial Z}\right)\\ \times\frac{\sinh\left[\beta_m\left(z+\frac{h}{2}\right)\right]}{\cosh(\beta_m h)}+A(\beta_m,z)\end{array}\right]$$

$$-\beta_m{}^3\sinh\left[\beta_m\left(z+\frac{h}{2}\right)\right]\ [(1+2v)C_m+B_m]$$

$$\left.-C_m\ \beta_m{}^4\left(z+\frac{h}{2}\right)\cosh\left[\beta_m\left(z+\frac{h}{2}\right)\right]\right\} \qquad (22)$$

$$\frac{\sigma_{rz}}{K}=2G\sum_{m=1}^{\infty}\beta_m{}^2J_1(\beta_m r)\left\{\frac{-\sqrt{2}}{a\,J_0(\beta_m a)}\right.$$

$$\times\left[\begin{array}{c}\frac{-A\left(\beta_m,-\frac{h}{2}\right)\sinh\left[\beta_m\left(z-\frac{h}{2}\right)\right]}{\cosh(h\beta_m)}+\left(\frac{-Q\,F(\beta_m)}{\beta_m\lambda}-\frac{1}{\beta_m}\frac{\partial A\left(\beta_m,\frac{h}{2}\right)}{\partial Z}\right)\\ \times\frac{\cosh\left[\beta_m\left(z+\frac{h}{2}\right)\right]}{\cosh(\beta_m h)}+A\left(\beta_m,-\frac{h}{2}\right)e^{\beta_m\left(z+\frac{h}{2}\right)}\end{array}\right]$$

$$\left.+(2v\,C_m+B_m)\ \beta_m\cosh\left[\beta_m\left(z+\frac{h}{2}\right)\right]-C_m\,\beta_m{}^2\left(z+\frac{h}{2}\right)\sinh\left[\beta_m\left(z+\frac{h}{2}\right)\right]\right\} \qquad (23)$$

In order to satisfy condition Eq. (12), solving Eqs. (20), (22) and (23) for B_m and C_m one obtains

Let $C_m=0$ (24)

$$B_m=\frac{\sqrt{2}}{a\ \beta_m J_0(\beta_m a)}\left[\begin{array}{c}\frac{A\left(\beta_m,-\frac{h}{2}\right)\sinh(\beta_m h)}{\cosh(\beta_m h)}+\left(\frac{-Q\,F(\beta_m)}{\beta_m\lambda}-\frac{1}{\beta_m}\frac{\partial A\left(\beta_m,\frac{h}{2}\right)}{\partial Z}\right)\\ \times\frac{1}{\cosh(\beta_m h)}+A\left(\beta_m,-\frac{h}{2}\right)\end{array}\right] \qquad (25)$$

Special Case and Numerical Calculations

Setting

(1) f(r) $=\delta(r-r_0)$

$a=1m,\ h=0.25m,\ r_0=1m$

where $\delta(r)$ is well known diract delta function of argument r.

$$F(\beta_m)=\frac{\sqrt{2}}{a}\,r_0\,J_0(\beta_m r_0)$$

(2) $q=\delta(r-0.5)\,z$

$$\bar{q}=\int_{r'=0}^{a}r'\,\frac{\sqrt{2}}{a}\,\frac{J_0(\beta_m r)}{J_0(\beta_m a)}\,\delta(r-0.5)\,z\,dr'$$

$$=\frac{\sqrt{2}}{a}\,\frac{J_0(\beta_m 0.5)(0.5)}{J_0(\beta_m a)}\,z$$

Material properties

The numerical calculation has been carried out for aluminum (pure) circular plate with the material properties defined as

Thermal diffusivity α = 84.18× 10^{-6} m^2s^{-1},

Specific heat $c_\rho=896$ J/kg,

Thermal conductivity k = 204.2 W/m K,

Shear modulus $G=25.5\ G$ pa,

Poisson ratio $\vartheta=0.281$.

Roots of transcendental equation

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The $\beta_1 = 3.8317,\ \beta_2 = 7.0156,\ \beta_3 = 10.1735,\ \beta_4 = 13.3237,\ \beta_5 = 16.4706, \beta_6 = 19.6159$ are the roots of transcendental equation $J_0{}'(\beta_m a) = 0$. The numerical calculation and the graph has been carried out with the help of mathematical software Mat lab.

DISCUSSION

In this paper a thick circular plate is considered and determined the expressions for temperature and stresses due to internal heat generation within it and we compute the effects of internal heat generation in terms of stresses along radial direction by substituting $q = 0$ in Eqs. (17), (19), (20), (21), (22), (23), (24) and (25) and we compare the results for $q = 0$ and $q \neq 0$. As a special case mathematical model is constructed for considering aluminum (pure) circular plate with the material properties specified above.

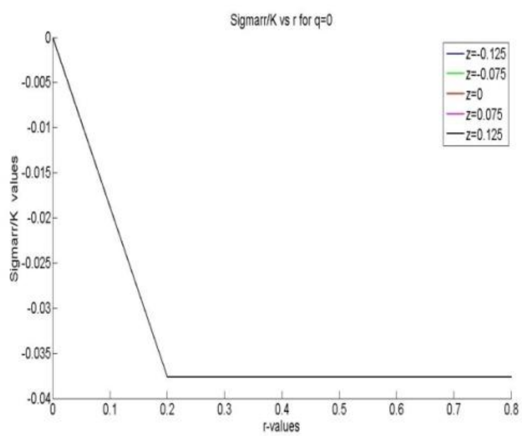

Figure 1: Radial stress function $\frac{\sigma_{rr}}{K}$ for (q = 0).

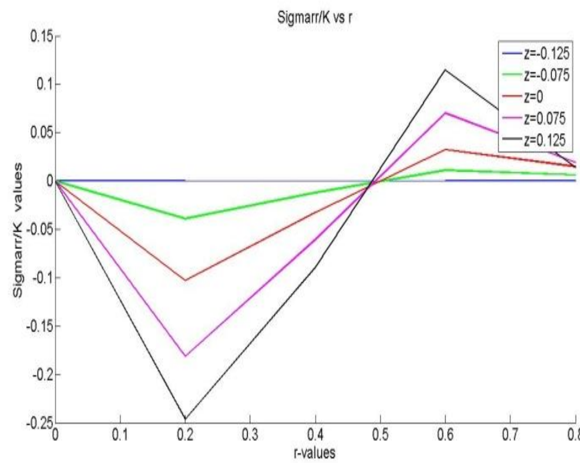

Figure 2: Radial stress function $\frac{\sigma_{rr}}{K}$ for ($q \neq 0$).

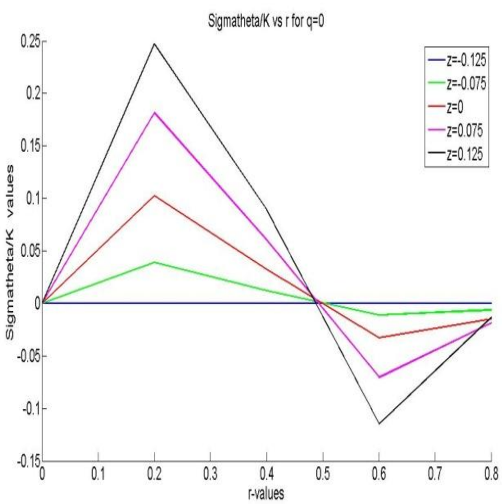

Figure 3: Angular stress function $\frac{\sigma_{\theta\theta}}{K}$ for (q = 0).

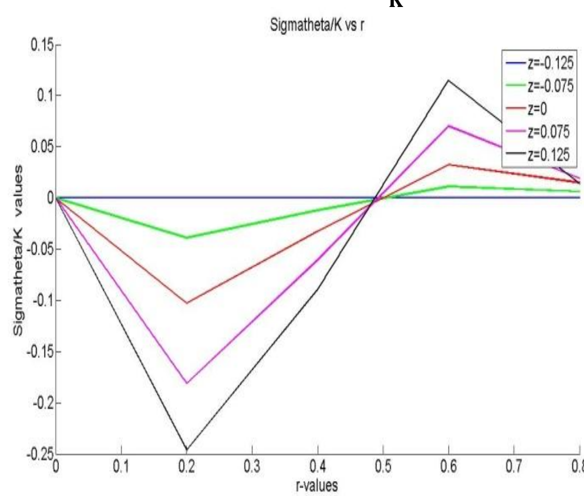

Figure 4: Angular stress function $\frac{\sigma_{\theta\theta}}{K}$ for ($q \neq 0$).

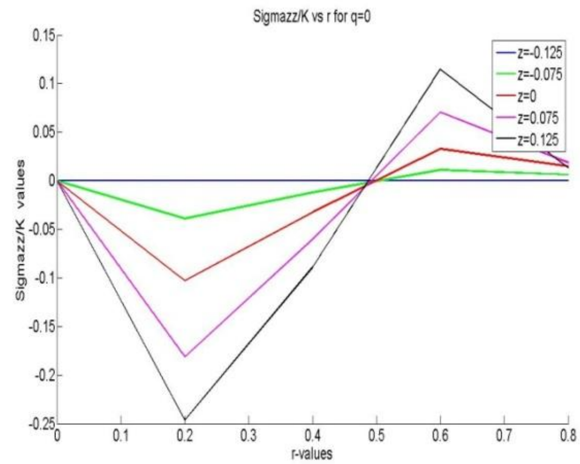

Figure 5: Axial stress function $\frac{\sigma_{zz}}{K}$ for (q = 0).

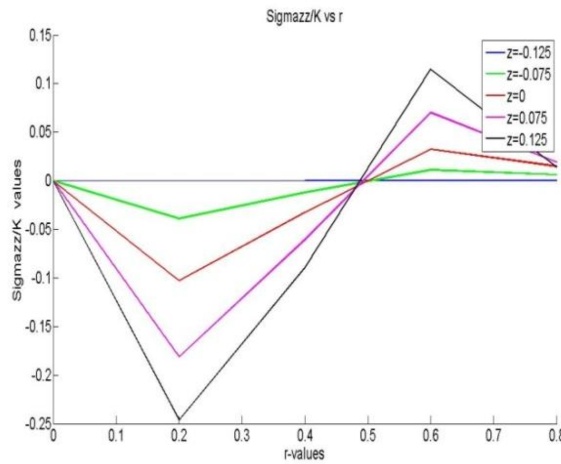

Figure 6: Axial stress function $\frac{\sigma_{zz}}{K}$ for ($q \neq 0$).

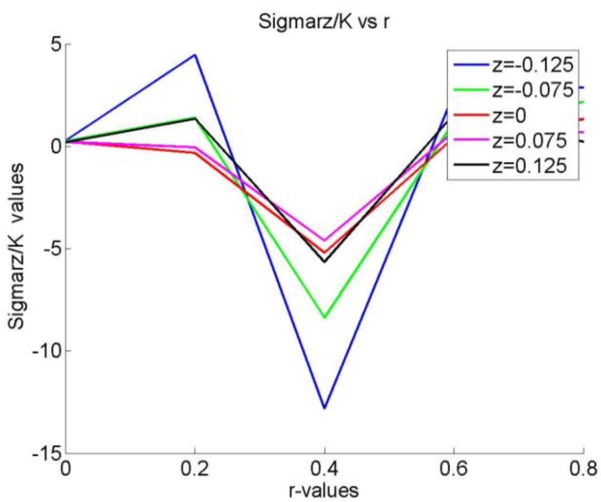

Figure 7: Stress function $\frac{\sigma_{rz}}{K}$ for (q=0).

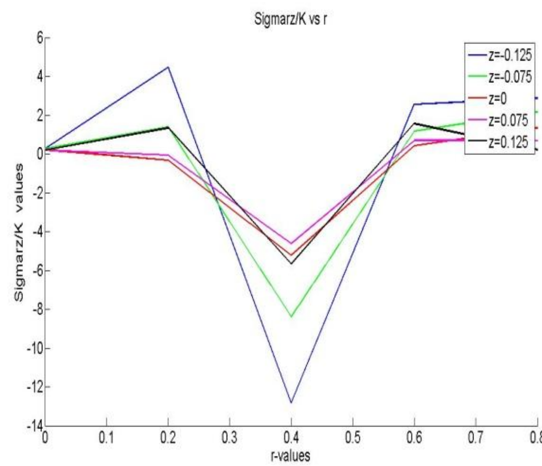

Figure 8: Stress function $\frac{\sigma_{rz}}{K}$ for ($q \neq$ 0).

From figure 1and 2, it is observed that the radial stress function $\frac{\sigma_{rr}}{K}$ develops compressive stress for (q= 0) in the radial direction whereas due to internal heat generation the radial stress function $\frac{\sigma_{rr}}{K}$ develops tensile stress in the radial direction.

From figure 3 and 4, it is observed that the angular stress function $\frac{\sigma_{\theta\theta}}{K}$ develops compressive stress for (q=0) in the radial direction whereas due to internal heat generation the angular stress function $\frac{\sigma_{\theta\theta}}{K}$ develops tensile stress in the radial direction.

From figure 5 and 6, it is observed that the axial stress function $\frac{\sigma_{zz}}{K}$ develops tensile stress for (q = 0) and ($q \neq$ 0) in the radial direction.

From figure 7 and 8, it is observed that the stress function $\frac{\sigma_{rz}}{K}$ develops tensile stress for (q = 0) and ($q \neq 0$) in the radial direction.

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CONCLUSION

We can conclude that due to internal heat generation in thick circular plate the radial stresses and the angular stresses are tensile. Also, it can be observed that there is no effect of internal heat generation on axial stress function $\frac{\sigma_{zz}}{K}$ and stress function $\frac{\sigma_{rz}}{K}$ in the radial direction.

The results obtained here are useful in engineering problems particularly in the determination of state of stress in a thick circular plate and base of furnace of boiler of a thermal power plant and gas power plant.

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