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Research Article

A STUDY ON MULTIPLE IMPROPER INTEGRALS USING MAPLE

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ABSTRACT

This paper takes the mathematical software Maple as the auxiliary tool to study two types of multiple improper integrals. We can obtain the closed forms of these two types of multiple improper integrals using differentiation with respect to a parameter and Leibniz differential rule. On the other hand, we propose some examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

Key Words: *Multiple Improper Integrals, Differentiation With Respect To A Parameter, Leibniz Differential Rule, Maple*

INTRODUCTION

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Beethoven, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. For the instructions and operations of Maple, we can refer to Abell and Braselton (2005), Dodson and Gonzalez (1995), Richards (2002), Stroeker and Kaashoek (1999).

In this paper, we study the multiple improper integral problems. This problem is closely related with probability theory and quantum field theory, and can refer to Ryder (1996), Streit (1970). For this reason, the evaluation and numerical calculation of multiple integrals is important. In this study, we mainly evaluate the following two types of n -tuple improper integrals

$$\int_{r_n}^{\infty}\cdots\int_{r_1}^{\infty}(x_1+\cdots+x_n)^m \exp[a(x_1+\cdots+x_n)]dx_1\cdots dx_n \tag{1}$$

$$\int_{s_n}^{\infty}\cdots\int_{s_1}^{\infty}(y_1\times\cdots\times y_n)^{a-1}[\ln(y_1\times\cdots\times y_n)]^m dy_1\cdots dy_n \tag{2}$$

Where m,n are positive integers, $a,r_1,\cdots,r_n,s_1,\cdots,s_n$ are real numbers, $a<0$, $s_1,\cdots,s_n>0$. We can obtain the closed forms of these two types of multiple improper integrals by using differentiation with respect to a parameter and Leibniz differential rule ; these are the major results of this paper (i.e., Theorems 1 and 2). As for the study of related multiple improper integrals can refer to Yu (2013a, 2013b, 2013c, 2013d, 2013e, 2012a, 2012b). On the other hand, we provide some examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

Main Results

Firstly, we introduce some notations used in this paper.

Notations

(i) The p -th order derivative of the function $u(x)$ is denoted by $u^{(p)}(x)$, where p is a non-negative integer.

(ii) Suppose s,t are real numbers, we define $(s)_t = s(s-1)\cdots(s-t+1)$, and $(s)_0 = 1$.

Next, we introduce two important theorems used in this study.

Differentiation with Respect to a Parameter (Lang, 1983)

Suppose $I=[\beta_1,\infty)\times[\beta_2,\infty)\times\cdots\times[\beta_n,\infty)\times[a_1,a_2]$, and the multivariable function $g(x_1,x_2,\cdots,x_n,a)$ is defined on I. Assume $g(x_1,x_2,\cdots,x_n,a)$ and its partial derivative $\frac{\partial g}{\partial a}(x_1,x_2,\cdots,x_n,a)$ is continuous function on I . If $\int_{\beta_n}^{\infty}\cdots\int_{\beta_1}^{\infty} g(x_1,\cdots,x_n,a)dx_1\cdots dx_n$ are convergent for all $a\in[a_1,a_2]$ and $\int_{\beta_n}^{\infty}\cdots\int_{\beta_1}^{\infty}\frac{\partial g}{\partial a}(x_1,\cdots,x_n,a)dx_1\cdots dx_n$ is uniformly convergent on $[a_1,a_2]$. Then

$$G(a)=\int_{\beta_n}^{\infty}\cdots\int_{\beta_1}^{\infty} g(x_1,\cdots,x_n,a)dx_1\cdots dx_n$$

is differentiable on the open interval (a_1,a_2), and its derivative

$$\frac{d}{da}G(a)=\int_{\beta_n}^{\infty}\cdots\int_{\beta_1}^{\infty}\frac{\partial g}{\partial a}(x_1,\cdots,x_n,a)dx_1\cdots dx_n$$

Leibniz Differential Rule (Apostol, 1975)

Suppose m is a non-negative integer, $f(x)$ and $g(x)$ are m-times differentiable functions. Then the m-th order derivative of the product function $f(x)\cdot g(x)$,

$$(f\cdot g)^{(m)}=\sum_{k=0}^{m}\binom{m}{k}f^{(k)}g^{(m-k)}$$

The following is the first major result in this study we obtain the closed form of the multiple improper integral (1).

Theorem 1: Suppose m,n are positive integers, $a,r_1,\cdots,r_n$ are real numbers, $a<0$. Then the n-tuple multiple improper integral

$$\int_{r_n}^{\infty}\cdots\int_{r_1}^{\infty}(x_1+\cdots+x_n)^m\exp[a(x_1+\cdots+x_n)]dx_1\cdots dx_n$$

$$=(-1)^n\exp[a(r_1+r_2+\cdots+r_n)]\sum_{k=0}^{m}\binom{m}{k}\frac{(-n)_k}{a^{n+k}}(r_1+r_2+\cdots+r_n)^{m-k} \quad (3)$$

Proof: Because

$$\int_{r_n}^{\infty}\cdots\int_{r_1}^{\infty}\exp[a(x_1+\cdots+x_n)]dx_1\cdots dx_n$$

$$=\int_{r_n}^{\infty}\cdots\int_{r_1}^{\infty}(\exp ax_1\times\exp ax_2\times\cdots\times\exp ax_n)dx_1\cdots dx_n$$

$$=\left(\int_{r_1}^{\infty}\exp ax_1 dx_1\right)\left(\int_{r_2}^{\infty}\exp ax_2 dx_2\right)\cdots\left(\int_{r_n}^{\infty}\exp ax_n dx_n\right)$$

$$=\left(\frac{-1}{a}\exp ar_1\right)\left(\frac{-1}{a}\exp ar_2\right)\cdots\left(\frac{-1}{a}\exp ar_n\right)$$

$$=(-1)^n\frac{1}{a^n}\cdot\exp[a(r_1+r_2+\cdots+r_n)] \quad (4)$$

By differentiation term by term and Leibniz differential rule, differentiating m times with respect to a on both sides of (4), we obtain

$$\int_{r_n}^{\infty}\cdots\int_{r_1}^{\infty}(x_1+\cdots+x_n)^m\exp[a(x_1+\cdots+x_n)]dx_1\cdots dx_n$$

$$=(-1)^n\sum_{k=0}^{m}\binom{m}{k}\left(\frac{1}{a^n}\right)^k[\exp a(r_1+r_2+\cdots+r_n)]^{(m-k)}$$

$$=(-1)^n\exp[a(r_1+r_2+\cdots+r_n)]\sum_{k=0}^{m}\binom{m}{k}\frac{(-n)_k}{a^{n+k}}(r_1+r_2+\cdots+r_n)^{m-k}$$

Next, we derive the second major result in this paper. We determine the closed form of the multiple improper integrals (2).

Theorem 2: Suppose m,n are positive integers, $a,s_1,\cdots,s_n$ are real numbers, $a<0$, $s_1,\cdots,s_n>0$. Then the n -tuple multiple improper integral

$$\int_{s_n}^{\infty}\cdots\int_{s_1}^{\infty}(y_1\times\cdots\times y_n)^{a-1}[\ln(y_1\times\cdots\times y_n)]^m dy_1\cdots dy_n$$

$$=(-1)^n(s_1\times s_2\times\cdots\times s_n)^a\sum_{k=0}^{m}\binom{m}{k}\frac{(-n)_k}{a^{n+k}}[\ln(s_1\times s_2\times\cdots\times s_n)]^{m-k} \tag{5}$$

Proof: In (3) of Theorem 1, taking $x_j=\ln y_j$ for all $j=1,\ldots,n$. Then we obtain

$$\int_{e^{r_n}}^{\infty}\cdots\int_{e^{r_1}}^{\infty}(y_1\times\cdots\times y_n)^{a-1}[\ln(y_1\times\cdots\times y_n)]^m dy_1\cdots dy_n$$

$$=(-1)^n\exp[a(r_1+r_2+\cdots+r_n)]\sum_{k=0}^{m}\binom{m}{k}\frac{(-n)_k}{a^{n+k}}(r_1+r_2+\cdots+r_n)^{m-k} \tag{6}$$

Let $r_j=\ln s_j$ for all $j=1,\ldots,n$. Then

$$\int_{s_n}^{\infty}\cdots\int_{s_1}^{\infty}(y_1\times\cdots\times y_n)^{a-1}[\ln(y_1\times\cdots\times y_n)]^m dy_1\cdots dy_n$$

$$=(-1)^n(s_1\times s_2\times\cdots\times s_n)^a\sum_{k=0}^{m}\binom{m}{k}\frac{(-n)_k}{a^{n+k}}[\ln(s_1\times s_2\times\cdots\times s_n)]^{m-k}$$

EXAMPLES

In the following, for the two types of multiple improper integrals in this study, we provide some examples and use Theorems 1, 2 to determine their closed forms. On the other hand, we employ Maple to calculate the approximations of these multiple improper integrals and their solutions.

Example 1: By Theorem 1, we obtain the following double improper integral

$$\int_{-4}^{\infty}\int_{2}^{\infty}(x_1+x_2)^8\exp[-3(x_1+x_2)]dx_1dx_2=e^6\cdot\sum_{k=0}^{8}\binom{8}{k}\frac{(-2)_k}{(-3)^{2+k}}(-2)^{8-k} \tag{7}$$

In the following, we use Maple to determine the approximations of this double improper integral and its closed form.

>evalf(Doubleint((x1+x2)^8*exp(-3*x1-3*x2),x1=2..infinity,x2=-4..infinity),14);

1865.3278279829

>evalf(exp(6)*sum(8!/(k!*(8-k)!)*product(-2-j,j=0..(k-1))/(-3)^(2+k)*(-2)^(8-k),k=0..8),14);

1865.3278279829

Example 2: Again by Theorem 1, we obtain the closed form of the following triple improper integral

$$\int_{2}^{\infty}\int_{-5}^{\infty}\int_{-3}^{\infty}(x_1+x_2+x_3)^9\exp[-4(x_1+x_2+x_3)]dx_1dx_2dx_3=-e^{24}\cdot\sum_{k=0}^{9}\binom{9}{k}\frac{(-3)_k}{(-4)^{3+k}}(-6)^{9-k} \tag{8}$$

Using Maple to calculate the approximations of both sides of (8) as follows:

>evalf(Tripleint((x1+x2+x3)^9*exp(-4*x1-4*x2-4*x3),x1=-3..infinity,x2=-5..infinity,x3=2..infinity),14);

$-1.5217661051433\cdot 10^{15}$

>evalf(-exp(24)*sum(9!/(k!*(9-k)!)*product(-3-j,j=0..(k-1))/(-4)^(3+k)*(-6)^(9-k),k=0..9),14);

$-1.5217661051433\cdot 10^{15}$

Example 3: Using Theorem 2, we can determine the following double improper integral

$$\int_{3}^{\infty}\int_{4}^{\infty}(y_1y_2)^{-3}[\ln(y_1y_2)]^6dy_1dy_2=\frac{1}{144}\sum_{k=0}^{6}\binom{6}{k}\frac{(-2)_k}{(-2)^{2+k}}(\ln 12)^{6-k} \tag{9}$$

Calculating the approximations of both sides of (9) as follows:

>evalf(Doubleint((y1*y2)^(-3)*(ln(y1*y2))^6,y1=4..infinity,y2=3..infinity),14);

6.4084366609725

>evalf(1/144*sum(6!/(k!*(6-k)!)*product(-2-j,j=0..(k-1))/(-2)^(2+k)*(ln(12))^(6-k),k=0..6),14);

6.4084366609726

Example 4: Also by Theorem 2, we obtain the closed form of the following triple improper integral

$$\int_{7}^{\infty}\int_{2}^{\infty}\int_{5}^{\infty}(y_1y_2y_3)^{-4}[\ln(y_1y_2y_3)]^9dy_1dy_2dy_3=\frac{-1}{70^3}\sum_{k=0}^{9}\binom{9}{k}\frac{(-3)_k}{(-3)^{3+k}}(\ln 70)^{9-k} \tag{10}$$

We also use Maple to obtain the approximations of both sides of (10).

>evalf(Tripleint((y1*y2*y3)^(-4)*(ln(y1*y2*y3))^9,y1=5..infinity,y2=2..infinity,y3=7..infinity),14);

0.55582738195151

>evalf(-1/70^3*sum(9!/(k!*(9-k)!)*product(-3-j,j=0..(k-1))/(-3)^(3+k)*(ln(70))^(9-k),k=0..9),14);

0.55582738195151

CONCLUSION

As mentioned, the multiple improper integral problems are important in probability theory and quantum field theory. In this study, we propose a new technique to solve two types of multiple improper integrals, and we hope this method can be applied in mathematical statistics or quantum physics. On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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